Graph Theory: Problem Set 7[1]

March 8 (2018)

First Mid Term Exam: 6th March
Room M2 (10:00-12:00) – Pages 1–46 (up to Ramsey)

1 Consider the complete bipartite $G = K_{5,5}$ together with $\alpha\colon E_G \to \{\text{red,blue}\}$. Show that there is a monochromatic $K_{2,2}$.

2 Let $p, q \geq 3$, and suppose that both $R(p, q-1)$ and $R(p-1, q)$ are even. Show that

$$R(p,q) \leq R(p,q-1) + R(p-1,q) - 1\,.$$

3 Determine $R_2(C_4)$.

4 Let us recall first how the simple **greedy colouring** works. Let $v_1, v_2, \ldots, v_n$ be an ordering of V_G. In order $i = 1, 2, \ldots, n$, assign

$$\alpha(v_i) = \min\{k \mid \alpha(v_j) \neq k \ \text{ for all } v_j \in N_G(v_i) \text{ with } j < i\}.$$

(a) Show that for each $n \geq 5$, there is a graph G together with an ordering $v_1, v_2, \ldots, v_n$ such that $\alpha(v_i) > \chi(G)$ for some i.

(b) Show that for all graphs G there exists an ordering $v_1, v_2, \ldots, v_n$ with $\alpha(v_i) \leq \chi(G)$ for all i.

Remark. As a special case we can prove (but won't) that *every greedy colouring is optimal for graphs that avoid subgraphs P_4.*

5 Let $\alpha\colon V_G \to [1,k]$ of G be a proper colouring. For a subset $S \subseteq G$, let $\alpha(S) = \{\alpha(u) \mid u \in S\}$. Assume $\chi(G) = k$, and $1 \leq i \leq k$. Show that there exists a vertex $v \in G$ with $\alpha(v) = i$ such that

$$\alpha(N_G(v)) = [1,k] \setminus \{i\}.$$

Remark. As a special case we can prove (but won't) that every greedy colouring is optimal for graphs that avoid P_4 as a subgraph.

6 Show that if $\chi(G) > 5$, then G has two cycles that have no common vertices.

[1] With a Ramsey result for trees.

Theorem (CHVÁTAL (1977)) *Let T be a tree of order m. Then*

$$R(T, K_n) = (m-1)(n-1) + 1 \,.$$

Proof. To see that $R(T, K_n) > (m-1)(n-1)$, let $G = K_{(n-1)(m-1)}$ of order $(n-1)(m-1)$. Choose any subgraphs K_{m-1} such that the vertices of these partition the vertex set. Let $\alpha(e) = 1$ for the edges of these, and $\alpha(e) = 2$ for the rest of the edges. It is clear that there is no 1-monochromatic tree T of order m. On the other hand, the edges coloured by 2 form a $(n-1)$-bipartite graph, and hence there are no 2-monochromatic K_n.

For the claim $R(T, K_n) \le (m-1)(n-1) + 1$, we first state

Claim $(*)$**.** Let T be a tree of order m. Then any graph G with $\delta(G) \ge m-1$ has a subgraph isomorphic to T.

For the rest of the claim, we use induction. The claim is trivial for $n = 1$. Let α be a 2-edge colouring of $K_{(m-1)(n-1)+1}$. If a vertex v is an end of more than $(m-1)(n-2)$ edges of colour 2, then, by the induction hypothesis, the other ends ($\neq v$) induce a subgraph that has a 1-monochromatic T or a 2-monochromatic K_{n-1}. The latter implies a 2-monochromatic K_n (when v is added).
So suppose all vertices are ends of at most $(m-1)(n-2)$ edges of colour 2. This means that every vertex is an end of at least

$$(m-1)(n-1) + 1 - 1 - (m-1)(n-2) = m-1$$

edges of colour 1. Now the graph G consisting of the edges of colour 1, has $\delta(G) \ge m-1$, and, by the claim $(*)$, it has a subgraph isomorphic to T. This proves the claim.