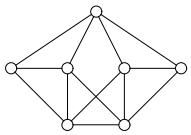
Graph Theory: Problem Set 8

March 15 (2018)

1 **(a)** Find all 3-critical graphs.

(b) Is the next graph critical?



2 Show that a 4-critical graph is either a wheel of even order, or it does not contain any wheels (even or odd) as subgraphs.

3 Compute the chromatic polynomial of the cycle C_n, for $n \geq 3$.

4 **(a)** Let $E_G = E_1 \cup E_2$ be a partition of the edge set of G, and write $G_i = G[E_i]$ for $i = 1, 2$. Show that

$$\chi(G) \leq \chi(G_1)\chi(G_2).$$

(b) A **block** of a graph G is a maximal 2-connected subgraph of G. Show that

$$\chi(G) = \max\{\chi(B) \mid B \text{ a block of } G\}.$$

1-2-3 Conjecture.[1] Let G be a connected graph of $n \geq 3$ vertices. Then there exists a weighting of the edges $\beta : E_G \to \{1, 2, 3\}$ such that the function $\alpha : V_G \to \mathbb{N}$ defined by

$$\alpha(v) = \sum_{uv \in G} \beta(uv)$$

is a proper vertex colouring.

5 Find a 5-regular planar graph.

6 **(a)** Let G be a triangle-free planar graph. Show that $\delta(G) \leq 3$.

(b) Does there exist a planar bipartite graph for which $\delta(G) = 3$?

Remark. Some simple looking results in graph theory lack short combinatorial proofs. The following result was proved by GRAHAM AND POLLAK (1971) using linear algebra.

Theorem. *Let $\alpha : E_{K_n} \to [1, k]$ be any edge colouring of K_n such that each subgraph $K_n[i]$ (of colour i) is a complete bipartite graph. Then $k \geq n - 1$.*

[1] M. Karoński, T. Łuczak, A. Thomason, *Edge weights and vertex colors*, J. Combin. Theory Ser. B, 91 (2004) 151–157. (The conjecture is true if the number of colours is 5.)