Graph Theory: Problem Set 9

March 22 (2018)

1 Let G be a maximal planar graph with $\nu_G \geq 4$, and denote $n_k = \#\{v \mid d_G(v) = k\}$, for all k. Show that

$$\sum_{k\geq 2}(6-k)n_k = 12\,.$$

2 Show that if $\nu_G \geq 11$, then G or its complement $\overline{G}$ is not planar.

Remark. HARARY, BATTLE AND KODOMA, and TUTTE (1962) showed that *if* $\nu_G \geq 9$*, then* G *or its complement* $\overline{G}$ *is not planar.* This seems to lack an elegant proof. (The result was obtained by looking through all graphs of order 9.)

3 For which values $k \geq 2$ is the complement $\overline{Q}_k$ of the k-cube planar?

4 The **girth** $\gamma(G)$ of a graph G is the length of a shortest cycle in G.

(a) Let G be a connected planar graph with $\gamma(G) \geq 3$. Show that

$$\varepsilon_G \leq \frac{\gamma(G)}{\gamma(G)-2}\cdot(\nu_G - 2).$$

(b) Let G be the Petersen graph. Show that $G{-}e$ is not planar for all edges $e \in G$.

5 Assume that G is planar **uniquely** 4**-colourable**, i.e., if $\alpha, \beta\colon V_G \to [1,4]$ are any two proper k-colourings of G, then there is a permutation π of $[1,4]$ of the colours such that $\beta = \pi\alpha$. Show that G is maximal planar.

6 Suppose G is a maximal planar graph having at least three vertices, and, moreover, $\chi(G) \leq 3$. Show that G is eulerian.