MONOTONE SUBSEQUENCES

Let $\sigma = (k_1, k_2, \ldots, k_{n^2+1})$ be a linear ordering of $\{1, \ldots, n^2+1\}$.
Claim: *σ contains a monotonic subsequence of length $n+1$.*

Example:

- $n = 3$ and $N = n^2 + 1 = 10$:

$$\sigma = (9, 5, 1, 7, 10, 6, 4, 3, 2, 8)$$

has no ascending subsequences of length 4, but has a descending subsequence of length 6.

$$\sigma = (k_1, k_2, \ldots, k_{n^2+1})$$

Let $P = \{(i, k_i) \mid i = 1, 2, \ldots, n^2 + 1\}$ with the pointwise order

$$(i, k_i) \leq_P (j, k_j) \iff i \leq j \text{ and } k_i \leq k_j\,.$$

A subsequence σ' of σ is ascending
$\iff$ σ' corresponds to a chain in P.

Thm 1.56: P has a partition to $w(P)$ chains.
$\implies$ σ has a partition of $w(P)$ ascending subsequences.
$\implies$ σ has an ascending subsequence of length

$$\geq (n^2 + 1)/w(P)$$

Suppose $(n^2+1)/w(P) \leq n$

(i.e., no ascending chains of length $n+1$)

then $w(P) \geq (n^2+1)/n \geq n + 1/n$, and so $w(P) \geq n+1$.

Hence there is an antichain A of size $t = w(P) \geq n+1$:

$$A = \{\ (i_1, k_{i_1}), (i_2, k_{i_2}), \ldots, (i_t, k_{i_t})\ \}$$

with $i_1 < i_2 < \ldots < i_t$ and so

$$k_{i_1} > k_{i_2} > \ldots > k_{i_t}$$

is descending.

SPERNER

Consider the poset $(2^N, \subseteq)$ for $N = \{1, 2, \ldots, n\}$.

Claim: *If* $\{A_1, A_2, \ldots, A_m\} \subseteq 2^N$ *is an antichain (i.e.,* $A_i \nsubseteq A_j$*, for all* $i \neq j$*), then*

$$m \leq \binom{n}{\lfloor n/2 \rfloor}.$$

Proof: Counting arguments on maximal chains in 2^N.

MARRIAGE THEOREM

Let

$$N = \{1, 2, \ldots, n\}.$$

Let $\mathscr{S} = \{S_1, S_2, \ldots, S_n\} \subseteq 2^X$ for a finite set $X = \bigcup_{i=1}^n S_i$.

A function $\sigma : N \to X$ is a distinct representative function (DRF) of $\mathscr{S}$ if for all i there exists j with $\sigma(j) \in S_i$ and $\sigma(j) \neq \sigma(k)$ for all $j \neq k$.

For an index set $I \subseteq N$, denote

$$S(I) = \bigcup_{i \in I} S_i .$$

Claim. *$\mathscr{S}$ has a DRF $\iff$ for all $I \subseteq N$,*

$$|S(I)| \geq |I| . \tag{*}$$

If $|I| < |S(I)|$ for some I, clearly no DRF can exist.

SUFFICIENCY

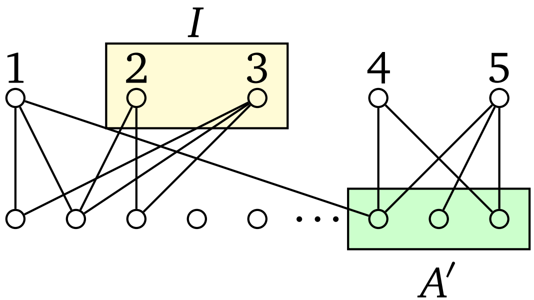
Define a partial order

$$x \le i \iff x \in S_i \,.$$

on $P = N \cup X$. Then $h(P) = 2$.

Let $A = I \cup A'$ be an *antichain* in P with $|A| = w(P)$, where $I \subseteq N$ and $A' \subseteq X$.

Thus $S(I) = \bigcup_{i \in I} S_i \subseteq X \setminus A'$.



Hence, by the condition $|S(I)| \geq |I|$,

$$|X| - |A'| \geq |S(I)| \geq |I|.$$

Since $A = I \cup A'$, we have $|A| \leq |X|$.

Now $w(P) = |X|$, since X is an antichain.

By Dilworth's theorem, P has a partition to $w(P)$ chains.
Each chain has two elements (from X to I), and
this gives a matching of $N = \{1, 2, \ldots, n\} \mapsto X$.

The matching is a DRF.