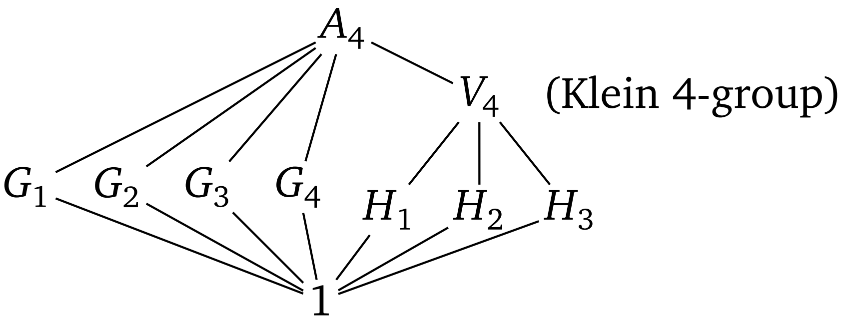
EXAMPLES

Chomp is a more than 50 years old game for two persons. There are simple starting posets for which the strategies of the game are unknown.

- Given a poset P with the minimum element 0.
- A move consists of picking an element $x \in P$ and removing all $y \geq_P x$.
- If you pick 0, you lose.

Page 8: SUBGROUPS

- The subgroup relation is a partial order.
  (The poset is a lattice.)



Alternating group A_4: $G_i \cong C_3$ and $H_i \cong C_2$

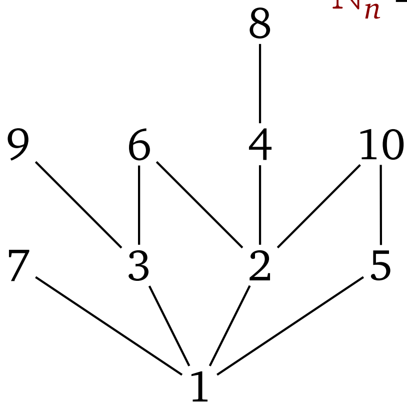
- The *normal subgroup relation* $\trianglelefteq$ is not a partial order in general.

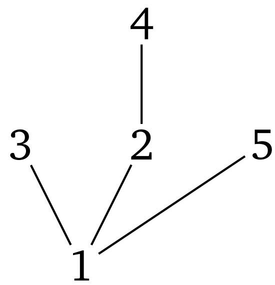
DIVISIBILITY POSET

$$T = (\mathbb{N} \setminus \{0\}, \mid)$$

Among finite subposets of T are the divisibility posets

$$\mathbb{N}_n = (\{1, 2, \ldots, n\}, \mid)$$

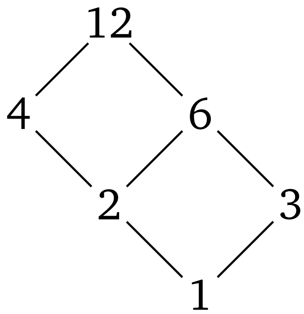




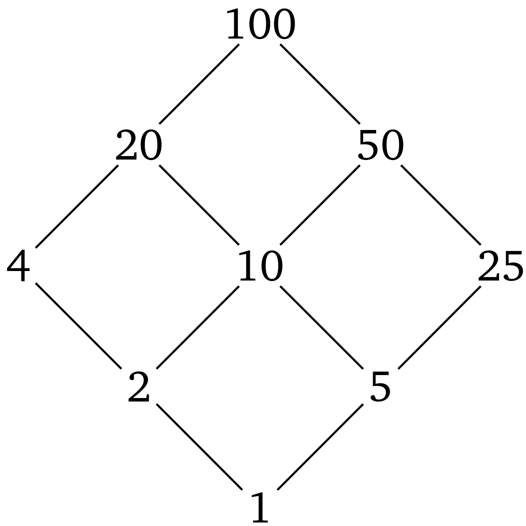
DIVISOR POSET

A subposet of $\mathbb{N}_n$: $T_n = \{m : m|n\}$

T_{12}:



T_{100}



Exercise. Draw T_{120}. More complicated that one.

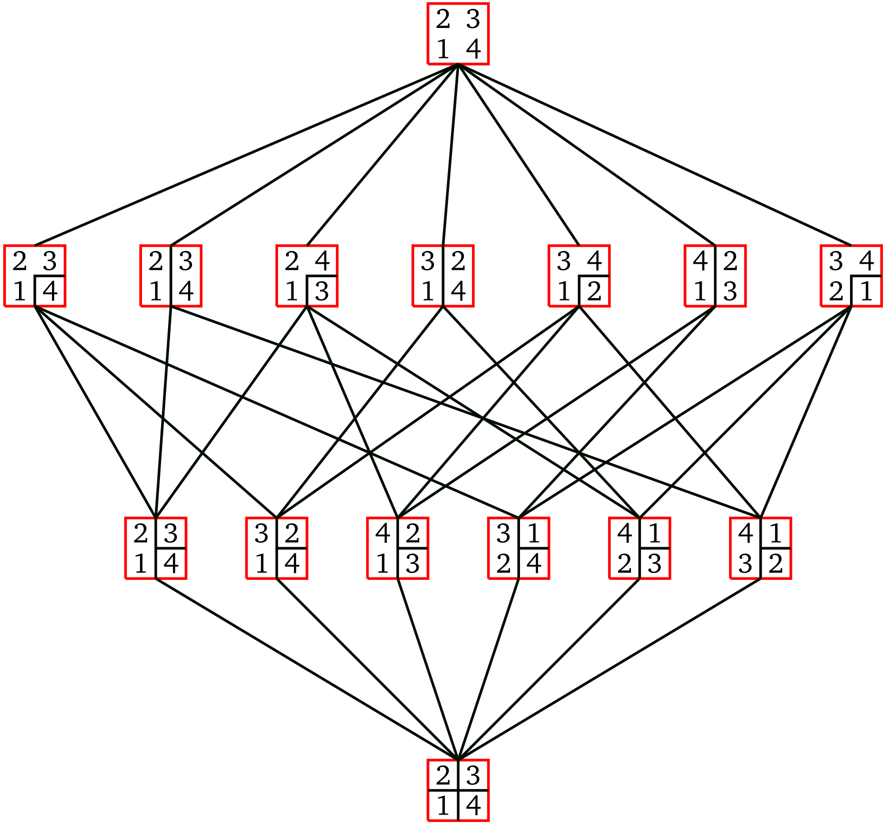
PARTITIONS

- The set $\Pi(X)$ of all partitions of X forms a poset under inclusion (partition order):

  $$\pi_1 \leq \pi_2 \text{ if each block of } \pi_2 \text{ is a union of blocks of } \pi_1$$

  i.e., each block of π_1 is contained in some block of π_2.

- The poset $\Pi(X)$ is related to the poset of equivalence relations $\mathrm{Eq}(X)$ on X forms a poset under inclusion.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4 2 / 1 3
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4 1 / 2 3
4 1 / 3 2
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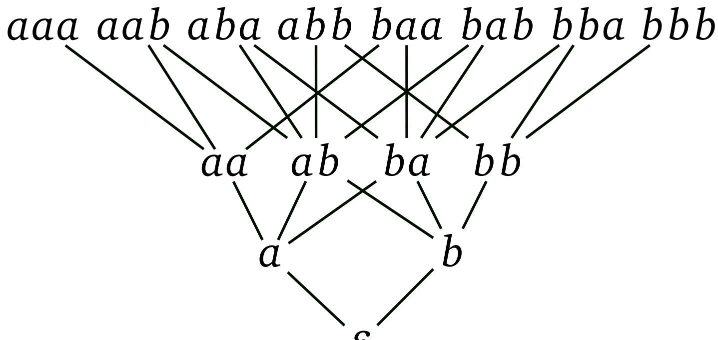
WORDS

- Let Σ be a set of symbols, called an alphabet. Each sequence of symbols $a_1a_2\cdots a_n$, for $n \geq 1$, is a word over Σ.
- Σ^* denotes the set of all words over Σ.
- A word u is a factor of v, $u \trianglelefteq v$, if $v = v_1uv_2$, where v_1 and v_2 can be empty. Then $(\Sigma^*, \trianglelefteq)$ is the factor poset of Σ^*.



LEXICOGRAPHIC POSETS

- Let P be a poset. Consider the set of P^* of all words over it. Even if P is finite, P^* is always infinite.
- The lexicographic order $\leq_P^\ell$ on P^* is defined as follows:

$$u \leq_P^\ell v \iff v = uw \text{ for some } w, \quad \text{or}$$
$$u = wau' \text{ and } v = wbv' \text{ where } a <_P b\,.$$

  Moreover, we put $\varepsilon \leq_P^\ell u$ for all $u \in P^*$.
- It has no maximal elements.
  The empty word is the bottom element.
  If P is linearly ordered, so is P^*.

For instance, when $a <^\ell b$ then $aababb <^\ell abab$.

SUBSEQUENCES

$P = \Sigma^*$ is a poset under the subsequence order:

$$\left.\begin{array}{l} x = x_1 x_2 \cdots x_n \\ y = y_1 x_1 y_2 x_2 \cdots y_n x_n y_{n+1} \end{array}\right\} \Longrightarrow x \leq_P^* y\,.$$

For instance, in $P = \{a, b\}^*$,

$$aab \leq_P^* abaaba$$

CODINGS

Let $\Sigma = \{-, +\}$ be a binary alphabet. If P is a poset then each sequence $s = (x_0, \ldots, x_n)$, where $x_i \bowtie x_{i+1}$, can be associated with a word $\sigma(s) = (\sigma_0, \ldots, \sigma_{n-1})$ where

$$\sigma_i = \begin{cases} + & \text{if } x_i \leq_P x_{i+1}, \\ - & \text{if } x_i \geq_P x_{i+1}. \end{cases}$$