WELL QUASI-ORDERS

A quasi-order (reflexive and transitive) P is a well quasi-order (WQO) if every infinite sequence $x_1, x_2, \dots$ has $x_n \leq_P x_m$ for some $n < m$.

Theorem. A quasi-order P is a WQO iff each infinite sequence $x_1, x_2, \dots$ has an infinite ascending subsequence $x_{i_1} \leq_P x_{i_2} \leq_P \dots$

Proof. As in Theorem 1.40.

THEOREM

If P and Q are WQOs, so is the direct product $P \times Q$.

Proof. Consider an infinite sequence $(x_1, y_1), (x_2, y_2) \ldots$ in $P \times Q$. Hence there exists an ascending chain $x_{i_1} \leq_P x_{i_2} \leq_P \ldots$. Applying the well quasi-order Q to the sequence $y_{i_1}, y_{i_2}, \ldots$, we obtain an infinite ascending subsequence $y_{j_1} \leq_Q y_{j_2} \leq_Q \ldots$ of this one. Hence

$$(x_{j_1}, y_{j_1}) \leq_{P \times Q} (x_{j_2}, y_{j_2}) \leq_{P \times Q} \ldots$$

is as required.

DICKSON'S THEOREM (1913)

Let $n \geq 1$ and let $x_1, x_2, \ldots$ be an infinite sequence from $\mathbb{N}^n$.
Then $x_i \leq x_j$ for some $i < j$.

Proof Done!

With a little bit of work we can derive Hilbert's basis theorem.
And then you go to Gröbner bases, and possibly never come back.