Ordered Sets

Problem Set 1 (Jan 16, 2015)

1 (a) Let $P = (X, R)$ be a poset. Show that also (X, R^{-1}) is a poset.

(b) Let (X, R) and (X, S) be posets. Is the union $(X, R \cup S)$ necessarily a poset?

2 Let S_n denote the set of all permutations (bijections) α on $\{1, 2, \ldots, n\}$. A pair (i, j) is an **inversion** in $\alpha \in S_n$ if $i < j$ and $\alpha(i) > \alpha(j)$. For instance, let

$$\alpha = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 1 & 4 \end{pmatrix} \in S_4$$

which means that $\alpha(1) = 2$, $\alpha(2) = 3$ and so forth. The inversions of α are $(1, 2)$ and $(1, 3)$. Define the relation $\leq$ on S_n by setting $\alpha \leq \beta$ if and only if all inversions of α are inversions of β. Show that $\leq$ is a partial order on S_n.

3 Let X be a finite set of n elements. Count the number of different
(a) relations $R \subseteq X \times X$, (b) reflexive relations on X.

4 Let P be a poset, and denote $\{x\}$ simply by x. Show that
(a) $x^{\mathrm{lu}} = x^{\mathrm{u}}$, (b) $x^{\mathrm{ul}} = x^{\mathrm{l}}$, (c) $x^{\mathrm{lul}} = x^{\mathrm{l}}$.

5 Let $\mathbb{E}_2(\mathbb{N})$ be the family of all **2-subsets** of $\mathbb{N}$, i.e., subsets $\{x, y\} \subset \mathbb{N}$ where $x \neq y$. Consider any partition $\{Z_1, Z_2, \ldots, Z_n\}$ of $\mathbb{E}_2(\mathbb{N})$ to n subsets for $n \geq 1$. Show that there exists an infinite subset $S \subseteq \mathbb{N}$ such that $\mathbb{E}_2(S) \subseteq Z_k$ for some k.

6 A **topology** on a set X consists of a set $\mathscr{T}$ of subsets, called **open sets**, that satisfy:

(i) $\emptyset$, $X \in \mathscr{T}$;

(ii) if $A_i \in \mathscr{T}$ for all $i \in I$ then also $\bigcup_{i \in I} A_i \in \mathscr{T}$;

(iii) if $A, B \in \mathscr{T}$ then also $A \cap B \in \mathscr{T}$.

Let X be a finite set. Show that there is a bijective correspondence between the topologies on X and the quasi-orders on X.

Solved problem. Let X be a finite set of n elements. Count the number of different symmetric relations on X.

Solution In a symmetric relation R, $x \neq y$ corresponds to the set $\{x, y\}$; so that $\{(x, y), (y, x)\} \subseteq R$ or $\{(x, y), (y, x)\} \cap R = \emptyset$. There are $\binom{n}{2}$ 2-element subsets of X, and R can contain any choice of those. Hence there are $2^{\binom{n}{2}}$ symmetric relations that do not contain any diagonal pairs (x, x). To each such relation one can add a choice of the pairs from ι_X. There are 2^n ways to choose those pairs. The total number of symmetric relations is thus $2^{\binom{n}{2}} \cdot 2^n = 2^{\binom{n}{2}+n}$. □