Ordered Sets (2015)

Problem Set 10 (March 27)

No lectures on Thursday 2nd April and no exercises on Friday 3rd April.

1 Let $\alpha\colon L \to L'$ be a surjective lattice homomorphism where L is a distributive lattice. Show that also L' is distributive.

2 Let L be a distributive lattice with $a, b \in L$. Show that the relation θ determined by

$$(x,y) \in \theta \iff \begin{cases} x \vee a \vee b = y \vee a \vee b \\ x \wedge a \wedge b = y \wedge a \wedge b. \end{cases} \tag{1}$$

is a congruence of L.

3 The congruence lattice $\mathrm{Con}(L)$ of a lattice is distributive.

4 Let L be a locally finite lattice that satisfies the Jordan-Dedekind condition, i.e., for all $x \leq_L y$, the maximal chains $x \to y$ have the same length. Show that there exists a mapping $d\colon L \to \mathbb{Z}$ that satisfies the condition

$$x \prec_L y \iff x <_L y \text{ and } d(y) = d(x) + 1.$$

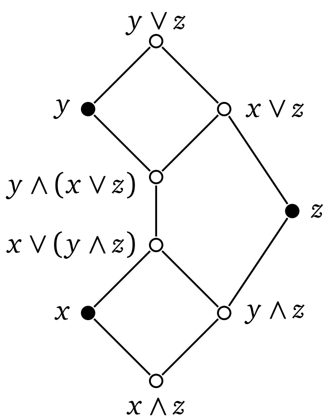
5 Let L be a lattice, and suppose $\kappa\colon L \to \mathbb{N}$ is a **Kolmogorov measure** on L, i.e., for all $x, y \in L$,

$$\kappa(x \vee y) + \kappa(x \wedge y) = \kappa(x) + \kappa(y).$$

Show that for all elements $x_1, x_2, \ldots, x_n \in L$, we have

$$\sum_{i=1}^{n} \kappa(x_i) = \sum_{j=2}^{n} \kappa\big(x_j \vee (\textstyle\bigwedge_{k=1}^{j-1} x_k)\big) + \kappa(x_1 \wedge \ldots \wedge x_n).$$

Solved problem. Let P be the three element poset where $x <_P y$ and $x \| z$ and $y \| z$. Show that on the right there is the largest lattice each element of which is obtained from x, y, z by a finite number of the operations $\vee$ and $\wedge$. (In general, the largest lattice 'generated' by three incomparable elements is infinite, but the largest distributive lattice generated by incomparable three elements is finite.)



Solution. From the conditions we deduce that $x \wedge z$ is the bottom element and $y \vee z$ is the top element. Also, $y \wedge (x \vee z)$ and $x \vee (y \wedge z)$ are in the interval $[x, y]$.
Details are called out for the conclusion. □