Ordered Sets (2015)

Problem Set 10 (March 27)

No lectures on Thursday 2nd April and no exercises on Friday 3rd April.

1 Let $\alpha\colon L \to L'$ be a surjective lattice homomorphism where L is a distributive lattice. Show that also L' is distributive.

Solution. The claim follows from the definitions. □

2 Let L be a distributive lattice with $a, b \in L$. Show that the relation θ determined by

$$(x,y) \in \theta \iff \begin{cases} x \vee a \vee b = y \vee a \vee b \\ x \wedge a \wedge b = y \wedge a \wedge b. \end{cases} \tag{1}$$

is a congruence of L.

Solution. Consider θ defined in (1). It is an equivalence relation.
Suppose that $(x,y) \in \theta$, and $z \in L$. Then

$$(x \vee z) \vee (a \vee b) = (x \vee a \vee b) \vee z = (y \vee a \vee b) \vee z = (y \vee z) \vee (a \vee b)$$

and, by distributivity,

$$\begin{aligned} (x \vee z) \wedge (a \wedge b) &= (x \wedge (a \wedge b)) \vee (z \wedge (a \wedge b)) \\ &= (y \wedge (a \wedge b)) \vee (z \wedge (a \wedge b)) \\ &= (y \vee z) \wedge (a \wedge b). \end{aligned}$$

The other case follows by duality.
(In fact, θ is the smallest congruence containing (a,b).) □

3 The congruence lattice $\mathrm{Con}(L)$ of a lattice is distributive.

Solution. We need to show that $\theta_1 \cap (\theta_2 \vee \theta_3) \subseteq (\theta_1 \cap \theta_2) \vee (\theta_1 \cap \theta_3)$ for congruences of L. Let $(x,y) \in \theta_1 \cap (\theta_2 \vee \theta_3)$. Then there is a sequence

$$x \wedge y = x_0 \leq_L x_1 \leq_L \cdots \leq_L x_n = x \vee y\,,$$

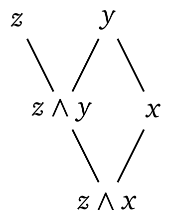
where $(x_i, x_{i+1}) \in \theta_i$ for $i = 2$ or $i = 3$. Since $(x,y) \in \theta_1$, it follows by Theorem 2.31 that also $(x_i, x_{i+1}) \in \theta_1$ for $i = 0, 1, \ldots, n-1$. The claim follows now by Theorem 2.36 when it is applied to the congruences $\theta_1 \cap \theta_2$ and $\theta_1 \cap \theta_2$. □

4 Let L be a locally finite lattice that satisfies the Jordan-Dedekind condition, i.e., for all $x \leq_L y$, the maximal chains $x \to y$ have the same length. Show that there exists a mapping $d\colon L \to \mathbb{Z}$ that satisfies the condition

$$x \prec_L y \iff x <_L y \text{ and } d(y) = d(x) + 1.$$

Solution. For $x \leq_L y$, let $m(x,y)$ denote the length of a maximal chain $x \to y$. Let $z \in L$ be a fixed element, and define $d(z) = 0$. For each $x \in L$, define

$$d(x) = m(z \wedge x, x) - m(z \wedge x, z).$$



Now,

$$d(y) - d(x) = m(z \wedge y, y) - m(z \wedge y, z) - m(z \wedge x, x) + m(z \wedge x, z)$$

and

$$m(z \wedge x, z) - m(z \wedge y, z) + m(z \wedge y, y) = m(z \wedge x, x) + m(x, y).$$

Combining these we have $d(y) - d(x) = m(x,y)$. Finally, $x \prec y$ if and only if $x <_L y$ and $m(x,y) = 1$. □

5 Let L be a lattice, and suppose $\kappa\colon L \to \mathbb{N}$ is a **Kolmogorov measure** on L, i.e., for all $x, y \in L$,

$$\kappa(x \vee y) + \kappa(x \wedge y) = \kappa(x) + \kappa(y).$$

Show that for all elements $x_1, x_2, \dots, x_n \in L$, we have

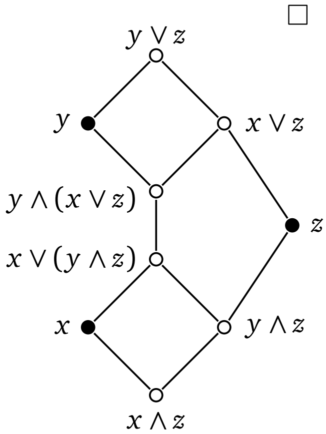
$$\sum_{i=1}^{n} \kappa(x_i) = \sum_{j=2}^{n} \kappa\big(x_j \vee (\textstyle\bigwedge_{k=1}^{j-1} x_k)\big) + \kappa(x_1 \wedge \dots \wedge x_n).$$

Solution. We prove the claim by induction. For $n = 2$ it holds by definition. Assume the claim holds for n, and add $\kappa(x_{n+1})$ to both sides. The right hand side ends with $\kappa(x_1 \wedge x_2 \wedge \dots \wedge x_n) + \kappa(x_{n+1})$ equals, by the definition, with

$$\kappa\big(x_{n+1} \vee (x_1 \wedge x_2 \wedge \dots \wedge x_n)\big) + \kappa(x_1 \wedge x_2 \wedge \dots \wedge x_{n+1})$$

proving the claim. □

Solved problem. Let P be the three element poset where $x <_P y$ and $x \| z$ and $y \| z$. Show that on the right there is the largest lattice each element of which is obtained from x, y, z by a finite number of the operations $\vee$ and $\wedge$. (In general, the largest lattice 'generated' by three incomparable elements is infinite, but the largest distributive lattice generated by incomparable three elements is finite.)



Solution. From the conditions we deduce that $x \wedge z$ is the bottom element and $y \vee z$ is the top element. Also, $y \wedge (x \vee z)$ and $x \vee (y \wedge z)$ are in the interval $[x, y]$.
Details are called out for the conclusion. □