Ordered Sets (2015)

Problem Set 11 (April 10)

1 Find a complete countably infinite lattice that is not algebraic.

2 Prove Lemma 3.30: If x and y are compact elements in L, then so is their join $x \vee y$. The meet of two compact elements need not be compact.

3 Show that if the lattice L has the bottom element 0, then $\mathrm{Id}(L)$ is algebraic. What are the compact elements of $\mathrm{Id}(L)$?

4 Let L be a complete lattice. Show that an element $c \in L$ is compact if and only if, for all directed subsets D,

$$c \leq_L \bigvee D \implies c \leq_L a \text{ for some } a \in D.$$

5 Prove that $C\colon 2^X \to 2^X$ is a closure operation if and only if it satisfies the single condition

$$A \subseteq C(B) \iff C(A) \subseteq C(B).$$

6 Let L be an algebraic lattice. Denote by K the set of all compact elements of L. Show that the operation C defined by

$$C(A) = \{x \in K \mid x \leq_L \bigvee A\}$$

is an algebraic closure operation on the set K.