Ordered Sets (2015)

Problem Set 11 (April 10)

1 Find a complete countably infinite lattice that is not algebraic.

Solution. Consider L with $0 \prec x \prec 1$ plus $0 \prec x_1 \prec x_2 \prec \ldots <_L 1$. Then x is not compact, and not a join of compact elements.
(Uncountable lattices: real interval $[0,1]$.) □

2 Prove Lemma 3.30: If x and y are compact elements in L, then so is their join $x \vee y$. The meet of two compact elements need not be compact.

Solution. Let $x \vee y \leq_L \bigvee A$ for a subset $A \subseteq L$. Then there are finite subsets F_0 and F_1 of A such that $x \leq_L \bigvee F_0$ and $y \leq_L \bigvee F_1$. It then follows that $x \vee y \leq_L \bigvee (F_0 \cup F_1)$. □

3 Show that if the lattice L has the bottom element 0, then $\mathrm{Id}(L)$ is algebraic. What are the compact elements of $\mathrm{Id}(L)$?

Solution. Notice that $\mathrm{Id}(L)$ is complete, since $0 \in L$. Also, $I = \bigvee_{x \in I}(x]$ for each ideal I, and hence $\mathrm{Id}(L)$ is generated by the principal ideals.
We show that each principal ideal $(x] \in \mathrm{Id}(L)$ is compact. Suppose that that $(x] \leq_L \bigvee_{j \in J} I_j$ for some index set J. Then, by Lemma 2.41, there are finitely many elements $x_1, x_2, \ldots, x_n \in \bigcup_{j \in J} I_j$ such that $x \leq_L x_1 \vee x_2 \vee \ldots \vee x_n$. Hence there exists a finite $J_0 \subseteq J$ such that $x_1, \ldots, x_n \in \bigcup_{j \in J_0} I_j \subseteq \bigvee_{j \in J_0} I_j$, and so $(x] \subseteq \bigvee_{j \in J_0} I_j$.
We show then that each compact element is a principal ideal. Let I be compact. Then $I = \bigvee_{x \in I}(x]$ implies that there exists $x_1, \ldots, x_n \in I$ with $I \subseteq \bigvee_{i=1}^{n}(x_i] = \left(\bigvee_{i=1}^{n} x_i\right]$, where $\bigvee_{i=1}^{n} x_i \in I$. Therefore $I = \left(\bigvee_{i=1}^{n} x_i\right]$. □

4 Let L be a complete lattice. Show that an element $c \in L$ is compact if and only if, for all directed subsets D,

$$c \leq_L \bigvee D \implies c \leq_L a \text{ for some } a \in D.$$

Solution. If c is compact, then $c \leq_L \bigvee D$ implies $c \leq_L \bigvee F$ for some finite subset $F \subseteq D$. Since D is directed, the claim follows.

In the converse direction, suppose $c \leq_L \bigvee A$ for a subset A. Now,

$$c \leq_L \bigvee A = \bigvee \left\{ \bigvee F \mid F \subseteq A,\ |F| < \infty \right\}$$

Here the set $\{\bigvee F \mid F \subseteq A,\ |F| < \infty\}$ is directed. Hence $c \leq_L \bigvee F$ for some finite $F \subseteq A$. □

5 Prove that $C \colon 2^X \to 2^X$ is a closure operation if and only if it satisfies the single condition

$$A \subseteq C(B) \iff C(A) \subseteq C(B).$$

Solution. Suppose first that C is a closure operation. Then (i)

$$A \subseteq C(B) \overset{(C3)}{\Longrightarrow} C(A) \subseteq C^2(B) \overset{(C2)}{=} C(B),$$

and, conversely,

$$C(A) \subseteq C(B) \overset{(C1)}{\Longrightarrow} A \subseteq C(B).$$

Suppose C satisfies the single condition. We derive the axioms of the closure operations. Then

$$\begin{aligned}
&C(A) \subseteq C(A) \implies A \subseteq C(A) \qquad \text{so } (C1),\\
&C(A) \subseteq C(A) \implies C^2(A) \subseteq C(A),\\
&C^2(A) \subseteq C^2(A) \implies C(A) \subseteq C^2(A) \qquad \text{so } (C2),\\
&A \subseteq B \implies C(A) \subseteq C^2(B) = C(B) \qquad \text{so } (C3).
\end{aligned}$$

□

6 Let L be an algebraic lattice. Denote by K the set of all compact elements of L. Show that the operation C defined by

$$C(A) = \{x \in K \mid x \leq_L \bigvee A\}$$

is an algebraic closure operation on the set K.

Solution. First of all C is a closure operation:

$$A \subseteq C(B) \iff C(A) \subseteq C(B).$$

Indeed, $x \leq_L \bigvee B$ for all $x \in A$ if and only if $\bigvee A \leq_L \bigvee B$.
It is algebraic: Let $x \in C(A) \cap K$. Then $x \leq_L \bigvee A$ and there exists finite $F_x \subset A$ such that $x \leq_L \bigvee F_x$. Therefore $x \in C(F_x)$, as required. □