Ordered Sets

Problem Set 1 (Jan 16, 2015)

1 (a) Let $P = (X,R)$ be a poset. Show that also (X,R^{-1}) is a poset.

(b) Let (X,R) and (X,S) be posets. Is the union $(X,R\cup S)$ necessarily a poset?

Solution. (a) Clear (b) No. Transitivity may fail: $(x,y)\in R$ and $(y,z)\in S$ does not imply that $(x,z)\in R\cup S$. □

2 Let S_n denote the set of all permutations (bijections) α on $\{1,2,\dots,n\}$. A pair (i,j) is an **inversion** in $\alpha\in S_n$ if $i<j$ and $\alpha(i)>\alpha(j)$. For instance, let

$$\alpha = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 1 & 4 \end{pmatrix} \in S_4$$

which means that $\alpha(1)=2$, $\alpha(2)=3$ and so forth. The inversions of α are $(1,2)$ and $(1,3)$. Define the relation $\le$ on S_n by setting $\alpha\le\beta$ if and only if all inversions of α are inversions of β. Show that $\le$ is a partial order on S_n.

Solution. Clear from the definition of partial orders □

3 Let X be a finite set of n elements. Count the number of different
(a) relations $R\subseteq X\times X$, (b) reflexive relations on X.

Solution. (a) A relation R is a subset of the set $X\times X$. There are n^2 pairs (x,y), and hence the number relations is 2^{n^2}, the number of subsets of pairs of elements of X.

(b) The reflexive relations are those subsets of $X\times X$ that contain the identity relation $\iota_X=\{(x,x)|x\in X\}$. There are n^2-n pairs (x,y) with $x\neq y$, and thus $2^{n^2-n}=2^{n(n-1)}$ subsets of $X\times X$ that do <u>not</u> intersect with ι_X. This is the number of reflexive relations: add ι_X to each such relation. □

4 Let P be a poset, and denote $\{x\}$ simply by x. Show that
(a) $x^{\mathrm{lu}}=x^{\mathrm{u}}$, (b) $x^{\mathrm{ul}}=x^{\mathrm{l}}$, (c) $x^{\mathrm{lul}}=x^{\mathrm{l}}$.

Solution. Now, $y\in x^{\mathrm{lu}}$ if and only if $y\ge_P z$ for all z with $z\le_P x$, i.e., if and only if $y\ge_P x$. The case (b) is dual to (a). The case (c) follows from these: $x^{\mathrm{lul}}=x^{\mathrm{ul}}=x^{\mathrm{l}}$. □

5 Let $\mathbb{E}_2(\mathbb{N})$ be the family of all **2-subsets** of $\mathbb{N}$, i.e., subsets $\{x,y\}\subset\mathbb{N}$ where $x\neq y$. Consider any partition $\{Z_1,Z_2,\dots,Z_n\}$ of $\mathbb{E}_2(\mathbb{N})$ to n subsets for $n\ge 1$. Show that there exists an infinite subset $S\subseteq\mathbb{N}$ such that $\mathbb{E}_2(S)\subseteq Z_k$ for some k.

Solution. Denote $x_0=0$. There exists an integer $r_0\le n$ such that Z_{r_0} has infinitely many 2-subsets containing 0. Let $S_0=\{y\mid\{x_0,y\}\in Z_{r_0}\}$. Assume that S_{i-1} has been already defined, and let $x_i=\min(S_{i-1})$. There are infinitely many 2-subsets of S_{i-1} containing x_i, and hence there exists an index r_i be such that Z_{r_i} has infinitely many of 2-subsets $\{x_i,y\}$ with $y\in S_{i-1}$. Let

$$S_i=\{y\in S_{i-1}\mid\{x_i,y\}\in Z_{r_i}\}\,.$$

Hence this set is infinite. We obtain sequences $x_0, x_1, \dots$ and $r_0, r_1, \dots$, where $r_i \leq n$. Now, there exists an index k such that $k = r_i$ for infinitely many i, say $r_{t_1}, r_{t_2}, \dots$ so that $Z_k = Z_{r_{t_1}} = Z_{r_{t_2}} = \dots$. It follows that $\{x_{t_i}, x_{t_j}\}$ belongs to Z_k for all $i \neq j$. □

6 A **topology** on a set X consists of a set $\mathcal{T}$ of subsets, called **open sets**, that satisfy:

(i) $\emptyset,\ X \in \mathcal{T}$;

(ii) if $A_i \in \mathcal{T}$ for all $i \in I$ then also $\bigcup_{i \in I} A_i \in \mathcal{T}$;

(iii) if $A, B \in \mathcal{T}$ then also $A \cap B \in \mathcal{T}$.

Let X be a finite set. Show that there is a bijective correspondence between the topologies on X and the quasi-orders on X.

Solution. A quasi-order $\leq$ on X defines a topology, where the open sets are the down-sets $\downarrow A$ for $A \subseteq X$. This mapping of quasi-orders to topologies is injective which is seen by considering the principal down-sets $\downarrow x$.
Conversely, a topology on X defines a quasi-order $\leq$ by

$$y \leq_{\mathcal{T}} x \iff \text{every open set that contains } x \text{ also contains } y.$$

Now, for all open sets A of $\mathcal{T}$, we have, by definition,

$$x \in A \text{ and } y \leq_{\mathcal{T}} x \implies y \in A,$$

and hence A is a down-set in the quasi-order. This proves that the correspondence is 1-to-1. □

Solved problem. Let X be a finite set of n elements. Count the number of different symmetric relations on X.

Solution In a symmetric relation R, $x \neq y$ corresponds to the set $\{x, y\}$; so that $\{(x, y), (y, x)\} \subseteq R$ or $\{(x, y), (y, x)\} \cap R = \emptyset$. There are $\binom{n}{2}$ 2-element subsets of X, and R can contain any choice of those. Hence there are $2^{\binom{n}{2}}$ symmetric relations that do not contain any diagonal pairs (x, x). To each such relation one can add a choice of the pairs from ι_X. There are 2^n ways to choose those pairs. The total number of symmetric relations is thus $2^{\binom{n}{2}} \cdot 2^n = 2^{\binom{n}{2}+n}$. □