Ordered Sets

Problem Set 3 (Jan 30, 2015)

1 Let P and Q be locally finite posets, and assume the mapping $\varphi \colon P \to Q$ preserves the cover relation: i.e., if $x \prec_P y$ then $\varphi(x) \prec_Q \varphi(y)$. Show that φ is isotone.

Solution. We prove the claim by induction on the size $s(x,y)$ of the intervals $[x,y]_P$ that $x \leq_P y$ implies $\varphi(x) \leq_Q \varphi(y)$. If $s(x,y) = 2$, then the claim follows from the assumption. Suppose that $s(x,y) \geq 3$. Hence there is an element z with $x <_P z <_P y$. Now $s(x,z) < s(x,y)$ and $s(z,y) < s(x,y)$, and so the induction hypothesis gives $\varphi(x) \leq_Q \varphi(z)$ and $\varphi(z) \leq_Q \varphi(y)$. Therefore also $\varphi(x) \leq_Q \varphi(y)$. □

2 (Part of Theorem 1.40) Prove that if P is partially well-ordered then in every infinite sequence $x_1, x_2, \ldots$ in P there are elements x_i and x_j such that $x_i \leq_P x_j$ with $i < j$.

Solution. Let $x_1, x_2, \ldots$ be such that $x_i \not\leq_P x_j$ for all $i < j$, i.e., for all $i < j$, either $x_i >_P x_j$ or $x_i \parallel x_j$. Let

$$M = \{x_i \ : \ x_i \parallel x_j \ \text{ for all } \ j > i\}.$$

Then M is an antichain and thus finite by FAC. Let m be the maximum index such that $x_m \in M$. Hence, for each $i > m$, there exists j with $j > i$ such that $x_i \geq_P x_j$. This gives an infinite descending chain; a contradiction. □

3 Let P be a finite poset such that the greatest lower bound exists for each pair $x, y \in P$. Show that, if P has a top element, also the least upper bound exists for all elements x and y.

Solution. It follows by induction that every subset $A \subseteq P$ has the greatest lower bound. Then for all $x, y \in P$, consider $A = \{x,y\}^{\mathrm{u}} = \{z \mid x \leq_P z, y \leq_P z\}$. Here A is nonempty since P has the (unique) maximum element. Now the greatest lower bound of the set A is the least upper bound of x and y. □

An isotone mapping $\alpha \colon P \to P$ is called a **retraction** of the poset P, if $\alpha^2 = \alpha$, i.e., $\alpha(\alpha(x)) = \alpha(x)$ for all $x \in P$. Then we say that the subposet $\alpha(P)$ is a **retract** of P.

4 Show that an isotone mapping $\alpha \colon P \to P$ is a retraction if and only if its restriction $\beta = \alpha \restriction \alpha(P)$ to the image $\alpha(P)$ is the identity mapping.

Solution. Assume that $\alpha^2 = \alpha$, and let $y \in \alpha(P)$, say $\alpha(x) = y$ or some $x \in P$. Then $\beta(y) = \alpha(\alpha(x)) = \alpha(x) = y$. Hence β is the identity.

Conversely, suppose β is the identity function. Then $\alpha(\alpha(x)) = \beta(\alpha(x)) = \alpha(x)$ for all $x \in P$. □

5 Assume every isotone mapping $\alpha \colon P \to P$ of the poset P has a fixed point (i.e., $\alpha(x) = x$ for some $x \in P$), and let Q be a retract of P. Show that every isotone mapping $\beta : Q \to Q$ has a fixed point.

Solution. Let $\alpha \colon P \to Q$ be a retraction of P onto Q $(= \alpha(P))$, and let $\beta : Q \to Q$ be isotone. Then $\beta\alpha \colon P \to Q$ is an isotone mapping (of P), and by assumption it has a fixed point x. Obviously $x \in Q$, and thus, by the previous exercise, $\alpha(x) = x$. Now, $\alpha(x) = x = \beta\alpha(x)$, and thus $\alpha(x)$ is a fixed point of β. □

6 Let P be a finite poset. A mapping $r\colon P \to \{1, 2, \dots, n\}$ is a **rank function** on P if it satisfies:

(1) $r(x) = 1$ if x is a minimal element of P;
(2) $r(x) = n$ if x is a maximal element of P;
(3) $r(y) = r(x) + 1$ if $x \prec_P y$.

We say that P is **graded of rank** n if all maximal chains of P have length n. (The *length* $|C|$ of a chain C is the number of its elements.)
Show that a finite poset P is graded of rank n if and only if it has a rank function $r\colon P \to \{1, 2, \dots, n\}$.

Solution. Suppose P is graded of rank n. For $x \in P$, let C_x be a maximal chain containing x, and define $r(x) = |\{y \in C_x \mid y <_P x\}|$. We show first that $r(x)$ is well defined, i.e., it does not depend on the choice of C_x. Indeed, let C and C' be any two maximal chains containing x, and consider the disjoint unions

$$C = C_L \cup \{x\} \cup C_U \quad \text{and} \quad C' = C'_L \cup \{x\} \cup C'_U$$

for $C_L = \{y \in C \mid y <_P x\}$ and $C'_L = \{y \in C' \mid y <_P x\}$. Also $C'_L \cup \{x\} \cup C_U$ is a chain containing x and thus $|C'_L| \le |C_L|$. By symmetry, there is equality here, and also $|C'_U| \le |C_U|$. This shows that r is well defined, and it follows from the maximality of the chains that r is rank function.

In converse, let $r\colon P \to \{1, 2, \dots, n\}$ be a rank function, and $x_1 <_P \dots <_P x_m$ a maximal chain. Then necessarily x_1 is a minimal element and x_m is a maximal element, and $x_i \prec_P x_{i+1}$ for $i = 1, 2, \dots, m-1$. Now, $r(x_1) = 1$ and $r(x_m) = n$, and $r(x_{i+1}) = r(x_i) + 1$ for $i = 1, 2, \dots, m-1$. Hence $m = n$, and this proves the claim. □

Solved problem. Assume that $\alpha\colon P \to P$ is an isotone mapping of the poset P of n elements. Show that the composition $\alpha^{n!}$ is a retraction of P.

Solution Since $|P| = n$, for each $x \in P$, among $x, \alpha(x), \dots, \alpha^n(x)$ there is a repeated element: $\alpha^{r(x)}(x) = \alpha^{s(x)}(x)$ with $t(x) = s(x) - r(x) > 0$. Now $\alpha^{s(x)+j}(x) = \alpha^{r(x)+j}$ for all $j \ge 0$, and thus for all $j \ge r(x)$, we have

$$\alpha^j(x) = \alpha^{(j-r(x))+r(x)}(x) = \alpha^{(j-r(x))+s(x)}(x) = \alpha^{j+t(x)}(x).$$

Therefore each $\alpha^j(x)$ for $j \ge r(x)$ is $t(x)$-*periodic*: $\alpha^j(x) = \alpha^{j+k\cdot t(x)}(x)$ for all $k \ge 0$. Consider then $y = \alpha^{n!}(x)$. By the above, $\alpha^{i+k\cdot t(x)}(x) = y$ for some $i \le n$ and all $k \ge 0$. Consequently, when $k = n!/t(x)$, then

$$\alpha^{n!}(y) = \alpha^{n!+i+t(x)\frac{n!}{t(x)}}(x) = \alpha^{i+m\cdot t(x)}(x) = y,$$

where $m = n!(1 + 1/t(x))$. The claim follows from this. (We need the bound $n!$ to ensure that we have an integer in each possible $n!/t(x)$.) □