Ordered Sets

Problem Set 4 (14:<u>00</u>! Feb 6, 2015)

1 **(a)** For positive integers r and s, denote $R(s,t)=(s-1)(t-1)+1$. Show that if a poset P has at least $R(s,t)$ elements then it has height $h(P)\geq s$ or width $w(P)\geq t$.

(b) Show that there is a poset P of $R(s,t)-1$ elements that has $h(P)<s$ and $w(P)<t$.

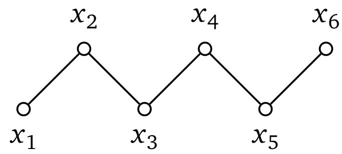
2 Let P be a finite poset. Show that there are equally many antichains in P as there are isotone mappings $\varphi\colon P\to \mathbf{C}_2$, where $\mathbf{C}_2$ is the two-element chain on $\{0,1\}$.

3 A finite poset P is a **fence**, if $P=\{x_1,x_2,\dots,x_n\}$ such that

$$x_{2i+1}<_P x_{2i} \quad\text{and}\quad x_{2i-1}<_P x_{2i}$$

for all $i=1,2,\dots$, and otherwise the elements are incomparable.

$n=6$:



Show that every isotone mapping $\varphi\colon P\to P$ of a fence has a fixed point.

4 Let P be a finite poset, and let $\mathscr{A}(P)$ be the poset of its antichains ordered as follows:

$$A\leq_{\mathscr{A}} B \iff (\forall x\in A)(\exists y\in B): x\leq_P y\,.$$

Show that $\mathscr{A}(P)$ is isomorphic to the poset of down-sets

$$\mathscr{D}(P)=\{{\downarrow}A\mid A\subseteq P, A\neq\emptyset\}$$

ordered by inclusion.

5 Prove the claim of Example 1.52: Let S be a subset of pairs of incomparable elements. If S contains no alternating cycles, then the transitive closure $(P\cup S)^+$ is antisymmetric.

6 Determine the width and height of the divisor poset T_{24}. Find also all partitions of T_{24} into $w(T_{24})$ chains.