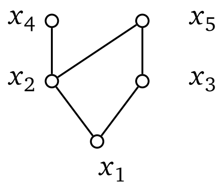
Ordered Sets (2015)

Problem Set 5 (Feb 13 at 14:00)

First midterm examination on Monday 23rd of February.

1 Find the Möbius function of the poset on the right.



2 Let P and Q be two posets, and consider their direct product $P \times Q$. Show that, for all pairs,

$$\zeta_{P\times Q}((x_1, y_1),(x_2, y_2)) = \zeta_P(x_1, x_2)\cdot\zeta_Q(y_1, y_2)$$

and

$$\delta_{P\times Q}((x_1, y_1),(x_2, y_2)) = \delta_P(x_1, x_2)\cdot\delta_Q(y_1, y_2).$$

3 Prove Theorem 1.67: The Möbius function $\mu_{P\times Q}$ of the direct product $P \times Q$ is the product of the Möbius functions μ_P and μ_Q of P and Q, that is,

$$\mu_{P\times Q}((x_1, y_1),(x_2, y_2)) = \mu_P(x_1, x_2)\cdot\mu_Q(y_1, y_2).$$

4 Compute the Möbius function of the poset $\mathbb{N}\times\mathbb{N}$, where $(x_1, y_1) \le (x_2, y_2)$ if and only if $x_1 \le x_2$ and $x_2 \le y_2$.

5 Prove Theorem 1.71: Let P be a finite poset. Denote $f^k = f * f * \cdots * f$ (k times) for each $f \in I(P)$.

(1) Show that $\zeta^2(x, y) = |[x, y]_P|$ (the number of elements in the interval $[x, y]$).

(2) Show that $\zeta^k(x, y)$ equals the number of chains of length k (with possible repetitions) from x to y, that is, the number of sequences $(x_0, x_1,\ldots, x_k)$ for which $x = x_0 \le_P x_1 \le_P \cdots \le_P x_{k-1} \le_P x_k = y$.

Here 'with possible repetitions' means that maybe $x_i = x_{i+1}$ in such a chain.

6 Let P be a locally finite poset. Define

$$\eta(x, y) = \begin{cases} 1 & x \prec_P y \\ 0 & otherwise. \end{cases}$$

Show that

$$\sum_{i=1}^{\infty}\eta^i(x, y) = |\ C\ :\ C\ \text{ a maximal chain in } [x, y]\ |.$$