Ordered Sets (2015)

Problem Set 6 (Feb 20 at 14:00)

- First midterm on Monday 23rd up to lattices.
- No lectures or exercises during 24-27 of February.

1 Let L be a lattice. Show that every closed interval $[x,y]$ with $x <_L y$ forms a sublattice.

Solution. Let $a,b \in [x,y]$. Now, $a \leq_L y$ and $b \leq_L y$, and hence $a \vee b \leq_L y$, and similarly $x \leq_L a \wedge b$. □

2 A subset $A \subseteq X$ is **cofinite** if its complement if finite. Show that the set of cofinite subsets of a set X is a lattice under inclusion.

Solution. Straightforward. □

3 Let L be any lattice. Show that

$$x \wedge (y \vee z) \geq_L (x \wedge y) \vee (x \wedge z)$$

for all $x,y,z \in L$.

Solution. Indeed,

$$\begin{aligned} x \wedge (y \vee z) &\geq_L x \wedge y && \text{(since } y \vee z \geq_L y\text{)} \\ x \wedge (y \vee z) &\geq_L x \wedge z && \text{(since } y \vee z \geq_L z\text{)} \end{aligned}$$

□

4 Let L be any lattice. Show that

$$x \leq_L y \implies x \vee (y \wedge z) \leq_L y \wedge (x \vee z)$$

for all $x,y,z \in L$.

Solution. Suppose that $x \leq_L y$. Then $x \leq_L y \wedge (x \vee z)$ and also $y \wedge z \leq_L y \wedge (x \vee z)$, since $z \leq_L x \vee z$. □

5 Show that the direct product of two lattices is a lattice.

Solution. Let L_1 and L_2 be lattices. Then $L_1 \times L_2$ is a poset, and

$$(x_1,y_1) \wedge (x_2,y_2) = (x_1 \wedge x_2, y_1 \wedge y_2) \text{ and } (x_1,y_1) \vee (x_2,y_2) = (x_1 \vee x_2, y_1 \vee y_2).$$

□

6 Let L be a lattice. An element $x \in L$ is **join-irreducible** if $x = \bigvee A$ for a finite subset A implies that $x \in A$. (Similarly, x is **meet-irreducile** if $x = \bigwedge A$ implies that $x \in A$.)

Suppose that L is finite.

(a) Show that an element $x \neq 0_L$ is join-irreducible if and only if there exists a unique $y \in L$ such that $y \prec_L x$.

(Analogous statement holds for the meet-irreducible elements: An element $x \neq 1_L$ is meet-irreducible if and only if there exists a unique $y \in L$ such that $x \prec_L y$.)

(b) Show that for each $x \in L$ with $x \neq 0_L$ there are join-irreducible elements $x_i \in \downarrow x$ with

$$x = x_1 \vee x_2 \vee \ldots \vee x_n\,.$$

Solution. (a) Since L is finite, the minimum element 0_L exists in L. Moreover, $0_L = \bigvee \emptyset$. Hence 0_L is not join-irreducible. For each $x \neq 0_L$, there exists at least one element $y \in L$ with $y \prec x$.

Assume that the set $A(x) = \{y \mid y \prec x\}$ has at least two elements. Then $x = \bigvee A(x)$ and $x \notin A(x)$. Hence x is not join-irreducible.

Conversely, let there be a unique y with $y \prec x$, and let $x = \bigvee A$ for a subset $A \subseteq L$. If $x \notin A$, then $A \subseteq \downarrow y$, and hence $\bigvee A = y$; a contradiction.

(b) Suppose x is a smallest counter-example, i.e., x is not a join of join-irreducible elements, but each $y <_L x$ is such a join. In particular, x is not join-irreducible. Let $y_i \prec_L x$ for $i = 1, 2, \ldots, k$. By assumption, each y_i is a join of join-irreducible elements, and since $x = y_1 \vee y_2 \vee \ldots \vee y_k$, the claim follows. □