Ordered Sets (2015)

Problem Set 7 (March 6)

1 Let L be a finite lattice, and L' any lattice. Suppose $\alpha, \beta \colon L \to L'$ are lattice homomorphisms satisfying $\alpha(x) = \beta(x)$ for all join-irreducible elements $x \in L$. Show that $\alpha = \beta$.

Solution. Let $x \neq 0_L$. Then $x = x_1 \vee x_2 \vee \ldots \vee x_n$ for some join-irreducible elements x_i (by the previous exercises). Hence

$$\alpha(x) = \alpha(x_1) \vee \alpha(x_2) \vee \ldots \vee \alpha(x_n) = \beta(x_1) \vee \beta(x_2) \vee \ldots \vee \beta(x_n) = \beta(x).$$

□

2 Let P be a poset and L a lattice. Assume that there are isotone mappings $\alpha \colon P \to L$ and $\beta \colon L \to P$ such that $\beta\alpha \colon P \to P$ is the identity map. Show that also P is a lattice.

Solution. The claim follows when we show that $\beta(\alpha(x) \vee \alpha(y))$ is the least upper bound of $x, y \in P$.

It is an upper bound: $\alpha(x) \vee \alpha(y)$ exists in the lattice L, and, for all x and y, we have that $x = \beta(\alpha(x)) \leq_P \beta(\alpha(x) \vee \alpha(y))$, since β is an isotone mapping. Similarly, $y \leq_P \beta(\alpha(x) \vee \alpha(y))$.

Least upper bound: Assume that $z \geq_P x$ and $z \geq_P y$. Hence $\alpha(z) \geq_L \alpha(x)$ and $\alpha(z) \geq_L \alpha(y)$, since α is isotone. So $\alpha(z) \geq_L \alpha(x) \vee \alpha(y)$. Also β is isotone, and hence $\beta\alpha(z) \geq_P \beta(\alpha(z) \vee \alpha(y))$. From $\beta\alpha = \iota$, we obtain $z = \beta\alpha(z) \geq_P \beta(\alpha(x) \vee \alpha(y))$. The claim follows from this. □

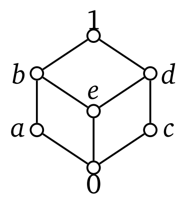
3 Let L be an infinite well-founded lattice (i.e., it satisfies the descending chain condition). Show that each element $x \in L$ is a join $x = \bigvee A_x$ for some finite set A_x of join-irreducible elements.

Solution. Assume the claim does not hold. Let $x \in L$ be a minimal element that is not a join of finitely many join-irreducible elements. In particular, x is not join-irreducible. Hence $x = \bigvee A$ for some finite set $A \subseteq \downarrow x \setminus \{x\}$. By the minimality of x, each $y \in A$ is a join $y = \bigvee A_y$ of finitely many join-irreducible elements. But then also $x = \bigvee \left(\bigcup_{y \in A} A_y \right)$; a contradiction. □

4 For which lattices L does there exist a surjective homomorphism $\alpha \colon M_5 \to L$, where M_5 is the 5-element lattice on page 50?

Solution. Adopt the notation of M_5 (p. 50). Assume that α is not injective. If $\alpha(b) = \alpha(c)$ then $\alpha(a) = \alpha(b \wedge c) = \alpha(b) \wedge \alpha(c) = \alpha(b)$ and similarly $\alpha(e) = \alpha(b)$, and we conclude that L is a singleton lattice. The same is true for all other of noninjective cases. Conclude that the only required lattices are M_5 and the singleton lattice. □

5 Consider the lattice on the right, and let $\theta \in \mathrm{Con}(L)$ be such that $a\theta b$. Show that also $c\theta d$ holds.



Solution. We have

$$a\theta b \implies 0=(a\wedge e)\theta(b\wedge e)=e \implies c=(0\vee c)\theta(e\vee c)=d$$

□

6 Let θ be an equivalence relation of a lattice L. Then $\theta \in \mathrm{Con}(L)$, if the following two conditions are satisfied:

(1) $x\theta y \iff (x\wedge y)\theta(x\vee y)$,

(2) $x\theta y,\ x\leq_L y,\ z\in L \implies (x\vee z)\theta(y\vee z)$ and $(x\wedge z)\theta(y\wedge z)$.

Solution. (A) We show first that

$$x\leq_L y,\ (x,y)\in\theta \ \text{ and } \ a,b\in[x,y]_L \implies (a,b)\in\theta.$$

Indeed, $a\wedge b=(a\wedge b)\vee x$ and $(a\wedge b)\vee y=y$, and so $a\wedge b=(a\wedge b)\vee x\theta(a\wedge b)\vee y=y$, i.e, $(a\wedge b)\theta y$. Here $a\wedge b\leq_L y$, and so (2) ensures (put $x=a\wedge b$ and $z=a\vee b$) that

$$\big((a\wedge b)\wedge(a\vee b)\big)\theta\big(y\wedge(a\vee b)\big),$$

and hence $(a\wedge b)\theta(a\vee b)$, wherefrom it follows that $a\theta b$ by (1).

(B) We show then that θ is a congruence: Assume that $x\theta y$.

$$x\theta y \overset{(1)}{\implies} (x\wedge y)\theta(x\vee y) \overset{(2)}{\implies} \big((x\wedge y)\vee z\big)\theta\big((x\vee y)\vee z\big),$$

where

$$x\vee z,\ y\vee z\in\big[(x\wedge y)\vee z,(x\vee y)\vee z\big]_L.$$

Therefore $(x\vee z)\theta(y\vee z)$ for all $z\in L$ by the first part (A) of the proof.

$$x\theta y \overset{(1)}{\implies} (x\wedge y)\theta(x\vee y) \overset{(2)}{\implies} \big((x\wedge y)\wedge z\big)\theta\big((x\vee y)\wedge z\big),$$

where

$$x\wedge z,\ y\wedge z\in\big[(x\wedge y)\wedge z,(x\vee y)\wedge z\big]_L.$$

Therefore also $(x\wedge z)\theta(y\wedge z)$ for all $z\in L$ by the first part (A) of the proof. □

Solved problem. We prove Dedekind's law for subgroups: if H_1,H_2,H_3 are subgroups of G and $H_1\subseteq H_2$ then

$$H_1(H_2\cap H_3)=H_2\cap H_1H_3.$$

Solution Assume $g=g_1g_2$ with $g_1\in H_1\subseteq H_2$ and $g_2\in H_2\cap H_3$. Then clearly $g\in H_2\cap H_1H_3$.

In the other direction, let $g=g_1g_2\in H_2$ with $g_1\in H_1\subseteq H_2$ and $g_2\in H_3$. Then $g_1\in H_2$ and $g_2=g_1^{-1}g\in H_2$ and so $g_2\in H_2\cap H_3$. Hence $g\in H_1(H_2\cap H_3)$ as required. □