Ordered Sets (2015)

Problem Set 8 (March 13)

There are no lectures on Tuesday 10th of March.

1 Let L be a lattice. Show that a subset $P \subseteq L$ is a prime ideal if and only if $L \setminus P$ is a prime filter of L.

Solution. Let $F = L \setminus P$.
Assume first that P is a prime ideal. Now $x, y \in F$ implies that $x \wedge y \notin P$, and so $x \wedge y \in F$. Also,

$$x \vee y \in F \iff x \vee y \notin P \iff x \notin P \text{ or } y \notin P \iff x \in F \text{ or } y \in F$$

Hence F is a prime filter.
On the other hand, if F is a prime filter, then the dual argument shows that P is a prime ideal. □

2 Prove Lemma 2.47: A proper subset I of a lattice L is a prime ideal if I is closed under finite joins and

(2.5') $$x \wedge y \in I \iff x \in I \quad\text{or}\quad y \in I\,.$$

Solution. Suppose that I is a prime ideal. Then it is closed under finite joins, and the implication '$\Longrightarrow$' holds by definition. Also, in each ideal the reverse implication holds, since $x \wedge y \in \downarrow x$ and $x \wedge y \in \downarrow y$.
Conversely, it is sufficient to show that if $x \in I$ and $y \leq_L x$, then also $y \in I$. This follows from the reverse implication: $x \in I$ implies $y = x \wedge y \in I$. □

3 Prove Lemma 2.39: Let I_i be a set of ideals of a lattice L for all $i \in A$. Then also $\bigcap_{i \in A} I_i$ is an ideal of L if it is nonempty. In particular, every subset $X \subseteq L$ has the smallest ideal containing X:

$$(X] = \{I \mid I \in \mathrm{Id}(L) \;\text{ and }\; X \subseteq I\}\,.$$

Solution. The claim follows from the definition of an ideal. □

4 Let L be a lattice that satisfies the ascending chain condition (ACC). Show that for each nonempty subset $A \subseteq L$, there exists a finite subset $F \subseteq A$ such that $\bigvee A = \bigvee F$.

Solution. Let A be a nonempty subset of L, and let

$$B = \{\bigvee F \mid \emptyset \neq F \subseteq A, F \text{ finite}\}\,.$$

Since L satisfies the ACC, the set B has a maximal element, say $z = \bigvee F$. For each $x \in A$, also $\bigvee(F \cup \{x\}) \in B$, and thus $z = \bigvee(F \cup \{x\})$, which implies that $x \leq_L z$, i.e., z is an upper bound of A. If also y is an upper bound of A, then $z = \bigvee F \leq_L y$, since $F \subseteq A$, and so $z = \bigvee A$. Hence $\bigvee A$ exists and it equals $\bigvee F$. □

5 Consider Theorem 2.36. Show that the relation Ψ is an equivalence relation.

Solution. Let $\theta_1, \theta_2 \in \mathrm{Con}(L)$. We say that a finite sequence $x_0 \leq_L x_1 \leq_L \ldots \leq_L x_n$ is **good**, and denote it by $x_0 \to x_n$, if $(x_i, x_{i+1}) \in \theta_1 \cup \theta_2$ for all i.

Recall the definition of Ψ:

$$(x, y) \in \Psi \iff \text{there is a good sequence } x \wedge y \to x \vee y$$

Now, since θ_1 and θ_2 are congruences, for all z,

$$\text{(1)} \qquad (x, y) \in \Psi \text{ with } x \leq_L y \implies (x \wedge z, y \wedge z) \in \Psi \text{ and } (x \vee z, y \vee z) \in \Psi$$

Indeed, insert $\wedge z$ (or $\vee z$, correspondingly) to each term in the given good sequence $x \wedge y \to x \vee y$ and then observe that $x = x \wedge y$ and $x \vee y = y$.

Clearly, Ψ is reflexive and symmetric. We need to show that it is transitive.
Let $(x, y) \in \Psi$ and $(y, z) \in \Psi$. Then we have some good sequences

$$\text{(3)} \qquad x \wedge y = x_0 \leq_L x_1 \leq_L \ldots \leq_L x_n = x \vee y\,,$$
$$\text{(4)} \qquad y \wedge z = y_0 \leq_L y_1 \leq_L \ldots \leq_L y_m = y \vee z\,.$$

We have

$$\begin{aligned} x \wedge y \wedge z &= (x \wedge y) \wedge (y \wedge z) \\ &= x_0 \wedge (y \wedge z) \ \leq_L \ \ldots \ \leq_L \ x_n \wedge (y \wedge z) \\ &= (x \vee y) \wedge (y \wedge z) = y \wedge z \qquad (\text{since } y \wedge z \leq_L y \leq_L x \vee y) \end{aligned}$$

and so we have a good sequence $x \wedge y \wedge z \to y \wedge z$.
Also, since $x \wedge y \leq_L y \vee z$,

$$\begin{aligned} y \vee z &= (x \wedge y) \vee (y \vee z) \\ &= x_0 \vee (y \vee z) \leq_L \ldots \leq_L x_n \vee (y \vee z) \\ &= x \vee y \vee z \end{aligned}$$

and this gives a good sequence $y \vee z \to x \vee y \vee z$.
In (4) there is a good sequence $y \wedge z \to y \vee z$. By combining good sequences, we obtain $x \wedge y \wedge z \to x \vee y \vee z$. Denote $a = x \wedge y \wedge z$ and $b = x \vee y \vee z$. By (1),

$$x \wedge z = a \vee (x \wedge z) \to b \vee (x \wedge z) = b\,.$$

By (1),

$$x \wedge z = (x \wedge z) \wedge (x \vee z) \to b \vee (x \vee z) = x \vee z.$$

Hence $(x, z) \in \Psi$. Thus Ψ is transitive. □

6 Prove Theorem 2.36.

Solution. Clearly $\theta_1 \cup \theta_2 \subseteq \Psi$. Also, if $\theta_1 \cup \theta_2 \subseteq \rho$ for some $\varrho \in \mathrm{Con}(L)$ then $\Psi \subseteq \rho$. Indeed, if $x \,\Psi\, y$ then there is a Ψ-sequence $x \wedge y \to x \vee y$, and hence $(x \wedge y)\rho(x \vee y)$. Since the congruence classes are convex, also $x \rho y$ holds.

We then employ Exercise 7.6. For Exercise 7.6(1), we use the defining sequence of $(x, y) \in \Psi$ by changing each term x_i to $x_i \wedge (x \vee y)$. We obtain a sequence

$$\begin{aligned}(x \wedge y) \wedge (x \vee y) = x_0 \wedge (x \vee y) \leq_L \cdots \leq_L x_n \wedge (x \vee y) \\ = (x \vee y) \wedge (x \vee y) = (x \vee y) = (x \wedge y) \vee (x \vee y),\end{aligned}$$

where the corresponding steps are in $\theta_1 \cup \theta_2$, since $\theta_1, \theta_2 \in \mathrm{Con}(L)$. Therefore we have $(x \wedge y)\Psi(x \vee y)$.

The converse follows from $x \wedge y = (x \wedge y) \wedge (x \vee y)$ and $x \vee y = (x \wedge y) \vee (x \vee y)$.

For Exercise 7.6(2), we have

$$x \leq_L y,\ (x, y) \in \Psi,\ z \in L \implies (x \wedge z)\Psi(y \wedge z) \text{ and } (x \vee z)\Psi(y \vee z).$$

The former condition is given in (1). □