Ordered Sets (2015)

Problem Set 9 (March 20)

No lectures on Thursday 2nd April and no exercises on Friday 3rd April.

1 Let P be a poset, Show that $D(P) = \{\downarrow A \mid A \subseteq P\}$ is a distributive lattice w.r.t. inclusion.

2 Let L be a lattice. Show that L is modular if and only if, for all $x, y, z \in L$,

$$\left.\begin{array}{r} y \leq_L x \\ x \wedge z = y \wedge z \\ x \vee z = y \vee z \end{array}\right\} \implies x = y\,.$$

3 Let L be a lattice together with a **valuation** $v\colon L \to \mathbb{R}$ satisfying

$$v(x) + v(y) = v(x \vee y) + v(x \wedge y).$$

Suppose v is order preserving, i.e., $x <_L y \implies v(x) < v(y)$. Show that L is modular.

4 Show that a lattice L is modular if and only if it satisfies the rule

$$x \wedge (y \vee z) = x \wedge ((y \wedge (x \vee z)) \vee z)$$

for all elements $x, y, z \in L$.

5 Let L be a modular lattice, and let $a \in L$. Let $x_1 \leq_L x_2 \leq_L \ldots$ be an ascending chain in L of infinitely many elements. Show that also one of the sequences $a \vee x_i$, for $i = 1, 2, \ldots$, or $a \wedge x_i$, for $i = 1, 2, \ldots$ is infinite ascending.

6 Consider the proof of Theorem 3.10. Show that $u \leq_L a \leq_L v$ (and similarly, $u \leq_L b \leq_L v$ and $u \leq_L c \leq_L v$.)