Discrete Boussinesq equations

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The continuous Boussinesq equation

The equation given by Boussinesq in his 1872 paper [page 75, Eq. (26)] can be written, after scaling, as

$$u_{xxxx} + 3(u^2)_{xx} + c\, u_{xx} - u_{yy} = 0. \tag{1}$$

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Let $u = v_x$ and integrate, this yields the first potential form

$$v_{xxxx} + 3({v_x}^2)_x + c\,v_{xx} - v_{yy} = 0. \tag{2}$$

Let $v = w_x$ and integrate, this yields the second potential form

$$w_{xxxx} + 3\,{w_{xx}}^2 + c\,w_{xx} - w_{yy} = 0. \tag{3}$$

Finally with $w = 2\log\tau$ we get the Hirota bilinear form

$$(D_x^4 + cD_x^2 - D_y^2)\tau\cdot\tau = 0. \tag{4}$$

Other Boussinesq equations

There are still other BSQ equations:

The modified BSQ

$$v_{xxxx} - 6v_{xx}v_x^2 + 12v_{xx}v_y + cv_{xx} + 12v_{yy} = 0. \tag{5}$$

The Schwarzian BSQ (Weiss 1984)

$$3\partial_y\left(\frac{u_y}{u_x}\right) + \partial_x\left(\frac{u_{xxx}}{u_x} + \frac{3}{2}\frac{u_y^2 - u_{xx}^2}{u_x^2}\right) = 0. \tag{6}$$

The classical BSQ (Kaup, PTP 1975)

$$v_t = cu_{xx} + u_{xxxx} + (u_x v)_x, \quad v = u_t - \tfrac{1}{2}u_x^2$$

or

$$u_{xxxx} + cu_{xx} - u_{tt} + 2u_{xt}u_x + u_t u_{xx} - \tfrac{3}{2}u_{xx}u_x^2 = 0. \tag{7}$$

Identify equations by their soliton solutions

The form of the two-soliton solution in the bilinear formalism is

$$F = 1 + e^{\eta_1} + e^{\eta_2} + A_{12} e^{\eta_1 + \eta_2},$$

and the multisoliton solution

$$F = \sum_{\mu_i \in \{0,1\}} \exp\left[\sum_{i>j}^{(N)} a_{ij}\, \mu_i\, \mu_j + \sum_{i=1}^{N} \mu_i\, \eta_i\right].$$

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The reason for this point of view is the when we go from continuous to discrete,

- the phase factor remains the same
- the plane wave factor changes in a predictable manner.
- the equation changes most.

The primary identifier is the phase factor, for KP it is

$$\exp(a_{ij}) \equiv A_{ij} = \frac{(p_i - p_j)(q_i - q_j)}{(p_i - q_j)(q_i - p_j)}. \qquad \text{(KP)}$$

Next the plane wave factor (for KP in continuous case)

$$e^{\eta_j} = e^{(p_j - q_j)x + (p_j^2 - q_j^2)y + (p_j^3 - q_j^3)t + \cdots},$$

and finally the equation

$$(D_x^4 + 3\,D_y^2 - 4\,D_x D_t) f \cdot f = 0.$$

where the *Hirota derivative operator D* is defined by

$$D_x^n\, f \cdot g = (\partial_{x_1} - \partial_{x_2})^n f(x_1) g(x_2)\big|_{x_2 = x_1 = x}$$

The BKP hierarchy

The phase factor is

$$A_{ij} = \frac{(p_i - p_j)(p_i - q_j)(q_i - p_j)(q_i - q_j)}{(p_i + p_j)(p_i + q_j)(q_i + p_j)(q_i + q_j)}$$

and the (continuous) plane-wave factor (containing only odd variables) is

$$e^{\eta_i} = \exp[(p_1 + q_1)x_1 + (p_1^3 + q_i^3)x_3 + (p_1^5 + q_i^5)x_5 + \cdots]$$

The first equation in the BKP hierarchy is

$$(D_1^6 - 5D_1^3 D_3 - 5D_3^2 + 9D_1 D_5)\tau \cdot \tau \quad = \quad 0.$$

Note: only odd variables.

KdV as 2-reduction KP

2+1 dimensional solitons have 2 soliton parameters p_i, q_i.

1+1 dimensional solitons have only one soliton parameter.

Apply the 2-reduction $q_i^2 = p_i^2$, i.e., $q_i = -p_i$ to KP results:
Get phase factor

$$A_{ij} = \frac{(p_i - p_j)(q_i - q_j)}{(p_i - q_j)(q_i - p_j)} \quad \longrightarrow \quad \frac{(p_i - p_j)^2}{(p_i + p_j)^2},$$

plane wave factor

$$e^{\eta_j} = e^{(p_j - q_j)x + \cancel{(p_j^2 - q_j^2)y} + (p_j^3 - q_j^3)t + \cdots} \quad \longrightarrow \quad e^{2p_j x + 2p_j^3 t + \cdots}$$

and equation

$$(D_x^4 + 3D_y^2 - 4\, D_x D_t) f \cdot f = 0 \quad \longrightarrow \quad (D_x^4 - 4\, D_x D_t) f \cdot f = 0.$$

BSQ as 3-reduction of KP

The 3-reduction means $q_i^3 = p_i^3$, or $p_i^2 + p_i q_i + q_i^2 = 0$ i.e., $q_i = \omega p_i$, where $\omega^3 = 1$, $\omega \neq 1$.

Apply to KP yields:

$$A_{ij} = \frac{(p_i - p_j)(q_i - q_j)}{(p_i - q_j)(q_i - p_j)} \quad \longrightarrow \quad \frac{(p_i - p_j)^2}{p_i^2 + p_i p_j + p_j^2},$$

$$e^{\eta_j} = e^{(p_j - q_j)x + (p_j^2 - q_j^2)y + \cancel{(p_j^3 - q_j^3)t} + \cdots} \quad \longrightarrow \quad e^{(1-\omega)p_j x + (1-\omega^2)p_j^2 y}$$

Now scale p and y by

$$p_i = k_1/(1-\omega), \quad y = i\sqrt{3}y' \quad \Rightarrow \quad e^{\eta_j} = e^{k_i x + k_j^2 y'}$$

and we get the Boussinesq equation

$$(D_x^4 + 3D_y^2 - 4\,D_x D_t) f \cdot f = 0 \quad \longrightarrow \quad (D_x^4 - D_{y'}^2) f \cdot f = 0.$$

KdV as 4-reduction BKP

Date, Jimbo, Kashiwara and Miwa, RIMS **18**, 1077 (1982):

Remark. The 4-reduced BKP hierarchy ($(\mathrm{BKP})_4$, "$D_2^{(2)}$") is equivalent to the 2-reduced KP hierarchy ($(\mathrm{KP})_2=\mathrm{KdV}$, $A_1^{(1)}$). In fact, by a change of variables

$$x_j \longmapsto \varepsilon_j\sqrt{2}x_j,\ D_j \longmapsto \varepsilon_j\sqrt{2}^{-1}D_j,\ j=1,3,5,\ldots$$
$$(\varepsilon_j\equiv 1,\ -1,\ -1,\ +1,\qquad \text{for}\quad j\equiv 1,3,5,7 \bmod 8,\ \text{respectively})$$

we see that the vertex operators and Hirota's bilinear equations for $(\mathrm{BKP})_4$ reduce to those for $(\mathrm{KP})_2$, respectively.

4-reduction means $q_j = ip_j$. When this is applied to A_{ij} of BKP it becomes A_{ij} of KdV. The plane-wave factor yields

$$\begin{aligned} e^{\eta_i} &= \exp[(p_1+q_i)x_1 + (p_1^3+q_i^3)x_3 + (p_1^5+q_i^5)x_5+\cdots] \\ &\longrightarrow \exp[(1+i)p_1x_1 + (1-i)p_1^3x_3 + (1+i)p_1^5x_5+\cdots] \end{aligned}$$

Then scale $p_j=(1-i)k_j$ and x_j as instructed $\to$ KdV PWF.

There are still other reduction possibilities:

3-pseudo-reduction applied to KP (Hirota, Physica D18 (1986))

$$p_j^2 + p_j q_j + q_j^2 = c.$$

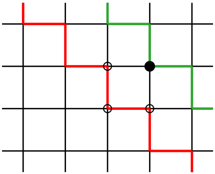
Or can start with 1st modified KP-equation and do

- 3-reduction,
- 3-pseudo-reduction,
- reduction $pq = c$, yielding the classical BSQ.

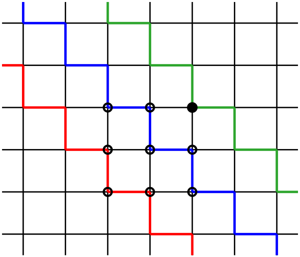
Discrete Boussinesq equation?

What would be the proper discrete Boussinesq equation?

BSQ is <u>second order in time</u>, therefore we need either <u>many points on the lattice</u> or <u>many components</u>.



First order evolution



Second order evolution

I consider two different approaches to this problem:

- Continuous BSQ is the 3-reduction of the KP equation.
  In the discrete case, apply 3-reduction to Hirota's or Miwa's equation.
- Use multi-component consistency-around-the-cube.
  I will also mention the corresponding 1-component forms and their continuum limits.

What is a discrete Hirota bilinear form?

[Notations: $f(n+1,m+2) \equiv \widehat{\widetilde{\widetilde{f}}} \equiv f_{12} = f_{nmm}$.]

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Equation must by gauge-invariant! Cont.: $f_i \to f_i' := e^{ax+bt} f_i$.

$$P(D)(e^{ax+bt} f) \cdot (e^{ax+bt} g) = e^{2(ax+bt)} P(D) f \cdot g$$

Discrete $f_j(n, m) \to f_j'(n, m) = A^n B^m f_j(n, m)$.

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$$P(D)(e^{ax+bt}\, f)\cdot(e^{ax+bt}\, g) = e^{2(ax+bt)}\, P(D) f \cdot g$$

Discrete $f_j(n,m) \to f_j'(n,m) = A^n B^m\, f_j(n,m)$.

We say an equation is in **discrete Hirota bilinear form** if it can be written as

$$\sum_j c_j\, f_j(n+\nu_j^+, m+\mu_j^+)\, g_j(n+\nu_j^-, m+\mu_j^-) = 0$$

where the sums of index shifts $\nu_j^+ + \nu_j^- = \nu^s, \mu_j^+ + \mu_j^- = \mu^s$ are the same in each component of the j-sum.

Hirota's DAGTE

Hirota's 3-term DAGTE equation (Hirota JPSJ 1981) with Miwa's (1982) parametrization (the Hirota-Miwa equation) is

$$a(b-c)\,\tau_{n+1,m,k}\tau_{n,m+1,k+1}+b(c-a)\,\tau_{n,m+1,k}\tau_{n+1,m,k+1}\\ +c(a-b)\,\tau_{n,m,k+1}\tau_{n+1,m+1,k}=0.$$

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and its soliton solutions are given by

$$\begin{aligned} e^{\eta_i} &= \left(\frac{1-aq_i}{1-ap_i}\right)^n \left(\frac{1-bq_i}{1-bp_i}\right)^m \left(\frac{1-cq_i}{1-cp_i}\right)^k, \\ A_{ij} &= \frac{(p_i-p_j)(q_i-q_j)}{(p_i-q_j)(q_i-p_j)}. \end{aligned}$$

This is a totally symmetric 3D equation.

2-reduction

Let $q = -p$, then

$$\begin{aligned} e^{\eta_i} &= \left(\frac{1+ap_i}{1-ap_i}\right)^n \left(\frac{1+bp_i}{1-bp_i}\right)^m \left(\frac{1+cp_i}{1-cp_i}\right)^k, \\ A_{ij} &= \frac{(p_i-p_j)^2}{(p_i+p_j)^2}. \end{aligned}$$

But in addition we must reduce the dimension.

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But in addition we must reduce the dimension.

The parameters a, b, c must be chosen so that the solution is invariant in some direction, i.e., $\tau_{n+\nu,m+\mu,k+\kappa} = \tau_{n,m,k}$

$$\left(\frac{1+ap_i}{1-ap_i}\right)^\nu \left(\frac{1+bp_i}{1-bp_i}\right)^\mu \left(\frac{1+cp_i}{1-cp_i}\right)^\kappa = 1.$$

For 2-reduction we take $\kappa = 0, \nu = 1, \mu = 1, \Rightarrow\ b = -a$.

Discrete bilinear KdV by 2-reduction

With $b = -a$,

$$e^{\eta_i} = \left(\frac{1+ap_i}{1-ap_i}\right)^{n-m} \left(\frac{1+cp_i}{1-cp_i}\right)^{k}.$$

We use the reduction condition

$$\tau_{n,m+1,k} = \tau_{n-1,m,k}, \forall n, k$$

to change all $m+1$ to m (after which we omit m)

The resulting equation is

$$(a+c)\,\tau_{n+1,k}\,\tau_{n-1,k+1} + (c-a)\,\tau_{n-1,k}\,\tau_{n+1,k+1} + 2c\,\tau_{n,k+1}\,\tau_{n,k} = 0.$$

This is then a bilinear discrete KdV and its doubly continuous limit is $(D_x^4 - 3D_xD_t)F \cdot F = 0$.

3-reduction

For the 3-reduction $p^3 - q^3 = 0$, let $q = \omega p$, with $\omega^3 = 1, \omega \neq 1$

$$
\begin{aligned}
e^{\eta_i} &= \left(\frac{1-\omega a p_i}{1-a p_i}\right)^n \left(\frac{1-\omega b p_i}{1-b p_i}\right)^m \left(\frac{1-\omega c p_i}{1-c p_i}\right)^k, \\
A_{ij} &= \frac{(p_i - p_j)^2}{p_i^2 + p_i p_j + p_j^2}.
\end{aligned}
$$

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$$\begin{aligned} e^{\eta_i} &= \left(\frac{1-\omega a p_i}{1-a p_i}\right)^n \left(\frac{1-\omega b p_i}{1-b p_i}\right)^m \left(\frac{1-\omega c p_i}{1-c p_i}\right)^k, \\ A_{ij} &= \frac{(p_i - p_j)^2}{p_i^2 + p_i p_j + p_j^2}. \end{aligned}$$

But in addition we must reduce the dimension.

We choose parameters a, b, c so that $\tau_{n+1,m+1,k+1} = \tau_{n,m,k}$, i.e.,

$$\left(\frac{1-\omega a p_i}{1-a p_i}\right) \left(\frac{1-\omega b p_i}{1-b p_i}\right) \left(\frac{1-\omega c p_i}{1-c p_i}\right) = 1.$$

for all p_i. This is accomplished with $b = \omega a$, $c = \omega^2 a$.

Discrete BSQ by 3-reduction

Thus

$$e^{\eta_i} = \left(\frac{1-\omega a p_i}{1-a p_i}\right)^{n-k} \left(\frac{1-\omega^2 a p_i}{1-\omega a p_i}\right)^{m-k}$$

We use the reduction condition

$$\tau_{n+1,m+1,k+1} = \tau_{n,m,k}$$

to change all $k+1$ to k (and then omit k)

The resulting equation is (Date Jimbo Miwa, JPSJ (1983))

$$\tau_{n+1,m}\,\tau_{n-1,m} + \omega^2\,\tau_{n,m+1}\,\tau_{n,m-1} + \omega\,\tau_{n-1,m-1}\,\tau_{n+1,m+1} = 0.$$

This is a bilinear discrete BSQ and its doubly continuous limit with

$$\tau_{n+\nu,m+\mu} = F(x+(\nu+\omega\mu)\epsilon, y+i\sqrt{3}(\nu+\tfrac{1}{3}(\omega-1)\mu)\epsilon^2), \quad \epsilon \to 0$$

is $(D_x^4 - 4D_y^2)F\cdot F = 0$.

Miwa's 4-term equation (BKP)

The equation is

$$
\begin{aligned}
&(a+b)(a+c)(b-c)\,\tau_{n+1,m,k}\tau_{n,m+1,k+1}\\
&+(b+c)(b+a)(c-a)\,\tau_{n,m+1,k}\tau_{n+1,m,k+1}\\
&+(c+a)(c+b)(a-b)\,\tau_{n,m,k+1}\tau_{n+1,m+1,k}\\
&+(a-b)(b-c)(c-a)\,\tau_{n+1,m+1,k+1}\tau_{n,m,k}=0.
\end{aligned}
$$

Its soliton solutions have the form (Miwa, 1982)

$$
\begin{aligned}
e^{\eta_i} &= \left(\frac{(1-ap_i)(1-aq_i)}{(1+ap_i)(1+aq_i)}\right)^n \left(\frac{(1-bp_i)(1-bq_i)}{(1+bp_i)(1+bq_i)}\right)^m \left(\frac{(1-cp_i)(1-cq_i)}{(1+cp_i)(1+cq_i)}\right)^k,\\
A_{ij} &= \frac{(p_i-p_j)(p_i-q_j)(q_i-p_j)(q_i-q_j)}{(p_i+p_j)(p_i+q_j)(q_i+p_j)(q_i+q_j)}.
\end{aligned}
$$

4-term discrete BSQ

A special case of the above, with reduction $\tau_{n+1,m+1,k+1} = \tau_{n,m,k}$ is [JH,D-j Zhang, JDEA (2012)]

$$f_{n-1,m+1} f_{n+1,m-1}\, o_1 o_3 - f_{n-1,m} f_{n+1,m}\, o_3(o_1 + o_3) - f_{n,m-1} f_{n,m+1}\, o_1(o_1 + o_3) + f_{n,m}^2 (o_1^2 + o_1 o_3 + o_3^2) = 0.$$

4-term discrete BSQ

A special case of the above, with reduction $\tau_{n+1,m+1,k+1} = \tau_{n,m,k}$ is [JH,D-j Zhang, JDEA (2012)]

$$f_{n-1,m+1} f_{n+1,m-1}\, o_1 o_3 - f_{n-1,m} f_{n+1,m}\, o_3(o_1 + o_3) \\ - f_{n,m-1} f_{n,m+1}\, o_1(o_1 + o_3) + f_{n,m}^2 (o_1^2 + o_1 o_3 + o_3^2) = 0.$$

In terms of Miwa's parameters, define symmetric functions

$$S_1 := a + b + c,\ S_2 := ab + bc + ca,\ S_3 := abc,\ \mathcal{S} := S_1 S_2 / S_3.$$

Then the above case corresponds to

$$\mathcal{S} = 9.$$

4-term discrete BSQ

A special case of the above, with reduction $\tau_{n+1,m+1,k+1} = \tau_{n,m,k}$ is [JH,D-j Zhang, JDEA (2012)]

$$\begin{aligned} &f_{n-1,m+1} f_{n+1,m-1}\, o_1 o_3 - f_{n-1,m} f_{n+1,m}\, o_3(o_1 + o_3) \\ &- f_{n,m-1} f_{n,m+1}\, o_1(o_1 + o_3) + f_{n,m}^2 (o_1^2 + o_1 o_3 + o_3^2) = 0. \end{aligned}$$

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Then the above case corresponds to

$$\mathcal{S} = 9.$$

The reduction condition for $P := p\sqrt{S_2}$, $Q := q\sqrt{S_2}$ reads

$$P^2 Q^2 + P^2 + PQ(\mathcal{S} - 1) + Q^2 + \mathcal{S} = 0.$$

For $\mathcal{S}=9$ (for which $S_2<0$) the reduction condition is solved by

$$P = i\sigma\frac{3\ell-1}{\ell+1},\quad Q = i\sigma\frac{3\ell+1}{\ell-1},\quad \sigma^2=1$$

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$$P=i\sigma\frac{3\ell-1}{\ell+1},\quad Q=i\sigma\frac{3\ell+1}{\ell-1},\quad \sigma^2=1$$

Let us further define

$$\ell^2=\frac{1}{3}\frac{k'+4r}{k'-4r},\quad r^2=o_1+o_1o_3+o_3^2$$

then the soliton solutions have the familiar form

$$\begin{aligned} e^{\eta_j} &= \left(\frac{k_j'+(o_1+2o_3)-3\sigma_j o_1}{k_j'+(o_1+2o_3)+3\sigma_j o_1}\right)^n\left(\frac{k_j'-(2o_1+o_3)+3\sigma_j o_3}{k_j'-(2o_1+o_3)-3\sigma_j o_3}\right)^m, \\ (A_{ij})^{\sigma_i\sigma_j} &= \frac{(k_i'-k_j')^2}{{k_1'}^2+k_1'k_2'+{k_2'}^2-12(o_1^2+o_1o_3+o_3^2)}. \end{aligned}$$

For $\mathcal{S} = 9$ (for which $S_2 < 0$) the reduction condition is solved by

$$P = i\sigma \frac{3\ell - 1}{\ell + 1}, \quad Q = i\sigma \frac{3\ell + 1}{\ell - 1}, \quad \sigma^2 = 1$$

Let us further define

$$\ell^2 = \frac{1}{3} \frac{k' + 4r}{k' - 4r}, \quad r^2 = o_1 + o_1 o_3 + o_3^2$$

then the soliton solutions have the familiar form

$$\begin{aligned} e^{\eta_j} &= \left(\frac{k'_j + (o_1 + 2o_3) - 3\sigma_j o_1}{k'_j + (o_1 + 2o_3) + 3\sigma_j o_1} \right)^n \left(\frac{k'_j - (2o_1 + o_3) + 3\sigma_j o_3}{k'_j - (2o_1 + o_3) - 3\sigma_j o_3} \right)^m, \\ (A_{ij})^{\sigma_i \sigma_j} &= \frac{(k'_i - k'_j)^2}{{k'_1}^2 + k'_1 k'_2 + {k'_2}^2 - 12(o_1^2 + o_1 o_3 + o_3^2)}. \end{aligned}$$

Its doubly continuous limit with

$$f_{n+\nu, m+\mu} = F(x + o_1 \nu - o_3 \mu, y + o_1^2 \nu + o_3^2 \mu), \quad o_i \to 0$$

is $(D_x^4 - 4D_y^2) F \cdot F = 0$.

The Pentagram map

The Pentagram map of R. Schwartz (1992) is defined by (Ovsienko, Schwartz, Tabachnikov (2011))

$$\begin{aligned} x_{n,m+1} &= x_{n,m}\,\frac{1-x_{n-1,m}y_{n-1,m}}{1-x_{n+1,m}y_{n+1,m}}, \\ y_{n,m+1} &= y_{n+1,m}\,\frac{1-x_{n+2,m}y_{n+2,m}}{1-x_{n,m}y_{n,m}}. \end{aligned}$$

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From this one variable can be eliminated.

$$Y_{N,M} = -1/(x_{n,m}y_{n,m}), \quad n=-\tfrac{1}{2}(M+N),\ m=M,$$

Then the equation becomes (Glick (2011))

$$Y_{N,M+1} = \frac{Y_{N-3,M}\,Y_{N+3,M}}{Y_{N,M-1}}\,\frac{(1+Y_{N-1,M})(1+Y_{N+1,M})}{(1+Y_{N-3,M})(1+Y_{N+3,M})}$$

Bilinearisation

(in collaboration with K. Maruno)
The Y-equation can be bilinearized with substitution

$$Y_{N,M} = \alpha \frac{F_{N-1,M} F_{N+1,M}}{F_{N-3,M} F_{N+3,M}}.$$

Without loss of generality we may assume that

$$F_{N-3,M} F_{N+3,M} + \alpha F_{N-1,M} F_{N+1,M} - A_{N,M} F_{N,M-1} F_{N,M+1} = 0$$

for some $A_{N,M}$. After eliminating Fs from the Pentagram Y-equation one finds that A must satisfy

$$\frac{A_{N-3,M}}{A_{N-1,M}} = \frac{A_{N+1,M}}{A_{N+3,M}}.$$

A simple solution is to take A a constant of function of M only.

Pentagram map as reduction of Hirota-Miwa

The Hirota bilinear form of the pentagram map has 3 terms, so we try reduction from Hirota-Miwa:

$$a(b-c)\,\tau_{n+1,m,k}\tau_{n,m+1,k+1}+b(c-a)\,\tau_{n,m+1,k}\tau_{n+1,m,k+1}$$
$$+c(a-b)\,\tau_{n,m,k+1}\tau_{n+1,m+1,k}=0.$$

Consider the periodic reduction by

$$\tau_{n,m,k+1}=\tau_{n+2,m+1,k}\,.$$

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Consider the periodic reduction by

$$\tau_{n,m,k+1}=\tau_{n+2,m+1,k}\,.$$

Use it to eliminate k-shifts, this yields

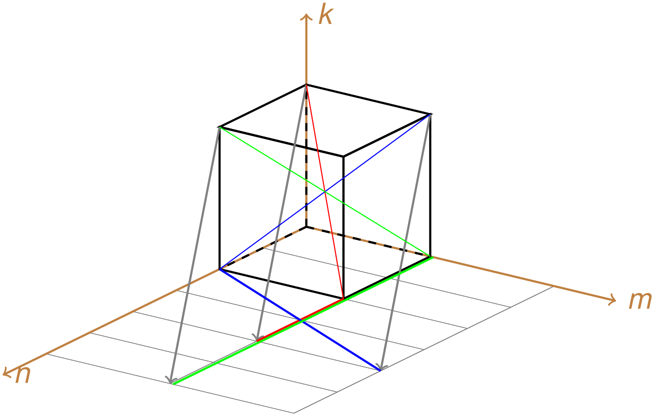
$$a(b-c)\,\tau_{n+1,m}\tau_{n+2,m+2}+b(c-a)\,\tau_{n,m+1}\tau_{n+3,m+1}$$
$$+c(a-b)\,\tau_{n+2,m+1}\tau_{n+1,m+1}=0.$$

The bilinear Y-equation is recovered after a change of variables

$$\tau_{n',m'}=F_{2n'-m'+1,m'-1},\quad N=2n-m+3,\ M=m$$

The projection

$$\tau_{n,m,k+1} = \tau_{n+2,m+1,k}$$



Continuum limit

The double continuum limit is BSQ equation, for a particular set of parameters, namely $\alpha = -9$, $\beta(m) = 8$
(in Miwa's parametrization, $b = -2a$, $c = -3a$)

Let

$$F_{n,m} = f(x + n\epsilon, y + i3m\epsilon^2)$$

then at order ϵ^4 we get

$$(D_x^4 - 12D_y^2) f \cdot f = 0.$$

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Open question: What about the continuum limits for other parameter values?

Reduction condition

The reduction condition for soliton solutions is

$$\left(\frac{1-aq_i}{1-ap_i}\right)^2\left(\frac{1-bq_i}{1-bp_i}\right)=\left(\frac{1-cq_i}{1-cp_i}\right).$$

For $b=-2a$, $c=-3a$ this simplifies to ($p\neq q$)

$$p^2+pq+q^2=-3apq(p+q).$$

In the continuum limit $a=x/n,\ n\to\infty$ so that for example

$$\left(\frac{1-aq}{1-ap}\right)^n=\left(1+\frac{(p-q)x}{n}+\ldots\right)^n\to e^{(p-q)x}=e^{(1-\omega)px}.$$

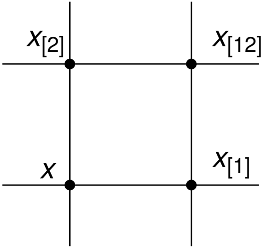
1-component Multilinearity

$$x_{n,m} = x_{00} = x$$

$$x_{n+1,m} = x_{10} = x_{[1]} = \widetilde{x}$$

$$x_{n,m+1} = x_{01} = x_{[2]} = \widehat{x}$$

$$x_{n+1,m+1} = x_{11} = x_{[12]} = \widehat{\widetilde{x}}$$



1-component Multilinearity

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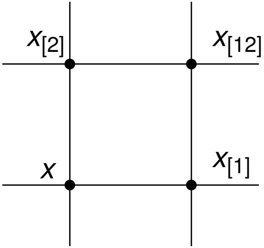


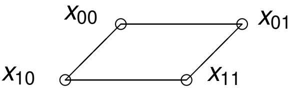
The four corner values are related by a multi-linear equation:

$$k\, x x_{[1]} x_{[2]} x_{[12]} + l_1\, x x_{[1]} x_{[2]} + l_2\, x x_{[1]} x_{[12]} + l_3\, x x_{[2]} x_{[12]} + l_4\, x_{[1]} x_{[2]} x_{[12]}$$
$$+ s_1\, x x_{[1]} + s_2\, x_{[1]} x_{[2]} + s_3\, x_{[2]} x_{[12]} + s_4\, x_{[12]} x + s_5\, x x_{[2]} + s_6\, x_{[1]} x_{[12]}$$
$$+ q_1\, x + q_2\, x_{[1]} + q_3\, x_{[2]} + q_4\, x_{[12]} + u \equiv Q(x, x_{[1]}, x_{[2]}, x_{[12]}; p_1, p_2) = 0$$

The p_i are some parameters associated with shift directions $[i]$, they may appear in the coefficients k, l_i, s_i, q_i, u.

CAC - Consistency Around a Cube

Consistency under extensions from 2D to 3D:
Adjoin a third direction $x_{n,m} \to x_{n,m,k}$ and construct a cube.



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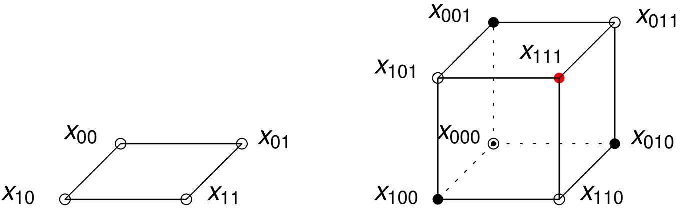

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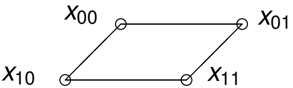

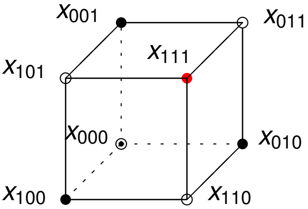

Map at the bottom $Q_{12}(x, \widetilde{x}, \widehat{x}, \widehat{\widetilde{x}}; p, q) = 0$,
on the sides $Q_{23}(x, \widehat{x}, \bar{x}, \bar{\widehat{x}}; q, r) = 0$, $Q_{31}(x, \bar{x}, \widetilde{x}, \bar{\widetilde{x}}; r, p) = 0$,
shifted maps on parallel shifted planes.

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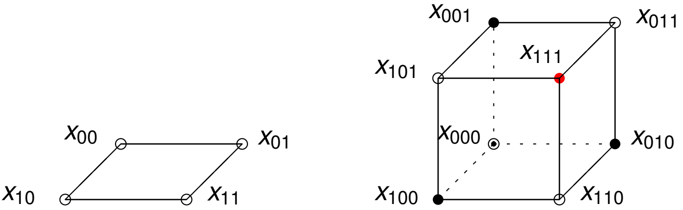

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shifted maps on parallel shifted planes.

Given x_{000}, x_{100}, x_{010}, x_{001} can compute x_{110}, x_{101}, x_{011} uniquely.
But x_{111} can be computed in 3 different ways, they must agree!

Lattice BSQ

What about multi-component extensions?

The following lattice Boussinesq equation (lBSQ) was proposed in Tongas and Nijhoff [Glasgow Math J, (2005), c.f. Nijhoff-Papageorgiou-Capel-Quispel, Inv. Probl. (1992)]

$$\widetilde{y} = x\widetilde{x} - z, \quad \widehat{y} = x\widehat{x} - z,$$
$$\widehat{\widetilde{z}} = x\widehat{\widetilde{x}} - y + \frac{p-q}{\widetilde{x} - \widehat{x}}.$$

This satisfies the 3D-consistency condition (CAC). Note that some equations live on the edges and some on surfaces.

There are also corresponding discrete versions of the modified and Schwarzian Boussinesq equations (Nijhoff, Walker).

A search using CAC

A search using CAC, inspired by the above example, was done in J. Phys. A **44**, 165204 (2011).

Starting point: The following properties were required of the edge equations:

1. The equations are affine linear in all variables, and separately in the set of shifted and in the set of the unshifted variables.
2. Precisely two shifted variables appear (say x and y).
3. The third (non-shifted) variable z appears in the equation.
4. The coefficients of the edge equations can be transformed to constants by a gauge transformation.

After further rational linear transformations there remained three types of edge equations.

Results

Integrable extensions:

$$\text{A-2} \qquad \widetilde{x}z = \widetilde{y} + x, \qquad \widehat{\widetilde{z}} = \frac{y}{x} + \frac{1}{x}\,\frac{p\,\widetilde{x} - q\,\widehat{x}}{\widetilde{z} - \widehat{z}},$$

$$\text{B-2} \qquad x\widetilde{x} = \widetilde{y} + z, \qquad \widehat{\widetilde{z}} + y = b_0(\widehat{\widetilde{x}} - x) + x\,\widehat{\widetilde{x}} + \frac{p - q}{\widetilde{x} - \widehat{x}},$$

$$\text{C-3} \qquad z\widetilde{y} = \widetilde{x} - x, \qquad \widehat{\widetilde{z}} = \frac{d_{23}x + d_1}{y} + \frac{z}{y}\,\frac{p\widetilde{y}\widehat{z} - q\widehat{y}\widetilde{z}}{\widetilde{z} - \widehat{z}},$$

$$\text{C-4} \qquad z\widetilde{y} = \widetilde{x} - x, \qquad \widehat{\widetilde{z}} = \frac{d_{24}x\widehat{\widetilde{x}} + d_1}{y} + \frac{z}{y}\,\frac{p\widetilde{y}\widehat{z} - q\widehat{y}\widetilde{z}}{\widetilde{z} - \widehat{z}}.$$

[The system given in Equations (4,5) of Zhang-Zhao-Nijhoff Stud. Appl. Math. (2012) is gauge equivalent to C-3.]

A-2

From the 3-component forms it is easy to eliminate one variable, but eliminating the second variable is sometimes difficult.

Here are some results:

$$\text{A-2} \qquad \widetilde{x} z = \widetilde{y} + x, \quad \widehat{\widetilde{z}} = \frac{y}{x} + \frac{1}{x}\,\frac{p^3\,\widetilde{x} - q^3\,\widehat{x}}{\widetilde{z} - \widehat{z}},$$

If one eliminates y, z the results is

$$\left(\frac{p^3\,x_{11} - q^3\,x_{02}}{x_{02} - x_{11}}\right)\frac{x_{12}}{x_{01}} - \left(\frac{p^3\,x_{20} - q^3\,x_{11}}{x_{11} - x_{20}}\right)\frac{x_{21}}{x_{10}} = \frac{x_{00}}{x_{10}} - \frac{x_{00}}{x_{01}} - \frac{x_{12}}{x_{22}} + \frac{x_{21}}{x_{22}}.$$

One can also derive 1-component equations in terms of z or $Y := y\,x$ and the result is as for B-2 in terms of x (next slide).

B-2

$$\begin{aligned} x_{10}\, x_{00} &= y_{10} + z_{00}, \quad x_{01}\, x_{00} = y_{01} + z_{00} \\ z_{11} &= -y_{00} + b_0\,[x_{11} - x_{00}] + x_{00}\, x_{11} + \frac{p^3 - q^3}{x_{10} - x_{01}} \end{aligned}$$

After eliminating y, z one gets

$$\begin{aligned} &\frac{p^3 - q^3}{x_{20} - x_{11}} - \frac{p^3 - q^3}{x_{11} - x_{02}} - x_{21}x_{10} + x_{12}x_{01} \\ &+ [x_{22} - b_0][x_{21} - x_{12}] + [x_{00} + b_0][x_{10} - x_{01}] = 0. \end{aligned}$$

B-2

$$x_{10}\, x_{00} = y_{10} + z_{00}, \quad x_{01}\, x_{00} = y_{01} + z_{00}$$

$$z_{11} = -y_{00} + b_0\,[x_{11} - x_{00}] + x_{00}\, x_{11} + \frac{p^3 - q^3}{x_{10} - x_{01}}$$

After eliminating y, z one gets

$$\frac{p^3 - q^3}{x_{20} - x_{11}} - \frac{p^3 - q^3}{x_{11} - x_{02}} - x_{21}x_{10} + x_{12}x_{01} + [x_{22} - b_0][x_{21} - x_{12}] + [x_{00} + b_0][x_{10} - x_{01}] = 0.$$

The double-continuum limit of the above is obtained by taking

$$x_{n+\nu,m+\mu} = (n+\nu)\,p + (m+\mu)\,q - u(x + 2(\nu/p + \mu/q), y + 2(\nu + \mu)/(pq))$$

with $p, q \to \infty$, $b_0 \to 0$, while p/q, pb_0 are fixed. The result is

$$u_{xxxx} + 6u_x u_{xx} + b_0(p+q)u_{xx} + 3u_{yy} = 0,$$

The meaning of b_0 for discrete solitons is discussed in JH and D-j Zhang, SIGMA **7**, 061 (2011).

C-3 and C-4

$$z_{00}\,y_{10} = x_{10} - x_{00}, \quad z_{00}\,y_{01} = x_{01} - x_{00},$$

$$\text{C-3} \qquad z_{11} = \frac{d_{23}\,x_{00} + d_1}{y_{00}} + \frac{z_{00}}{y_{00}}\,\frac{p^3\,y_{10}\,z_{01} - q^3\,y_{01}\,z_{10}}{z_{10} - z_{01}}$$

The C-4 model is a variant of the above with

$$z_{11} = \frac{d_{24}\,x_{00}x_{11} + d_1}{y_{00}} + \frac{z_{00}}{y_{00}}\,\frac{p^3\,y_{10}\,z_{01} - q^3\,y_{01}\,z_{10}}{z_{10} - z_{01}}$$

Eliminating z,y results with the lattice SBSQ (with extra terms)

$$\frac{(x_{22}-x_{12})(x_{02}-x_{11})(x_{01}-x_{00})}{(x_{22}-x_{21})(x_{20}-x_{11})(x_{10}-x_{00})}$$

$$= \frac{(-d_1 - d_2\,X)(x_{11}-x_{02}) + p^3\,(x_{12}-x_{02})(x_{11}-x_{01}) - q^3\,(x_{12}-x_{11})(x_{02}-x_{01})}{(-d_1 - d_2\,Y)(x_{11}-x_{20}) + q^3\,(x_{21}-x_{20})(x_{11}-x_{10}) - p^3\,(x_{21}-x_{11})(x_{20}-x_{10})}$$

for C-3 $d_2X = d_{23}x_{01},\ d_2Y = d_{23}x_{10}$ and
for C-4 $d_2X = d_{24}x_{12}x_{01},\ d_2Y = d_{24}x_{21}x_{10}$.

C-3,C-4 Continuum limits

The double-continuum limits of C-3 and C-4 are obtained with

$$x_{n+\nu,m+\mu} = u(x + 2(\nu/p + \mu/q), y + 2(\nu/p^2 + \mu/q^2))$$

with $p, q \to \infty$ while keeping p/q fixed. The result is

$$3\partial_y \left(\frac{u_y}{u_x}\right) + \partial_x \left(\frac{u_{xxx}}{u_x} + \frac{3}{2}\frac{u_y^2 - u_{xx}^2}{u_x^2} - \frac{1}{2}\frac{d_1 + d_{23}u + d_{24}u^2}{u_x}\right) = 0.$$

This is a generalization of the usual Schwarzian Boussinesq equation.

C-3 in terms of z

$$z_{00}\, y_{10} = x_{10} - x_{00}, \quad z_{00}\, y_{01} = x_{01} - x_{00},$$

C-3

$$z_{11} = \frac{d_{23}\, x_{00} + d_1}{y_{00}} + \frac{z_{00}}{y_{00}} \frac{p^3\, y_{10}\, z_{01} - q^3\, y_{01}\, z_{10}}{z_{10} - z_{01}}$$

If we eliminate x, y the result is, in terms of $Z = 1/z$

$$\left(\frac{(p^3+d_{23})\, Z_{11} - (q^3+d_{23})\, Z_{02}}{Z_{02} - Z_{11}}\right) \frac{Z_{12}}{Z_{01}} - \left(\frac{(p^3+d_{23})\, Z_{20} - (q^3+d_{23})\, Z_{11}}{Z_{11} - Z_{20}}\right) \frac{Z_{21}}{Z_{10}} = \frac{Z_{00}}{Z_{10}} - \frac{Z_{00}}{Z_{01}} - \frac{Z_{12}}{Z_{22}} + \frac{Z_{21}}{Z_{22}}.$$

which is same as mBSQ of A-2, but with additional parameter.

$$\left(\frac{(p^3+d_{23})\,Z_{11}-(q^3+d_{23})\,Z_{02}}{Z_{02}-Z_{11}}\right)\frac{Z_{12}}{Z_{01}}-\left(\frac{(p^3+d_{23})\,Z_{20}-(q^3+d_{23})\,Z_{11}}{Z_{11}-Z_{20}}\right)\frac{Z_{21}}{Z_{10}}$$
$$=\frac{Z_{00}}{Z_{10}}-\frac{Z_{00}}{Z_{01}}-\frac{Z_{12}}{Z_{22}}+\frac{Z_{21}}{Z_{22}}.$$

The double continuum limit with

$$p,q,d_{23}\to\infty,\quad p/q,\ p/d_{23}\,\text{const.}$$

$$Z_{n+\nu,m+\mu}=\exp\{v(x+\nu/p+\mu/q,y+\nu/p^2+\mu/q^2)\}/(p^{n+\nu}q^{m+\mu})$$

is mBSQ

$$v_{xxxx}-6v_{xx}v_x^2+12v_{xx}v_y+4d_{23}(1/p+1/q)v_{xx}+12v_{yy}=0.$$

Conclusions

In this talk I have discussed various equations that may be considered discretizations of the Boussinesq equation. These were obtained by

- 3-reduction $\tau_{n+1,m+1,k+1} = \tau_{n,m,k}$ from Hirota-Miwa (DAGTE) 3-term equation
- 3-reduction $\tau_{n+1,m+1,k+1} = \tau_{n,m,k}$ from Miwa's 4-term equation (BKP)
- Reduction $\tau_{n+2,m+1,k-1} = \tau_{n,m,k}$ from Hirota-Miwa (pentagram map)
- Multicomponent consistency search.

It is expected that there are still other possibilities.