Hirota's bilinear method and integrability

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Hirota: Let us define a new dependent variable F by

$$u = 2\partial_x^2 \log F. \tag{1}$$

With F it should be easy to construct soliton solutions.

Bilinear form for KdV

How do soliton equations look in terms of F?

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Example: KdV

$$u_{xxx} + 6uu_x + u_t = 0. \qquad (2)$$

The first two terms should have the same number of derivatives, so introduce v by

$$u = \partial_x v. \qquad (3)$$

After this (2) can be written as

$$\partial_x [v_{xxx} + 3v_x^2 + v_t] = 0,$$

which can be integrated to the *potential KdV*.

$$v_{xxx} + 3v_x^2 + v_t = 0, \qquad (4)$$

Now substituting

$$v = \alpha \partial \log F,$$

into $v_{xxx} + 3v_x^2 + v_t = 0$ results with

$$F^2 \times (\text{something quadratic}) + 3\alpha(2-\alpha)(2FF'' - F'^2)F'^2 = 0.$$

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Thus we get a quadratic equation if we choose $\alpha = 2$:

$$F_{xxxx}F - 4F_{xxx}F_x + 3F_{xx}^2 + F_{xt}F - F_xF_t = 0.$$

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$$F_{xxxx}F - 4F_{xxx}F_x + 3F_{xx}^2 + F_{xt}F - F_xF_t = 0.$$

This can be written as

$$(D_x^4 + D_xD_t)F \cdot F = 0,$$

where the Hirota's derivative operator D is defined by

$$\begin{aligned} D_x^n f \cdot g &= (\partial_{x_1} - \partial_{x_2})^n f(x_1)g(x_2)\big|_{x_2=x_1=x} \\ &\equiv \partial_y^n f(x+y)g(x-y)\big|_{y=0}. \end{aligned}$$

We say that an equation is in the Hirota bilinear form is all its derivatives appear trough Hirota's D-operator, defined before

$$D_x^n f \cdot g = (\partial_{x_1} - \partial_{x_2})^n f(x_1) g(x_2)\big|_{x_2=x_1=x}.$$

Thus D operates on a product of two functions like the Leibniz rule, except for a crucial sign difference. For example

$$\begin{aligned} D_x f \cdot g &= f_x g - f g_x, \\ D_x D_t f \cdot g &= f g_{xt} - f_x g_t - f_t g_x + f g_{xt} \\ P(D) f \cdot g &= P(-D) g \cdot f. \end{aligned}$$

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For later use note also that

$$\begin{aligned} P(D) f \cdot 1 &= P(\partial) f, \\ P(D) e^{px} \cdot e^{qx} &= P(p-q) e^{(p+q)x} \end{aligned}$$

Bilinear form of KP

Another example: the Kadomtsev-Petviashvili equation:

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The Hirota-Satsuma shallow water-wave equation

$$u_{xxt} + 3uu_t - 3u_x v_t - u_x = u_t, \; v_x = -u, \tag{5}$$

becomes with (1) and one integration

$$(D_x^3 D_t - D_x^2 - D_t D_x) F \cdot F = 0, \tag{6}$$

which actually has an integrable $(2+1)$-dimensional extension

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The Sawada-Kotera equation (SK)

$$u_{xxxxx} + 15uu_{xxx} + 15u_x u_{xx} + 45u^2 u_x + u_t = 0, \tag{8}$$

bilinearizing with (1) and one integration to

$$(D_x^6 + D_x D_t) F \cdot F = 0, \tag{9}$$

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$$(D_x^6 + 5D_x^3 D_t - 5D_t^2 + D_x D_y) F \cdot F = 0. \tag{10}$$

Soliton solutions

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The trivial (vacuum) solution $u = 0$ corresponds to $F = 1$.

Therefore we assume $P(0, 0, \dots) = 0$.

Soliton solutions are built perturbatively on top of this vacuum.

$$F = 1 + \varepsilon f_1 + \varepsilon^2 f_2 + \varepsilon^3 f_3 + \cdots$$

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Note also the gauge invariance of bilinear equations:

$$P(D)(e^{\kappa}F \cdot e^{\kappa}G) = e^{2\kappa}P(D)F \cdot G, \quad \text{if } \kappa = \vec{c} \cdot \vec{x}.$$

For the 1SS try

$$F = 1 + \varepsilon f_1. \tag{11}$$

This implies

$$P(D_x, \dots)\{1 \cdot 1 + \varepsilon\, 1 \cdot f_1 + \varepsilon\, f_1 \cdot 1 + \varepsilon^2\, f_1 \cdot f_1\} = 0.$$

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Since P is even next order yields

$$P(\partial_x, \partial_y, \dots) f_1 = 0. \tag{12}$$

which is solved by

$$f_1 = e^{\eta}, \quad \eta = px + qy + \omega t + \cdots + \text{ const}, \tag{13}$$

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Then order ε^2 term vanishes: $P(\vec{D})e^\eta \cdot e^\eta = e^{2\eta} P(\vec{p} - \vec{p}) = 0$.

The solution

$$F = 1 + e^{\eta}, \quad \eta = \vec{x} \cdot \vec{p} + \eta^0, \quad P(\vec{p}_i) = 0,$$

corresponds to a soliton:

$$\begin{aligned} u &= 2\partial_x^2(\log(F)) \\ &= \frac{2p^2 e^{\eta}}{(1+e^{\eta})^2} = \frac{p^2/2}{\cosh(\frac{1}{2}\eta)^2} \end{aligned}$$

Ansatz for the two-soliton solution (perturbatively!):

$$F = 1 + \varepsilon\left(e^{\eta_1} + e^{\eta_2}\right) + \varepsilon^2 A_{12} e^{\eta_1+\eta_2}, \quad \eta_i = \vec{x}\cdot\vec{p}_i + \eta_i^0,$$

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Example, KdV: $\eta = px + \omega t + \eta_0$, and DR $\omega = -p^3$:

$$A_{12} = -\frac{(p_1 - p_2)^4 + (p_1 - p_2)(\omega_1 - \omega_2)}{(p_1 + p_2)^4 + (p_1 + p_2)(\omega_1 + \omega_2)} = \frac{(p_1 - p_2)^2}{(p_1 + p_2)^2}.$$

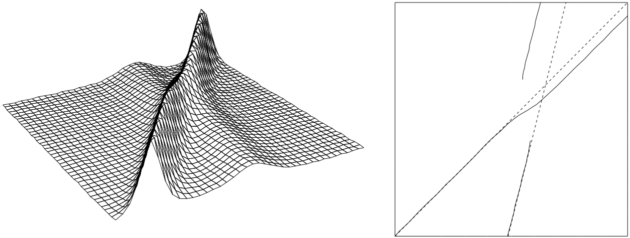

Figure: Scattering of Korteweg–de Vries solitons. On the left a profile view, on the right the locations of the maxima, along with the free soliton trajectory as a dotted line. ($p_1 = \frac{1}{2}$, $p_2 = 1$.)

Result: Any equation of type

$$P(\vec{D}_x)F \cdot F = 0$$

has two-soliton solutions

$$F = 1 + e^{\eta_1} + e^{\eta_2} + A_{12}e^{\eta_1+\eta_2}, \quad \text{where} \quad A_{ij} = -\frac{P(p_i - p_j, \dots)}{P(p_i + p_j, \dots)}$$

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This is a level of partial integrability: we can have elastic scattering of two solitons, for any dispersion relation, if the nonlinearity is suitable.

Clearly all of these equations cannot be integrable.
What distinguished integrable equations?

Hirota integrability:

If the 1SS is given by

$$F = 1 + \varepsilon e^{\eta}, \quad \eta_i = \vec{x} \cdot \vec{p}_i + \eta_i^0, \quad P(\vec{p}_i) = 0,$$

then there should be an NSS of the form

$$F = 1 + \varepsilon \sum_{j=1}^{N} e^{\eta_j} + \text{(finite number of h.o. terms)}$$

without any further conditions on the parameters $\vec{p}_i$ of the individual solitons.

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Almost all equations has multisoliton solutions for some restricted set of parameters, it does not imply integrability.

Apply this principle to the three-soliton solution:

$$\begin{aligned} F_{3SS} =& 1 + \varepsilon \left(e^{\eta_1} + e^{\eta_2} + e^{\eta_3} \right) \\ & + \varepsilon^2 \left(A_{12} e^{\eta_1+\eta_2} + A_{23} e^{\eta_2+\eta_3} + A_{31} e^{\eta_3+\eta_1} \right) \\ & + \varepsilon^3 A_{123} e^{\eta_1+\eta_2+\eta_3} \end{aligned}$$

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Condition on NSS: If any soliton goes far away, the rest should look like the (N-1)SS.

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No freedom left: parameters restricted only by the DR, phase factors given already.

Existence of a 3SS is a condition on the equation, i.e., on P !

Substituting F_{3SS} in $P(D)F \cdot F = 0$ yields the "three-soliton-condition"

$$\sum_{\sigma_i=\pm} P(\sigma_1\vec{p}_1+\sigma_2\vec{p}_2+\sigma_3\vec{p}_3)P(\sigma_1\vec{p}_1-\sigma_2\vec{p}_2) \times P(\sigma_2\vec{p}_2-\sigma_3\vec{p}_3)P(\sigma_1\vec{p}_1-\sigma_3\vec{p}_3)=0.$$

or

$$\sum_{\sigma_i=\pm} \frac{P(\sigma_1\vec{p}_1+\sigma_2\vec{p}_2+\sigma_3\vec{p}_3)}{P(\sigma_1\vec{p}_1+\sigma_2\vec{p}_2)P(\sigma_2\vec{p}_2+\sigma_3\vec{p}_3)P(\sigma_1\vec{p}_1+\sigma_3\vec{p}_3)}=0. \quad (16)$$

This can be taken as a search problem:

Find all polynomials P, such that (16) holds on the affine variety $\{(\vec{p}_1,\vec{p}_2,\vec{p}_3)|P(\vec{p}_i)=0\}$.

Use the 3-soliton condition as a indicator of integrability.

We are only interested in polynomials P that have a nonlinear irreducible factor.

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Complete set of solutions is (JH, J. Math. Phys. (1987-1988)):

$$
\begin{aligned}
(D_x^4 - 4D_xD_t + 3D_y^2)F\cdot F &= 0, && (17)\\
(D_x^3D_t + aD_x^2 + D_tD_y)F\cdot F &= 0, && (18)\\
(D_x^4 - D_xD_t^3 + aD_x^2 + bD_xD_t + cD_t^2)F\cdot F &= 0, && (19)\\
(D_x^6 + 5D_x^3D_t - 5D_t^2 + D_xD_y)F\cdot F &= 0. && (20)
\end{aligned}
$$

and their reductions.

These equations also have 4SS and pass the Painlevé test.

(19) was new, it is non-evolutionary and its Lax pair is unknown.

Other types of soliton equations

The modified KdV equation (mKdV)

$$u_{xxx} + \epsilon 6u^2 u_x + u_t = 0, \tag{21}$$

with traveling wave solutions

$$u = \frac{\pm p}{\cosh(px - p^3 t + c)}, \text{ if } \epsilon = 1, u = \frac{\pm p}{\sinh(px - p^3 t + c)}, \text{ if } \epsilon = -1.$$

We consider only $\epsilon = +1$.

Make the equation scale invariant with

$$u = \partial_x w, \tag{22}$$

after which we get from (21)

$$\partial_x [w_{xxx} + 2w_x^3 + w_t] = 0,$$

integrate once to get the *potential mKdV equation*.

In this case a good substitution is given by

$$w = 2\arctan(G/F), \text{ i.e., } u = 2\frac{D_x G \cdot F}{F^2 + G^2}, \tag{23}$$

and then the potential mKdV becomes

$$\begin{aligned} &(F^2 + G^2)[(D_x^3 + D_t)G \cdot F] \\ &+3(D_x F \cdot G)[D_x^2(F \cdot F + G \cdot G)] = 0 \end{aligned} \quad . \tag{24}$$

Two unknowns F, G means we can have two equations:

$$\begin{cases} (D_x^3 + D_t + 3\lambda D_x)(G \cdot F) &= 0, \\ (D_x^2 + \lambda)(F \cdot F + G \cdot G) &= 0, \end{cases}$$

where λ is an arbitrary function of x, t.
Vacuum solution $F = 1$, $G = 0$ works iff $\lambda = 0$.

For the sine-Gordon (sG) equation

$$\phi_{xx} - \phi_{tt} = \sin\phi, \tag{25}$$

the substitution

$$\phi = 4\arctan(G/F), \tag{26}$$

yields

$$\begin{aligned}[(D_x^2 - D_t^2 - 1)G\cdot F](F^2 - G^2)\\ -FG[(D_x^2 - D_t^2)(F\cdot F - G\cdot G)] = 0.\end{aligned}$$

There is again some ambiguity in splitting this equation, because the term $\lambda FG(F^2 - G^2)$ could be in either part. We use

$$\begin{cases} (D_x^2 - D_t^2 - 1)G\cdot F &= 0,\\ (D_x^2 - D_t^2)(F\cdot F - G\cdot G) &= 0.\end{cases} \tag{27}$$

Multisoliton solutions for the mKdV/sG class

The mKdV and sG equations belong to the class

$$\begin{cases} B(D_{\vec{x}})\, G \cdot F = 0, \\ A(D_{\vec{x}})(F \cdot F + \epsilon G \cdot G) = 0, \end{cases} \tag{28}$$

where A is even and B either odd (mKdV) or even (sG). If B is odd one can also rotate to

$$\begin{cases} B(D_{\vec{x}})\, g \cdot f = 0, \quad (B \text{ odd }) \\ A(D_{\vec{x}})\, g \cdot f = 0. \end{cases} \tag{29}$$

For the vacuum we choose $F = 1, G = 0$ and therefore we must have $A(0) = 0$. For the 1SS we may try

$$F = 1 + \alpha e^{\eta}, \; G = \beta e^{\eta}.$$

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Direct calculation yields from (28) the conditions

$$\alpha A(\vec{p}) = 0, \; \beta B(\vec{p}) = 0, \; \alpha\beta B(0) = 0.$$

For the vacuum we choose $F = 1, G = 0$ and therefore we must have $A(0) = 0$. For the 1SS we may try

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Direct calculation yields from (28) the conditions

$$\alpha A(\vec{p}) = 0,\ \beta B(\vec{p}) = 0,\ \alpha\beta B(0) = 0.$$

Now we can in principle have *two different kinds of solitons*

$$\begin{array}{lll} \text{type a:} & F = 1 + e^{\eta_A},\ G = 0, & \text{DR: } A(\vec{p}) = 0, \\ \text{type b:} & F = 1,\ G = e^{\eta_B}, & \text{DR: } B(\vec{p}) = 0. \end{array} \tag{30}$$

(For mKdV and SG the A polynomial is too trivial.)

Two-soliton solutions

We have three different combinations

a+a: The starting point must be

$$F = 1 + \varepsilon e^{\eta_1} + \varepsilon e^{\eta_2} + O(\varepsilon^2), \quad G = O(\varepsilon^2),$$

with $A(\vec{p}_1) = A(\vec{p}_2) = 0$

$G = 0, \Longrightarrow$ the A-equation is KdV type for F

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a+b: Now the starting point is

$$F = 1 + \varepsilon e^{\eta_1} + O(\varepsilon^2), \quad G = \varepsilon e^{\eta_2} + O(\varepsilon^2),$$
$$\text{with } A(\vec{p}_1) = B(\vec{p}_2) = 0.$$

This leads to

$$F = 1 + e^{\eta_1}, \quad G = e^{\eta_2} + L_{12} e^{\eta_1 + \eta_2}, \tag{31}$$
$$\text{with } L_{12} = -\frac{B(\vec{p}_2 - \vec{p}_1)}{B(\vec{p}_1 + \vec{p}_2)}. \tag{32}$$

b+b: In this case we start with

$$F = 1 + O(\varepsilon^2), \quad G = \varepsilon e^{\eta_1} + \varepsilon e^{\eta_2} + O(\varepsilon^2),$$
$$\text{with } B(\vec{p}_1) = B(\vec{p}_2) = 0,$$

and the 2SS turns out to have the form

$$F = 1 - K_{12} e^{\eta_1 + \eta_2}, \quad G = e^{\eta_1} + e^{\eta_2}, \tag{33}$$

$$\text{with } K_{12} = \epsilon \frac{A(\vec{p}_1 - \vec{p}_2)}{A(\vec{p}_1 + \vec{p}_2)}. \tag{34}$$

Results from search:
(Assume both A and B are nonlinear enough to support solitons:)

$$\begin{cases} (D_x^3 + D_y)\, g \cdot f &= 0, \\ (D_x^3 D_t + aD_x^2 + D_t D_y)\, g \cdot f &= 0, \end{cases}$$

$$\begin{cases} (D_x^3 + D_y)\, g \cdot f &= 0, \\ (D_x^6 + 5D_x^3 D_y - 5D_y^2 + D_t D_x)\, g \cdot f &= 0. \end{cases}$$

$$\begin{cases} (D_x D_t + b)\, G \cdot F &= 0, \\ (D_x^3 D_t + 3bD_x^2 + D_t D_y)(F \cdot F + G \cdot G) &= 0, \end{cases}$$

The nonlinear Schrödinger equation is given by

$$iu_t + u_{xx} + 2\epsilon|u|^2 u = 0, \tag{35}$$

where the function u is complex.

The substitution that bilinearizes (35) is

$$u = g/f, \quad g \text{ complex}, f \text{ real},$$

yielding

$$f\,[(iD_t + D_x^2)g\cdot f] - g\,[D_x^2 f\cdot f - \epsilon 2|g|^2] = 0,$$

For normal (bright) solitons we split this into

$$\begin{cases} (iD_t + D_x^2)g\cdot f &= 0, \\ D_x^2 f\cdot f &= \epsilon 2|g|^2. \end{cases} \tag{36}$$

Soliton solutions for the nonlinear Schrödinger (nlS) type (F is real and G complex):

$$\begin{cases} B(D_{\vec{x}})\,G\cdot F &= 0, \\ A(D_{\vec{x}})\,F\cdot F &= |G|^2. \end{cases} \tag{37}$$

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Formal expansion the 1SS is

$$f = 1 + \varepsilon f_1 + \varepsilon^2 f_2 + \dots, \quad g = \varepsilon g_1 + \dots$$

With this on finds that there can be two kinds of solitons,

$$\begin{cases} F = 1 + e^{\eta_A},\ G = 0, & \text{with dispersion relation } A(\vec{p}) = 0, \\ F = 1 + K e^{\eta_B + \eta_B^*},\ G = e^{\eta_B}, & \text{with dispersion relation } B(\vec{p}) = 0. \end{cases}$$

A search can be based on the existence of 2SSs.

Three equations were found (JH, J. Math. Phys. (1988))

$$\begin{cases} (D_x^2 + iD_y + c)\, G \cdot F &= 0, \\ (a(D_x^4 - 3D_y^2) + D_x D_t)\, F \cdot F &= |G|^2, \end{cases}$$

$$\begin{cases} (i\alpha D_x^3 + 3cD_x^2 + i(bD_x - 2dD_t) + g)\, G \cdot F &= 0, \\ (\alpha D_x^3 D_t + aD_x^2 + (b + 3c^2) D_x D_t + dD_t^2)\, F \cdot F &= |G|^2, \end{cases}$$

$$\begin{cases} (i\alpha D_x^3 + 3D_x D_y - 2iD_t + c)\, G \cdot F &= 0, \\ (a(\alpha^2 D_x^4 - 3D_y^2 + 4\alpha D_x D_t) + bD_x^2)\, F \cdot F &= |G|^2. \end{cases}$$

The last equation combines the two most important $(2+1)$-dimensional equations, Davey-Stewartson and Kadomtsev-Petviashvili equations.

We will next take a closer look on this equation.

Bilinear form of DS

The Davey-Stewartson (DS) equation (the long-wave limit of the Benney-Roskes equation) is given by ($\sigma = \pm 1$)

$$\begin{cases} i\phi_t + (-\sigma_1\partial_X^2 + \partial_Y^2)\phi &= \sigma_2|\phi|^2\phi + 2\sigma_1\sigma_2 Q\phi, \\ (+\sigma_1\partial_X^2 + \partial_Y^2)Q &= -\partial_X^2|\phi|^2. \end{cases} \tag{38}$$

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$$\phi = G/F, \quad Q = 2\sigma_2\partial_X^2 \log F. \quad F \text{ real}, G \text{ complex}.$$

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We convert this into Hirota's bilinear form using the substitution

$$\phi = G/F, \quad Q = 2\sigma_2\partial_X^2 \log F. \quad F \text{ real, } G \text{ complex.}$$

After integrating the second equation twice w.r.t x we get

$$\begin{cases} (iD_t - \sigma_1 D_X^2 + D_Y^2)G \cdot F = 0, \\ (+\sigma_1 D_X^2 + D_Y^2)F \cdot F = -\sigma_2|G|^2. \end{cases}$$

What about the signs σ_i?

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The DSI variant corresponds to $\sigma_1 = -1$

$$\begin{cases} (iD_t + D_X^2 + D_Y^2) G \cdot F & = & 0, \\ (-D_X^2 + D_Y^2) F \cdot F & = & -\sigma_2 |G|^2. \end{cases}$$

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It is usually presented in 45^o rotated form and is known to have dromion solutions.

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Later we will consider the DSII variant ($\sigma_1 = 1$)

$$\begin{cases} (iD_t + (-D_X^2 + D_Y^2)) G \cdot F &= 0, \\ (D_X^2 + D_Y^2) F \cdot F &= -\sigma_2 |G|^2. \end{cases} \tag{39}$$

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after a 45 degree rotation.

Reductions

Let us consider some special cases of

$$\begin{cases} \left(-2iD_t + 3D_xD_y + i\alpha D_x^3 + c\right) G\cdot F &= 0, \\ \left[a(\alpha^2 D_x^4 - 3D_y^2 + 4\alpha D_x D_t) + bD_x^2\right] F\cdot F &= 2|G|^2, \end{cases} \tag{40}$$

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1) If $\alpha = 1, b = 0,\ G = 0$ we get KPI

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2) If $\alpha = c = 0$ we recover the rotated DSII,

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Thus (40) combines of the two most important $(2+1)$-dim equations, but only their DSII and KPI variants.

Place in the Jimbo-Miwa classification

The first equations of KP and second modified KP hierarchy are

$$\begin{aligned}(D_1^3+2D_3+3D_1D_2)\sigma\cdot\tau &= 0,\\ (D_1^3+2D_3-3D_1D_2)\bar\sigma\cdot\tau &= 0,\\ (D_1^4-4D_1D_3+3D_2^2)\tau\cdot\tau &= 24\bar\sigma\sigma.\end{aligned} \tag{41}$$

Note scaling!

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with $D_1=D_x$, $D_2=iD_y$, $D_3=-D_t$, $\tau=F$, $\sigma=G$, $\bar\sigma=G^*$,
$a=1/12$, $\alpha=1$, $b=0$, $c=0$

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Solutions to (41) have been studied by Hirota and Ohta (1991) and Isojima, Willox and Satsuma (2002), Kodama (2005).

Bilinear to nonlinear

F, G are polynomials of exponentials like $e^{\gamma x+\delta y+\kappa t}$.

Soliton-like expressions obtained, e.g., from
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the substitution above gives

$$\begin{cases} i\alpha(q_{xxx} + 6q_x w_{xx}) + 3(q_{xy} + 2w_{xy}q) - 2iq_t + cq &= 0, \\ a2[\alpha^2(w_{xxxx} + 6{w_{xx}}^2) - 3w_{yy} + 4\alpha w_{xt}] + 2bw_{xx} &= |q|^2. \end{cases}$$

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To get something looking like KP and DS we need $u = \partial^2 w$, but which derivatives?

Take an x-derivative of the second equation and define $v = w_x$:

$$\begin{cases} i\alpha(q_{xxx} + 6q_x v_x) + 3(q_{xy} + 2v_y q) - 2iq_t + cq & = 0, \\ 2a[\alpha^2(v_{xxxx} + 12v_x v_{xx}) - 6v_{yy} + 8\alpha v_{xt}] + 2bv_{xx} & = (|q|^2)_x. \end{cases}$$

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Note that we had to use different substitutions for KP and DS. Connection is simpler at the bilinear level!

KP-type solutions

$$\begin{cases} (i\alpha D_x^3 + 3D_xD_y - 2iD_t + c)G\cdot F & = 0, \\ \left[a(\alpha^2 D_x^4 - 3D_y^2 + 4\alpha D_xD_t) + bD_x^2\right]F\cdot F & = |G|^2. \end{cases}$$

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2SS:

$$G = 0, \quad F = 1 + e^{\eta_1} + e^{\eta_2} + A_{12} e^{\eta_1 + \eta_2} \quad A_{12} = \ldots$$

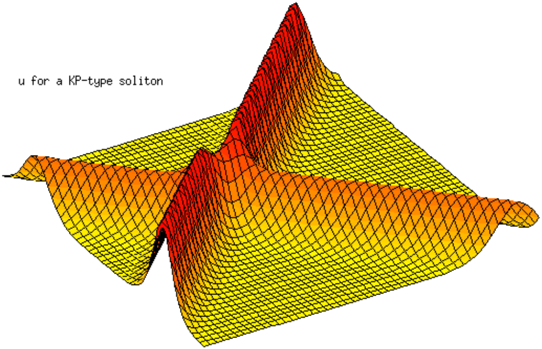

Figure: *xy*-plot of a KP-type solution for the full equation, the plot shows $u = 2\partial_x^2 \log(f)$ at a fixed time.

nlS-type solutions

$$\begin{cases} (i\alpha D_x^3 + 3D_x D_y - 2iD_t + c)G \cdot F & = 0, \\ \left[a(\alpha^2 D_x^4 - 3D_y^2 + 4\alpha D_x D_t) + bD_x^2\right]F \cdot F & = |G|^2. \end{cases}$$

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$$G = \kappa_1 e^{\eta_1}, \quad F = 1 + a_{11} e^{\eta_1+\eta_1^*}, \quad \eta_j = p_j x + q_j y + \omega_j t + \eta_j^0$$

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2SS is given by:

$$G = \kappa_1\, e^{\eta_1} + \kappa_2\, e^{\eta_2} + s_{112}e^{\eta_1+\eta_1^*+\eta_2} + s_{122}e^{\eta_1+\eta_2^*+\eta_2},$$
$$F = 1 + e^{\eta_1+\eta_1^*} + a_{12}e^{\eta_1+\eta_2^*} + a_{21}e^{\eta_2+\eta_1^*} + a_{22}e^{\eta_2+\eta_2^*} + Ae^{\eta_1+\eta_1^*+\eta_2+\eta_2^*}$$

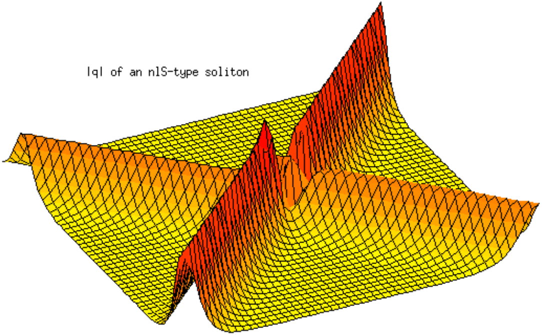

Figure: xy-plot of a nlS-type solution for the full equation, the plot shows $|q| = |G|/F$ at a fixed time

Combinations

$$\begin{cases} (i\alpha D_x^3 + 3D_xD_y - 2iD_t + c)G \cdot F &= 0, \\ \left[a(\alpha^2 D_x^4 - 3D_y^2 + 4\alpha D_x D_t) + bD_x^2\right] F \cdot F &= |G|^2. \end{cases}$$

We can also have combinations of KP- and nls-type solutions:

$$\begin{aligned} F &= 1 + e^{\eta_1} + a_{22}e^{\eta_2+\eta_2^*} + a_{122}e^{\eta_1+\eta_2+\eta_2^*}, \\ G &= \kappa_2 e^{\eta_2} + b_{12}e^{\eta_1+\eta_2}. \end{aligned}$$

Here index 1 corresponds to a KP-soliton with DR from second equation, 2 to a nls-type soliton with DR from the first equation.

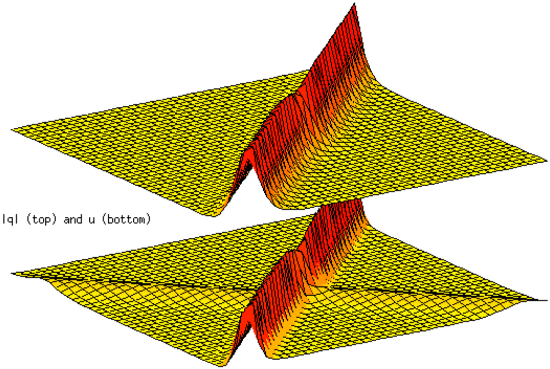

Figure: *xy*-plot of the combination of one nlS-type solution and one KP-type solution for the full equation, at a fixed time. The top part shows $|q| = |G|/F$, the bottom part $u = 2\partial_x^2 \log(f)$.

Dromion solutions

Recall the standard DSI

$$\begin{cases} (iD_t + D_X^2 + D_Y^2)G \cdot F &= 0, \\ D_X D_Y F \cdot F &= |G|^2. \end{cases}$$

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Dromion solutions are built up from ghosts:

$$F = 1 + e^{\eta_i + \eta_i^*},\ G = 0, \quad \eta_i = p_i x + q_i y + \omega_i t + \eta_0, \quad \text{DR: } p_i q_i = 0,\ \forall i$$

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$$F = 1 + e^{\eta_1 + \eta_1^*} + e^{\eta_2 + \eta_2^*} + A e^{\eta_1 + \eta_1^* + \eta_2 + \eta_2^*}, \quad G = \kappa e^{\eta_1 + \eta_2}$$

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but now we have five dispersion relations:

$$p_1^{(r)} = 0, \quad q_2^{(r)} = 0,$$

$$\omega_1^{(r)} = -2q_1^{(r)}(q_1^{(i)} + q_2^{(i)}), \quad \omega_2^{(r)} = -2p_2^{(r)}(p_1^{(i)} + p_2^{(i)}),$$

$$\Im(\omega_1 + \omega_2) = \Re((p_1 + p_2)^2 + (q_1 + q_2)^2), A = 1 + \frac{|\kappa|^2}{8 q_1^{(r)} p_2^{(r)}}.$$

Dromion for the generalized system

$$F = 1 + e^{\eta_1+\eta_1^*} + e^{\eta_2+\eta_2^*} + Ae^{\eta_1+\eta_1^*+\eta_2+\eta_2^*}, \quad G = \kappa e^{\eta_1+\eta_2}$$

with five dispersion relations:

$$\begin{aligned}
\frac{q_1^{(r)}}{p_1^{(r)}} + \frac{q_2^{(r)}}{p_2^{(r)}} =& 2\alpha\left(p_1^{(i)} + p_2^{(i)}\right),\\
\left(\frac{q_1^{(r)}}{p_1^{(r)}}\right)^2 + \left(\frac{q_2^{(r)}}{p_2^{(r)}}\right)^2 =& 4\alpha^2\left([p_1^{(r)}]^2 + [p_2^{(r)}]^2\right) + 4\alpha(q_1^{(i)} + q_2^{(i)}) + \frac{2b}{3a},\\
4\alpha p_j^{(r)}\omega_j^{(r)} =& -4\alpha^2[p_j^{(r)}]^4 + 3[q_j^{(r)}]^2 - \frac{b}{a}[p_j^{(r)}]^2,\\
\omega_1^{(i)} + \omega_2^{(i)} =& -\frac{3}{2}(q_1^{(r)}p_1^{(r)} + q_2^{(r)}p_2^{(r)}) - \frac{\alpha}{2}(p_1^{(i)} + p_2^{(i)})^3\\
&+ \frac{3\alpha}{2}(p_1^{(i)} + p_2^{(i)})([p_1^{(r)}]^2 + [p_2^{(r)}]^2)\\
&+ \frac{3}{2}(p_1^{(i)} + p_2^{(i)})(q_1^{(i)} + q_2^{(i)}) - \frac{c}{2}
\end{aligned}$$

where

$$
\begin{aligned}
A =& 1 + \left(4a\alpha^2[p_1^{(r)}p_2^{(r)}]^2 - |\kappa|^2/96\right) / \\
& \Big[ap_1^{(r)}p_2^{(r)}[\alpha^2(p_1^{(i)} + p_2^{(i)})^2 - \alpha^2(3[p_1^{(r)}]^2 + 2p_1^{(r)}p_2^{(r)} + 3[p_2^{(r)}]) \\
& - 2\alpha(q_1^{(i)} + q_2^{(i)}) - \tfrac{b}{3a}]\Big]
\end{aligned}
$$

An illustration of this dromion solution is next

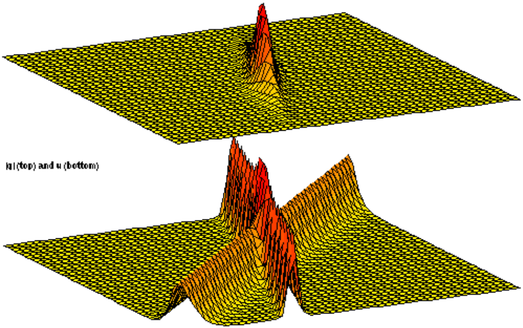

Figure: xy-plot of a dromion solution for the full equation, at a fixed time. The top part shows $|q| = |G|/F$, the bottom part $u = 2\partial_x^2 \log(f)$.

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Advantages of the bilinear formalism:

- Multisoliton solutions easy to construct.
- The dependent variables are usually tau-functions, with good properties.
- Natural for the Sato theory, which explains hierarchies of integrable equations (Jimbo and Miwa)
- Suitable for classification: the bilinear form strongly restricts the freedom of changing dependent variables.

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