Definitions and Predictions of Integrability for Difference Equations

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Symmetries and Integrability of Difference Equations
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Although complete integrability is structurally unstable, many properties persist in nearby non-integrable systems.

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- Need to discretize continuous equations for numerical analysis
- Interesting mathematics in the background, e.g., elliptic functions.
- Continuum integrability is well established, all easy things have already been done. Discrete integrability relatively new, still new things to be discovered.

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$$x_{n+1} + x_{n-1} = f(x_n).$$

Given x_0, x_1 we can compute x_n for all $n \in \mathbb{Z}$.
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In these lectures: we take a look on various meanings of integrability for difference equations, and the possible associated algorithmic methods to identify (partial) integrability.

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Different settings bring in different properties, tools and results.

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Sensitive dependence on the initial value:

$$\frac{dy_n}{dc_0} = \tfrac{1}{2}\, 2^n \, \sin(2^n c_0)$$

Thus error grows exponentially: “chaotic”.

Integrable discretization? (OΔE)

Example: ODE

$$\frac{du}{dt} = \alpha u(1-\beta u), \qquad (*)$$

with solution

$$u(t) = \frac{u_0}{\beta u_0 + (1-\beta u_0)e^{-\alpha t}}.$$

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Naive discretization:

$$\frac{du}{dt} \approx \frac{u(t+\Delta t) - u(t)}{\Delta t} \Rightarrow$$

$$u(t+\Delta t) - u(t) = \Delta t\, \alpha u(t)(1-\beta u(t)). \qquad (d1)$$

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Let $u(t) = u(n\Delta t) = \frac{a}{\alpha\beta\Delta t}x_n$, $a = 1+\alpha\Delta t$, then we get

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This is the logistic equation which can be chaotic.

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Second attempt:

$$u(t + \Delta t) - u(t) = \Delta t\, \alpha u(t + \Delta t)(1 - \beta u(t)). \qquad (d2)$$

or after solving for $u(t + \Delta t)$

$$u(t + \Delta t) = \frac{u(t)}{(1 - \alpha \Delta t) + \alpha\beta\Delta t\, u(t)}$$

Why should we even consider this?

The original equation

$$\frac{du}{dt} = \alpha u(1 - \beta u), \qquad (*)$$

can be linearized with $u = 1/(w + \beta)$ to

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Discretize the linearized equation as

$$w(t + \Delta t) - w(t) = -\alpha \, \Delta t \, w$$

and then substituting $w = -\beta + 1/u$ we get

$$u(t + \Delta t) - u(t) = \alpha \, \Delta t \, u(t + \Delta t)(1 - \beta u(t)). \qquad (d2)$$

The difference equation for w

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is solved by

$$w(t+n\Delta t) = (1-\alpha\,\Delta t)^n w(t)$$

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Now the discrete solution samples the continuum solution.

$$u(t) = \frac{u_0}{\beta u_0 + (1-\beta u_0)e^{-\alpha t}}.$$

Examples and continuum limits

The discrete Painlevé I equation (d-PI) is given by

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$$\epsilon n = z,\ x_n = f(z),\ x_{n\pm 1} = f(z \pm \epsilon), \quad \epsilon \to 0,\ n \to \infty,\ \epsilon n \text{ fixed}$$

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This yields

$$3f + \epsilon^2 f'' = \frac{\alpha + \beta z/\epsilon}{f} + b.$$

The get rid of the denominator we must take

$$f(z) = c_1 + c_2 \epsilon^\kappa y(z),$$

and expand. The power $\kappa > 0$ is to determined.

$$3c_1+3c_2\epsilon^\kappa y(z)+3c_2\epsilon^{2+\kappa}y'' = b+\frac{1}{c_1}(\alpha+\beta z/\epsilon)(1-\frac{c_2}{c_1}\epsilon^\kappa y+(\frac{c_2}{c_1})^2\epsilon^{2\kappa}y^2\ldots$$

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To balance terms we must take $\kappa=2$, then we get

ϵ^0: $3c_1=b+\frac{1}{c_1}\alpha$

ϵ^2: $3c_2=-\frac{c_2}{c_1^2}\alpha$

leading to

$$c_1=\frac{b}{6},\quad \alpha=-\frac{b^2}{12}$$

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Finally at ϵ^4 we get the first Painleve equation

$$y''=6y^2+z,$$

if we choose

$$c_2=-\frac{b}{3},\quad \beta=-\frac{b^2}{18}\epsilon^5.$$

Constants of motion for continuous ODE

Definition of Liouville integrability:

A Lagrangian $L(\dot{q}, q)$, where q is N-dimensional, is integrable if there are N constants of motion (CM) $I_k(\dot{q}, q)$ (L one of them) such that the I_k

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The role of a CM (in continuous and discrete world): it restricts the available phase space and thereby makes the motion more predictable.

Relation of CM to the equation:

$$\frac{dI(\dot{q},q)}{dt} = \sum_i \frac{\partial I}{\partial \dot{q}_i}\ddot{q}_i + \sum_i \frac{\partial I}{\partial q_i}\dot{q}_i.$$

The RHS should vanish when we impose the equations of motion of the type

$$\ddot{q}_i = \dots$$

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N=1: Any given $I(\dot q, q)$ is a CM for some equation $\ddot q = ...$.

The basic difficulty in the discrete case

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Consider the discrete equivalent, a 3-point equation in $x \equiv u_{n+1}$, $y \equiv u_n$, $z \equiv u_{n-1}$.

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How could this produce an equation linear in x, z if K is nonlinear?

The lack of Liebnitz rule bites us again!

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Then we may try a biquadratic K:

$$K(x,y) := c_5 x^2 y^2 + c_4 xy(x+y) + c_3 xy + c_2(x^2+y^2) + c_1(x+y). \qquad (*)$$

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We get

$$\frac{K(x,y) - K(y,z)}{x-z} = c_1 + c_2(x+z) + c_3 y + c_4 y(x+y+z) + c_5 y^2 (x+z),$$

from which we get an equation having $(*)$ as CM.

$$x + z = \frac{c_4 y^2 + c_3 y + c_1}{c_5 y^2 + c_4 y + c_2}$$

The QRT map

Can we generalize?

Yes: take a rational biquadratic:

$$K(x,y) = \frac{c_5x^2y^2 + c_4xy(x+y) + c_3xy + c_2(x^2+y^2) + c_1(x+y)}{d_5x^2y^2 + d_4xy(x+y) + d_3xy + d_2(x^2+y^2) + d_1(x+y)}$$

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Direct computation shows that this is a CM for the symmetric version of the Quispel-Roberts-Thomson (QRT) map:

$$x = \frac{f_1(y) - f_2(y)\,z}{f_2(y) - f_3(y)\,z}$$

where f_i are certain specific quartic polynomials.

This contains almost all 3-point maps.

Some examples of QRT

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One of the discrete Painlevé equation is $\mathrm{dP_{III}}$ ($f_2 = 0$):

$$x_{n+1}x_{n-1} = \frac{cd\,(x_n - a\lambda^n)(x_n - b\lambda^n)}{(x_n - c)(x_n - d)}.$$

This is a nonautonomous equation,
i.e., it contains explicit n-dependence.

The HKY generalization

The Hirota-Kimura-Yahagi (HKY) generalization: Quartic CM

Consider

$$K(x,y) = \frac{2xy}{x^2+y^2+\beta^2},$$

Then we have

$$K(x,y)-K(y,z) = \frac{-2y(x-z)[xz-(y^2+b^2)]}{(x^2+y^2+b^2)(y^2+z^2+b^2)}$$

leading to the 3-point equation

$$xz = y^2+b^2.$$

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But we also have

$$K(x,y)+K(y,z)=\frac{2y(x+z)[xz+(y^2+b^2)]}{(x^2+y^2+b^2)(y^2+z^2+b^2)}$$

How can this be interpreted?

It seems that in the second case K is conserved "up to sign".

Then $K(x,y)^2$, which is quartic, should be a genuine invariant.

Indeed:

$$K(x,y)^2 - K(y,z)^2 = \frac{-4y^2(x+z)(x-z)[xz+(y^2+b^2)][xz-(y^2+b^2)]}{(x^2+y^2+b^2)^2(y^2+z^2+b^2)^2}$$

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Thus

- $u_{n+1}u_{n+1} = u_n^2 + b^2$ has a quadratic invariant
- $u_{n+1}u_{n+1} = -(u_n^2 + b^2)$ has a quartic invariant

It seems that in the second case K is conserved “up to sign”.

Then $K(x,y)^2$, which is quartic, should be a genuine invariant.

Indeed:

$$K(x,y)^2 - K(y,z)^2 = \frac{-4y^2(x+z)(x-z)[xz+(y^2+b^2)][xz-(y^2+b^2)]}{(x^2+y^2+b^2)^2(y^2+z^2+b^2)^2}$$

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Other HKY-type invariants are known.

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What about growth analysis?
Recall that difference equations can trivially be solved step by step, what is the growth of the resulting expression?

Singularity analysis for difference equations

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Using this principle it has been possible to find discrete analogies of Painlevé equations. [Ramani, Grammaticos and JH, *Phys. Rev. Lett.* 67 (1991) 1829, and many others]

Singularity confinement in practice

Consider first the autonomous case of dPI

$$x_{n+1} = -x_n - x_{n-1} + \frac{a}{x_n} + b.$$

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To resolve "$\infty - \infty$":
assume $x_0 = \epsilon$ (small) and redo the calculations.

Detailed singularity confinement calculation

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The singularity is confined and initial information **u** is recovered.
The singularity pattern is $\dots, 0, \infty, -\infty, 0, \dots$.

Non-confined singularity

A worst case example:

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We obtain

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In general

$$x_k = k\,\tfrac{a}{\epsilon} + \dots,$$

and the singularity is not confined, ever.
Furthermore: there are no ambiguities.

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Problem: x_4 should start like $\mathbf{u} + \dots$!

$$x_4 = -\frac{a_3-a_2-a_1+a_0}{a_2+a_1+a_0}\,\frac{a_0}{\epsilon} + \dots$$

x_4 should start like **u** $+ \dots \Longrightarrow$
The condition for singularity confinement at this same step is:

$$a_{n+3} - a_{n+2} - a_{n+1} + a_n = 0,\ \forall n$$

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with solution

$$a_n = \alpha + \beta n + \gamma (-1)^n. \qquad (*)$$

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Recall the form of the discrete Painlevé equation (d-PI)

$$x_{n+1} + x_n + x_{n-1} = \frac{\alpha + \beta n}{x_n} + b.$$

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Recall the form of the discrete Painlevé equation (d-PI)

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In general, with a_n as in (*) the singularity is confined, and

$$x_4 := \frac{\mathbf{u}(\alpha+\gamma)+2b\beta}{\alpha+3\beta-\gamma} + O(\epsilon),$$

in particular, if $\beta = \gamma = 0$ (i.e., $a_n = \alpha$), $x_4 = \mathbf{u} + \dots$

Singularity confinement in projective space

The singularities reveal their nature best in projective space, where $(u, v, f) \approx (\lambda u, \lambda v, \lambda f)$, $\lambda \neq 0$

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Write it as a first order system

$$\begin{cases} x_{n+1} & = & -x_n - y_n + \frac{a_n}{x_n} + b, \\ y_{n+1} & = & x_n, \end{cases}$$

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$$\begin{cases} x_{n+1} &= -x_n - y_n + \frac{a_n}{x_n} + b, \\ y_{n+1} &= x_n, \end{cases}$$

Then homogenize by substituting $x_n = u_n/f_n$, $y_n = v_n/f_n$:

$$\begin{cases} \frac{u_{n+1}}{f_{n+1}} &= -\frac{u_n}{f_n} - \frac{v_n}{f_n} + a_n \frac{f_n}{u_n} + b, \\ \frac{v_{n+1}}{f_{n+1}} &= \frac{u_n}{f_n}, \end{cases}$$

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Then clearing denominators yields a polynomial map in $\mathbb{P}^2$

$$\begin{cases} u_{n+1} & = & -u_n(u_n + v_n) + f_n(a_n f_n + b u_n), \\ v_{n+1} & = & u_n^2, \\ f_{n+1} & = & f_n u_n. \end{cases}$$

Note: default growth of degree (= complexity): $\deg(u_n) = 2^n$

The sequence that led to a singularity was

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In projective space we have

$$\begin{pmatrix} 0 \\ \mathbf{u} \\ 1 \end{pmatrix} \to \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \to \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} \to \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \to \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix},$$

The last term is a true singularity, since it is not in $\mathbb{P}^2$.

For the detailed ϵ study with $x_{-1} = \mathbf{u}$, $x_0 = \epsilon$ we have

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$$\begin{pmatrix} x_2 \\ x_1 \\ 1 \end{pmatrix} \approx \begin{pmatrix} u_2 \\ v_2 \\ f_2 \end{pmatrix} = \begin{pmatrix} -a_0^2 + \epsilon a_0(2\mathbf{u} - b) + \dots \\ a_0^2 + 2\epsilon a_0(-\mathbf{u} + b) + \dots \\ \epsilon a_0 + \epsilon^2(-\mathbf{u} + b) + \dots \end{pmatrix}.$$

For the detailed ϵ study with $x_{-1} = \mathbf{u}$, $x_0 = \epsilon$ we have

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$$\begin{pmatrix} x_3 \\ x_2 \\ 1 \end{pmatrix} \approx \begin{pmatrix} u_3 \\ v_3 \\ f_3 \end{pmatrix} = \begin{pmatrix} \epsilon^2 a_0^2(-a_0 + a_1 + a_2) + \dots \\ a_0^4 + 2\epsilon a_0^3(-2\mathbf{u} + b) \dots \\ -\epsilon a_0^3 + \epsilon^2 a_0^2(3\mathbf{u} - 2b) + \dots \end{pmatrix}$$

$$\begin{pmatrix} u_4 \\ v_4 \\ f_4 \end{pmatrix} = \begin{pmatrix} \epsilon^2 a_0^6 A_3 + \epsilon^3 a_0^5 (b(4A_3 + a_0 - a_2) - \mathbf{u}(6A_3 + a_0)) + \dots \\ \epsilon^4 a_0^4 A_2^2 + \dots \\ -\epsilon^3 a_0^5 A_2 + \dots \end{pmatrix}$$

$(A_2 = a_2 + a_1 - a_0, A_3 = a_0 - a_1 - a_2 + a_3.)$

This is the crucial point of singularity confinement.

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If $A_3 = 0$, $A_2 \neq 0$ then ϵ^3 is a common factor and can be divided out and then the $\epsilon \to 0$ limit yields

$$\begin{pmatrix} u_4 \\ v_4 \\ f_4 \end{pmatrix} = \begin{pmatrix} (a_0(\mathbf{u} - b) + a_2 b) \\ 0 \\ a_3 \end{pmatrix}.$$

Thus we have emerged from the singularity and in particular recovered the initial data **u**.

- The cancellation of the common factor ϵ^3 removes the singularity.
- Any cancellation also reduces growth of complexity, as defined by the degree of the iterate.

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- growth is linear in $n \Rightarrow$ equation is linearizable.
- growth is polynomial in $n \Rightarrow$ equation is integrable.
- growth is exponential in $n \Rightarrow$ equation is chaotic.

Singularity confinement is not sufficient

Counterexample (JH and C Viallet, PRL 81, 325 (1999))

$$x_{n+1} + x_{n-1} = x_n + \frac{1}{x_n^2}.$$

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Epsilon analysis of singularity confinement:
Assume $x_{-1} = \mathbf{u}$, $x_0 = \epsilon$ and then

$$\begin{aligned}
x_1 &= \epsilon^{-2} - \mathbf{u} + \epsilon, \\
x_2 &= \epsilon^{-2} - \mathbf{u} + \epsilon^4 + O(\epsilon^6), \\
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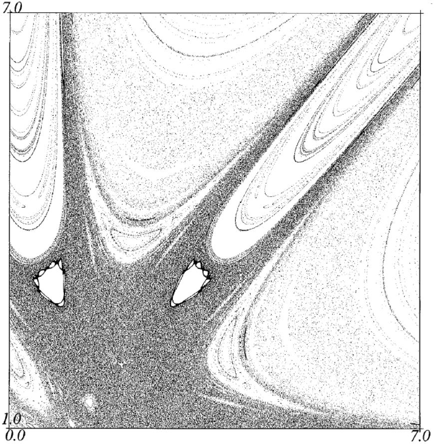
Thus singularity is confined with pattern $\ldots, 0, \infty, \infty, 0, \ldots$.
Furthermore, the initial information **u** is recovered in x_4. OK?

No! The HV map shows numerical chaos

$$x_{n+1} + x_{n-1} = x_n + \frac{7}{x_n^2}$$



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For the previous Painlevé equation the degrees grow as

$$1, 2, 4, 8, 13, 20, 28, 38, 49, 62, 76, \dots$$

which is fitted by $d_n = \frac{1}{8}(9 + 6n^2 - (-1)^n)$. [JH and Viallet, Chaos, Solitons and Fractals, **11**, 29-32 (2000).]

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- Diophantine integrability (numerically fast) (Halburd)