Deriving the 17 wallpaper groups

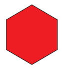
Jarkko Kari, University of Turku

Properties of wallpaper groups:

(1) The group has a shortest non-trivial translation τ.

(2) Crystallographic restriction: All rotations are 2-, 3- , 4- or 6-fold rotations.

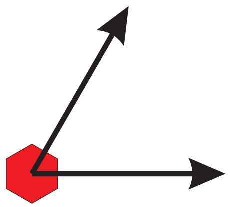
Also, 4-fold rotations cannot co-exist with 3- or 6-fold rotations (because together they would generate a rotation by 30°.)



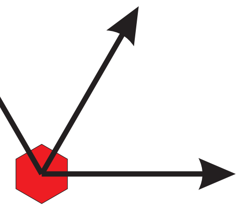
Case (1): Suppose G contains 60° rotations.

Let P be the center of some sixfold rotation $\rho \in G$.

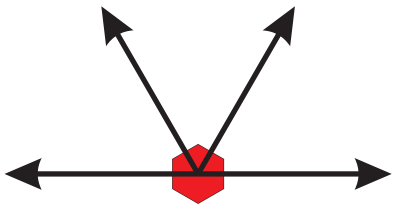
Let τ be a shortest translation in G.



The conjugate $\rho\tau\rho^{-1}$ of τ is the translation that moves point P to point $\rho\tau(P)$. The conjugate is also in group G.



Conjugating τ by rotations around P gives more translations.



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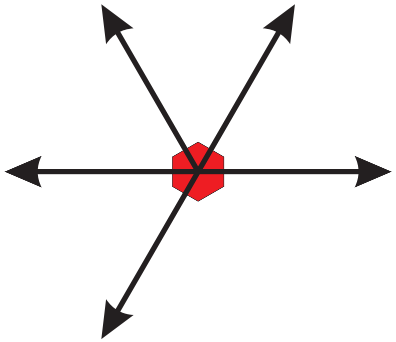

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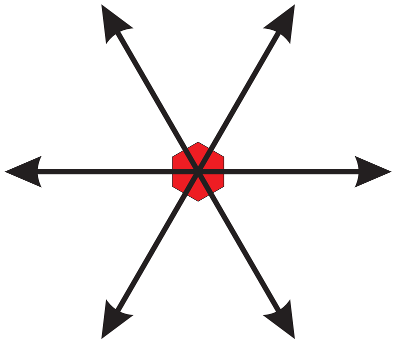

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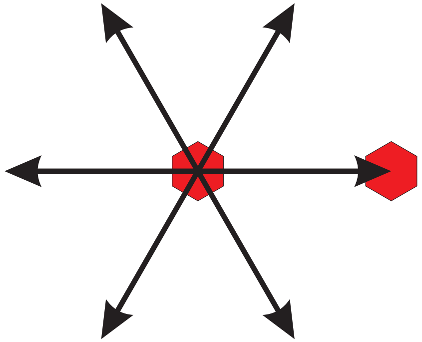
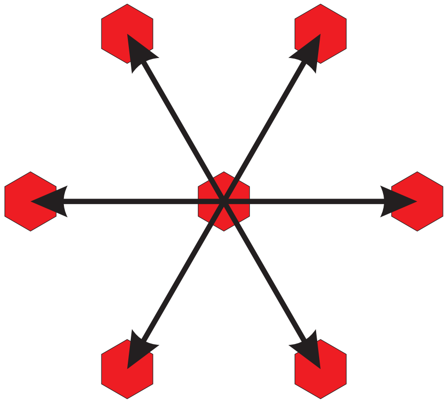
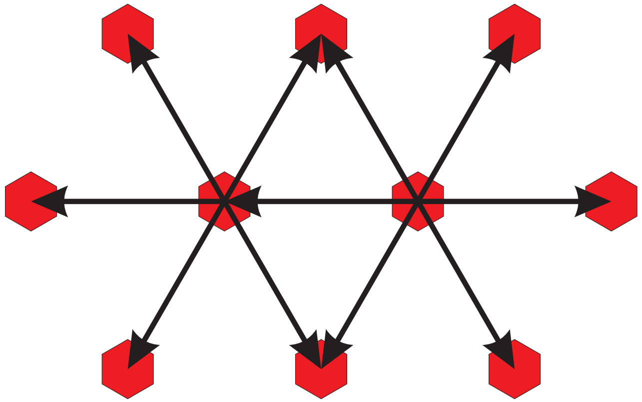


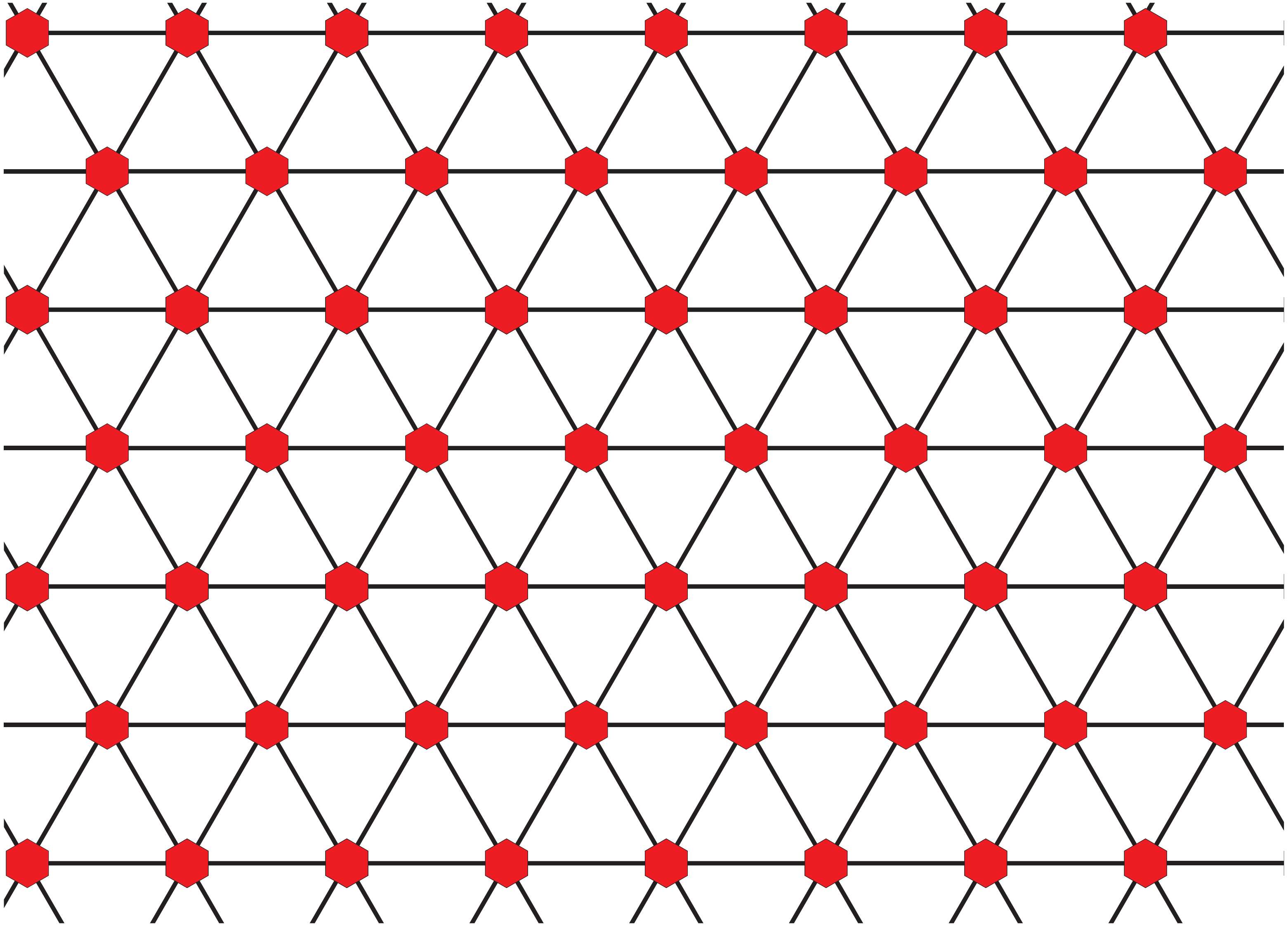
The conjugate $\tau\rho\tau^{-1}$ of ρ is a 60° rotation around point $\tau(P)$.



Analogously, conjugates of ρ by the six translations provide six centers of sixfold rotation.

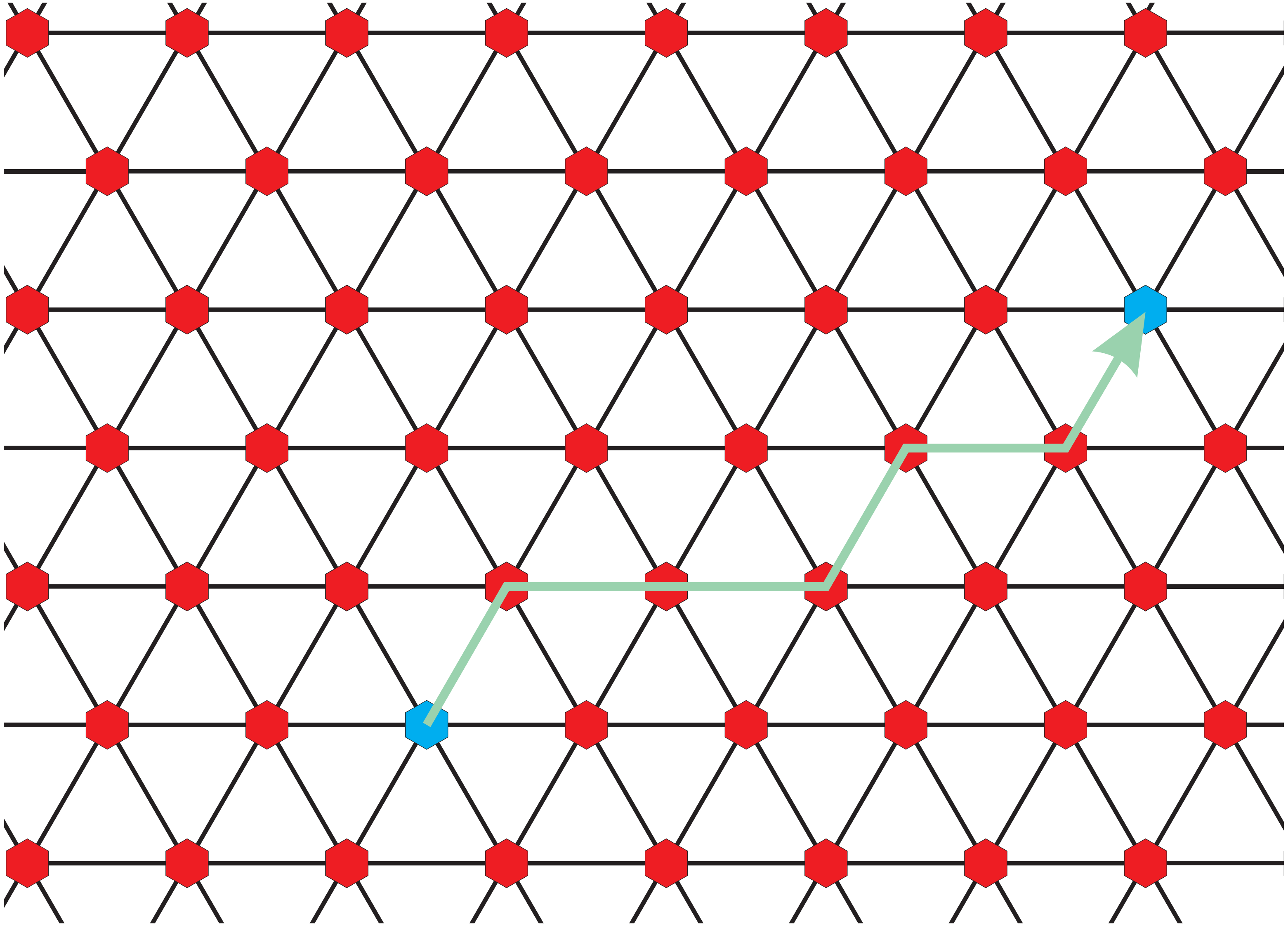


The same reasoning around the rotation center $\tau(P)$ provides new centers of rotation.

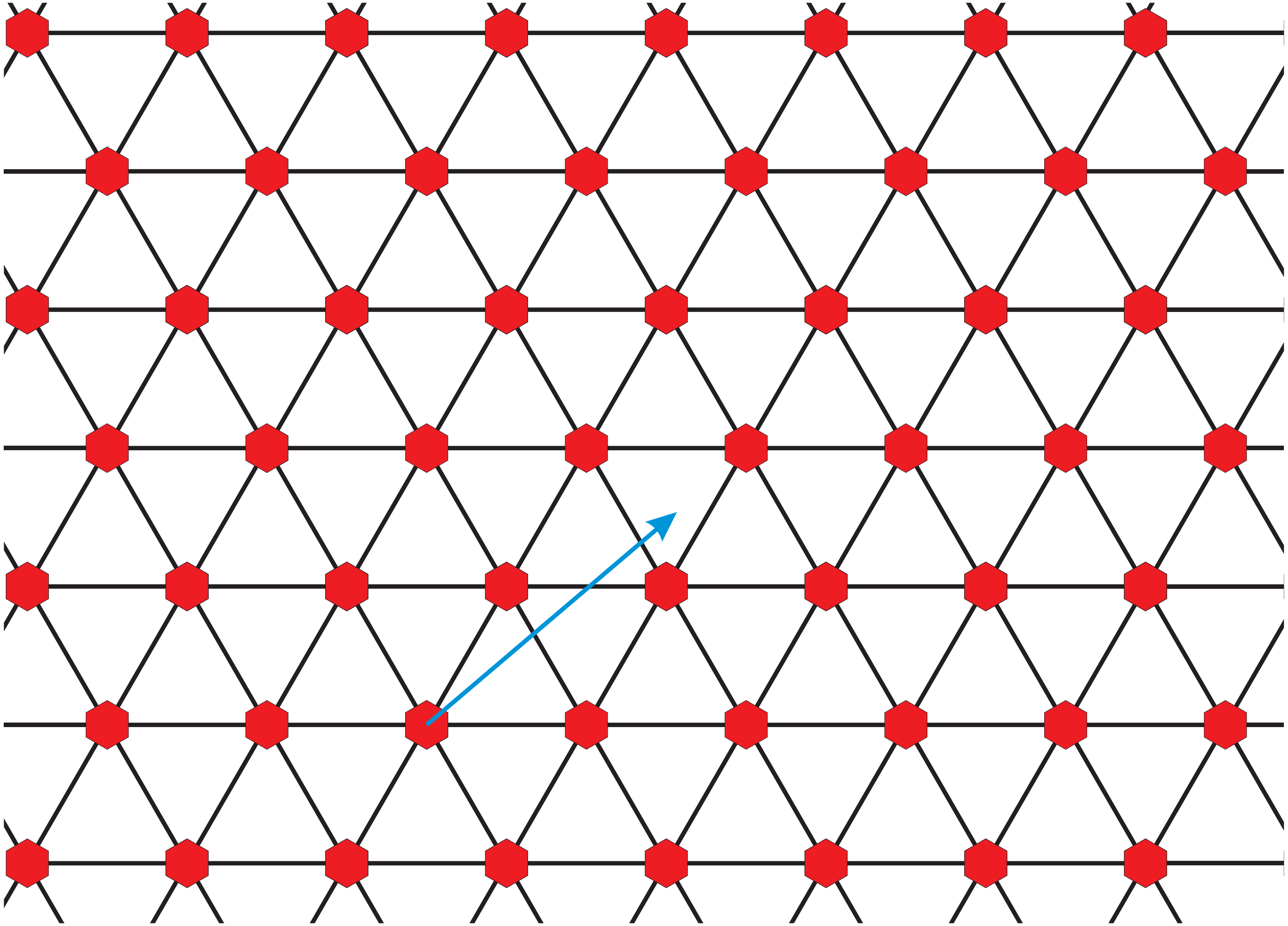


This reasoning can be repeated for all centers of rotation, which provides us with an infinite triangular lattice of rotation centers.

Let S_6 be the set of these rotation centers.

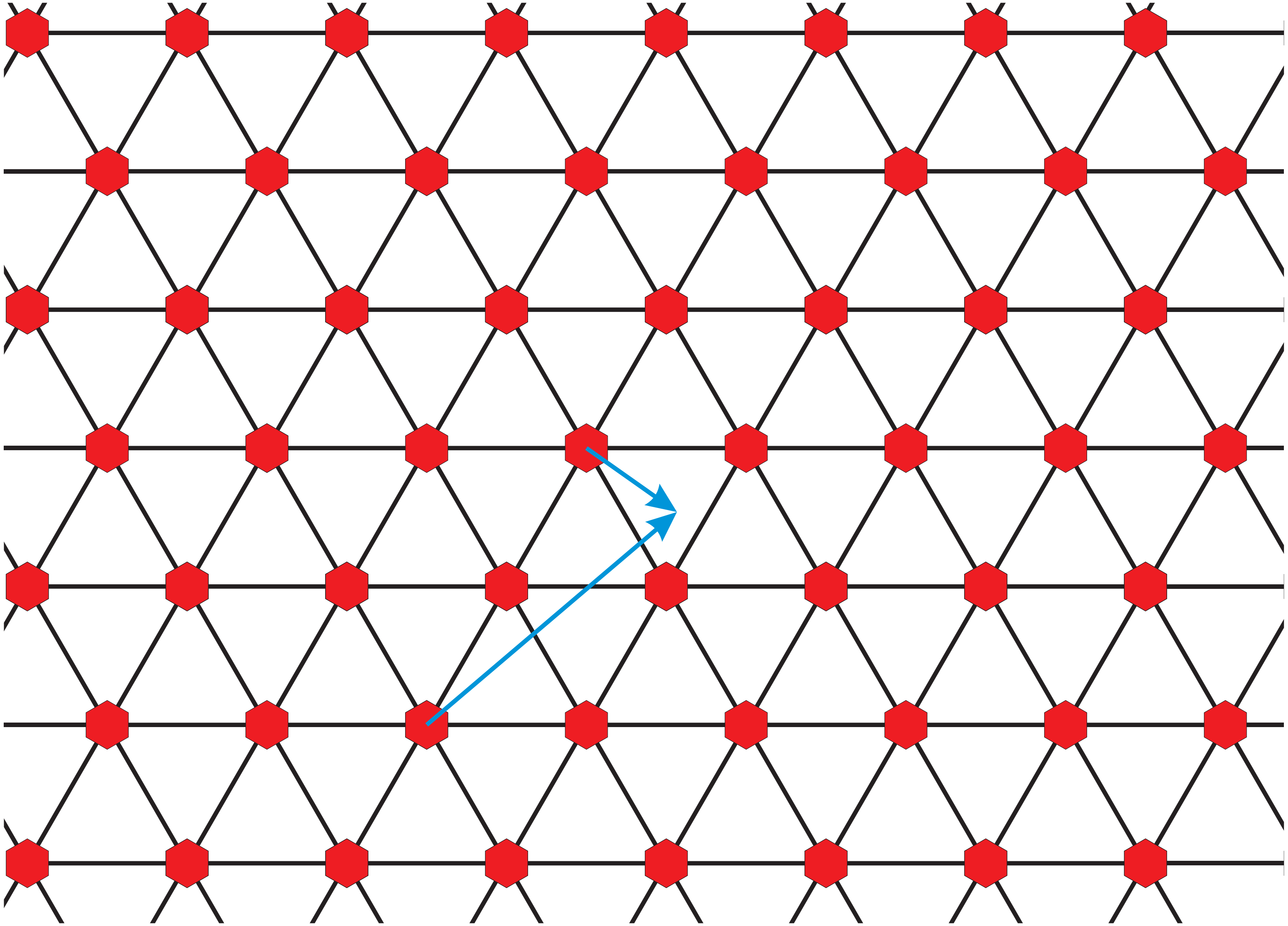


Any translation moving a lattice point into another lattice point is in group G, as it is a composition of the one step translations in the lattice.



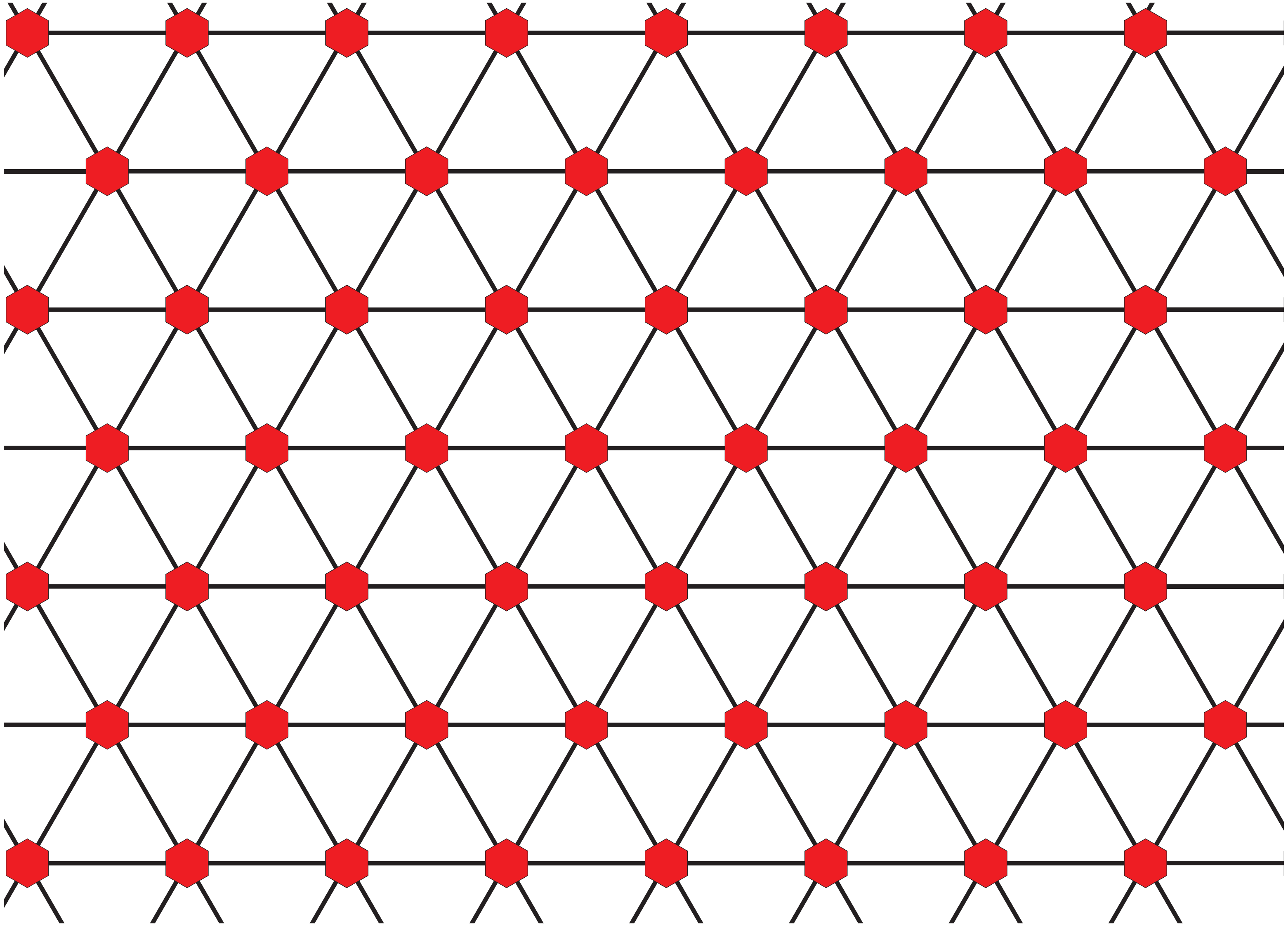
Conversely, all translations in G must take lattice points to lattice points.

Suppose the contrary: some translation takes a lattice point into a non-lattice point.



Then G also contains a translation taking a vertex of one of the equilateral triangles of the lattice into a non-vertex point of the triangle.

But this contradicts the minimality of translation τ.



We have proved that the translations in G are exactly the translations that map the lattice S_6 into itself.

Let us prove the same for rotations.

If $\rho' \in G$ is any rotation then – by the crystallographic restriction – the rotation angle is a multiple of 60°.

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It follows that $\rho'\rho^i = \tau'$ is a translation, for some i, where ρ is our initial rotation around a lattice point.

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Then $\rho' = \tau'\rho^{-i}$ takes lattice points to lattice points since both ρ and τ' are symmetries of the lattice.

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We have proved that the rotations in G are also symmetries of the lattice S_6.

Conversely, let us show that all rotational symmetries of S_6 are in G. So let ρ' be a rotation that is a symmetry of S_6.

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The symmetry group of S_6 is a wallpaper group that contains a sixfold rotation, so by the crystallographic restriction, the rotation angle of ρ' is a multiple of $60°$.

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The symmetry group of S_6 is a wallpaper group that contains a sixfold rotation, so by the crystallographic restriction, the rotation angle of ρ' is a multiple of $60°$.

Then, $\rho'\rho^i = \tau'$ is a translation, for some i. This translation maps lattice points to lattice points and hence $\tau' \in G$.

Conversely, let us show that all rotational symmetries of S_6 are in G. So let ρ' be a rotation that is a symmetry of S_6.

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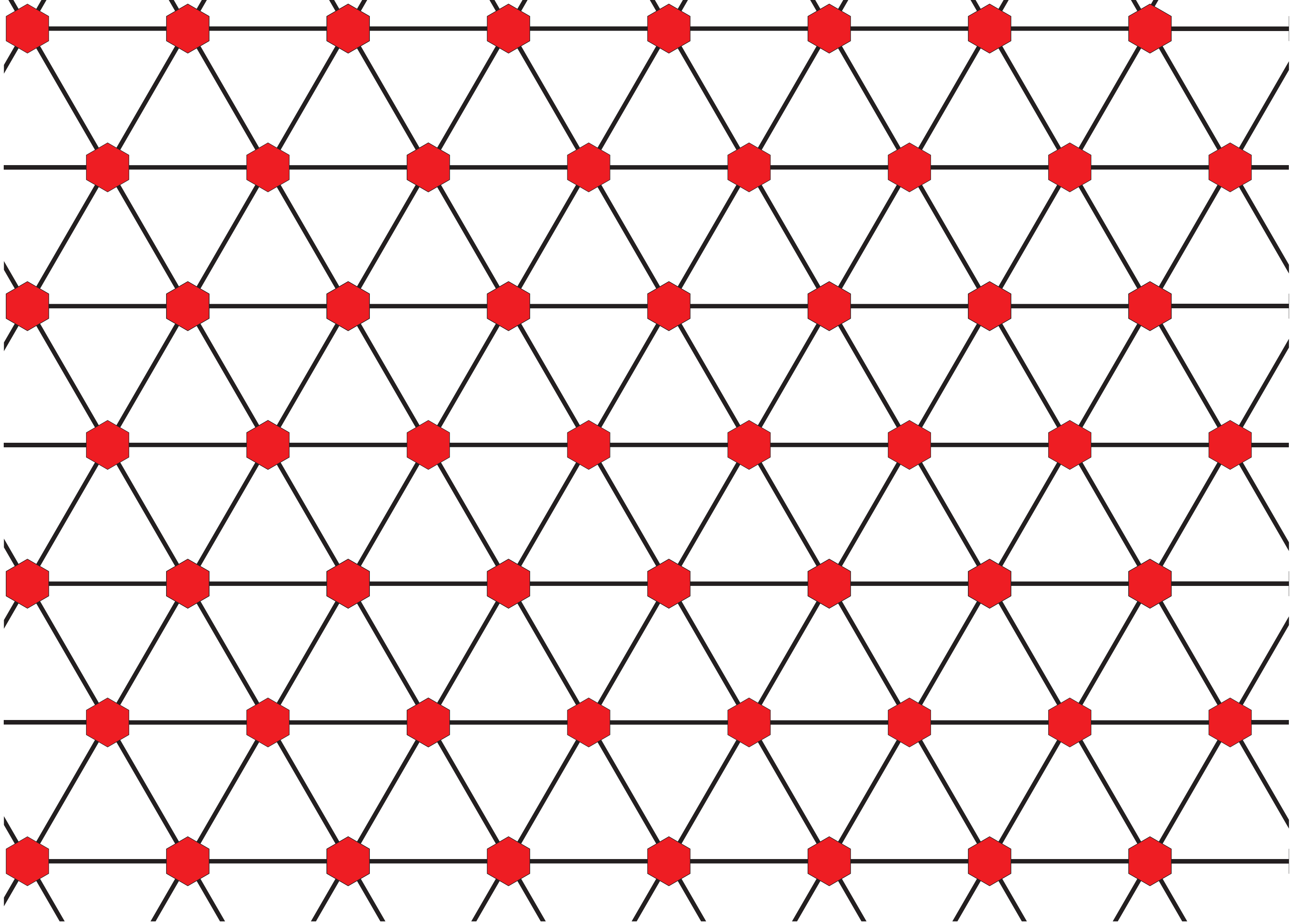
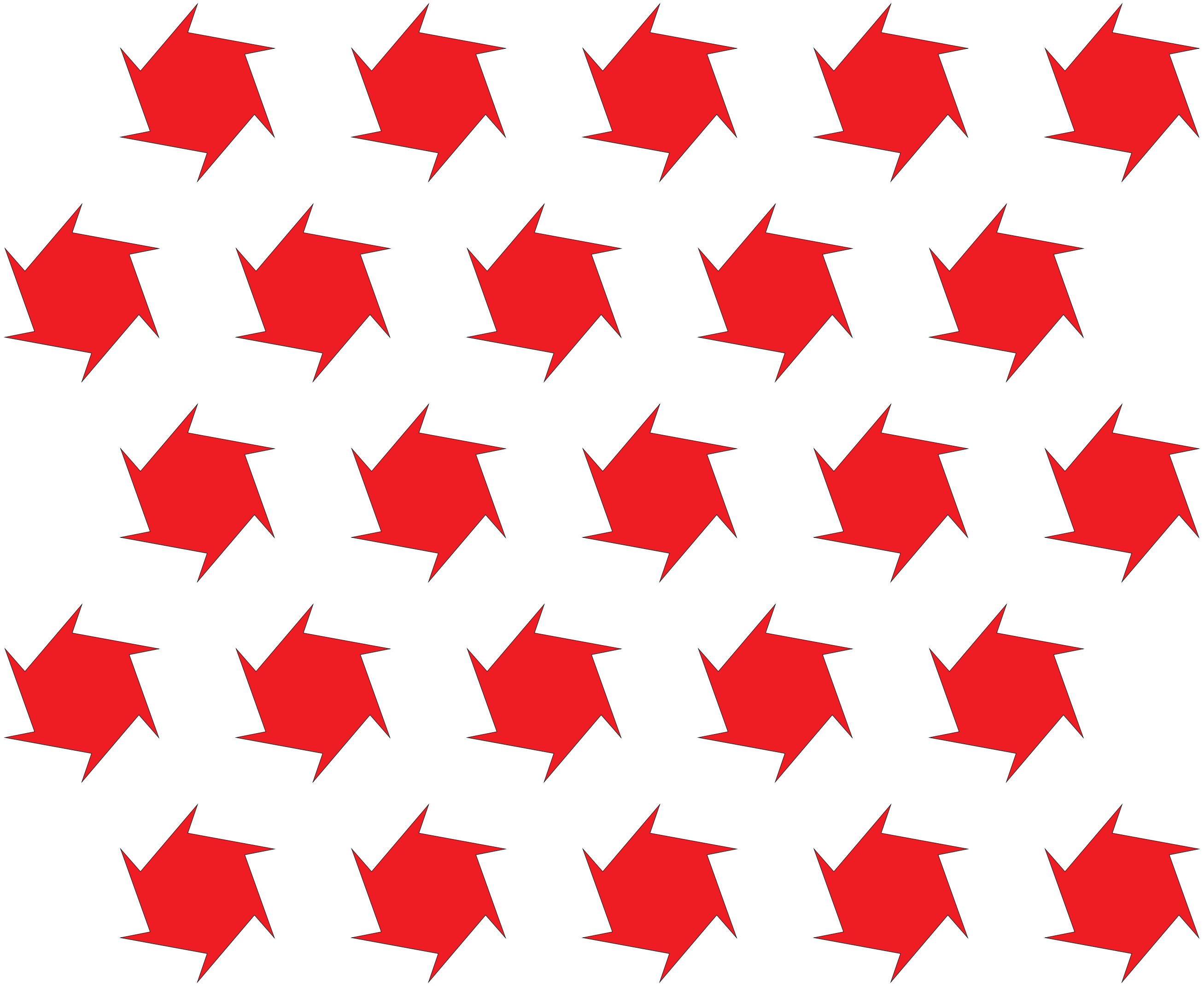
We have $\rho' = \tau'\rho^{-i} \in G$.



We have proved that the even isometries in G are exactly the even symmetries of the lattice S_6.

If G contains no odd isometries then G is uniquely determined: It is the wallpaper group

$$W_6 = \langle \rho, \tau \rangle.$$



An example of a pattern whose symmetry group is

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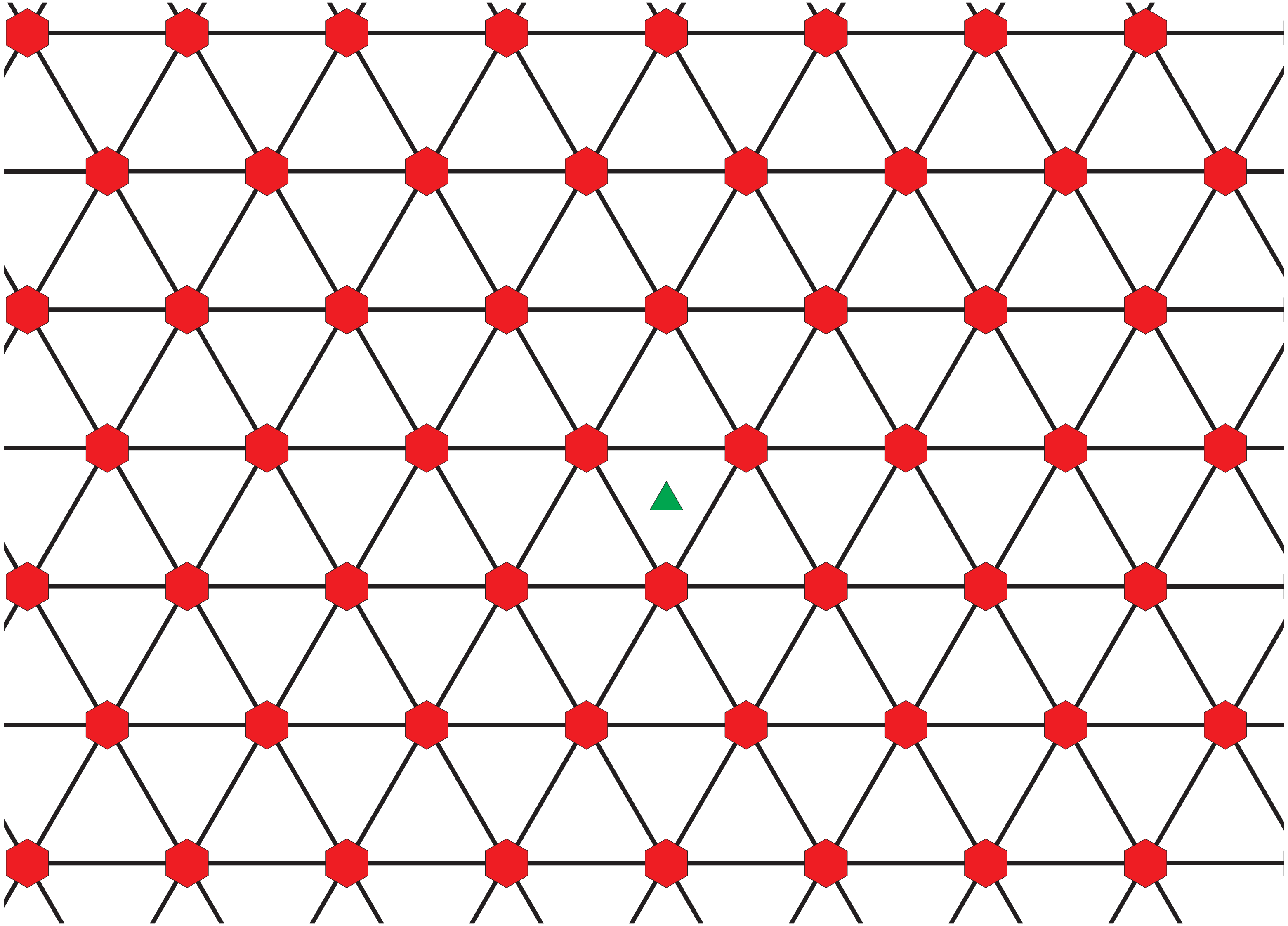
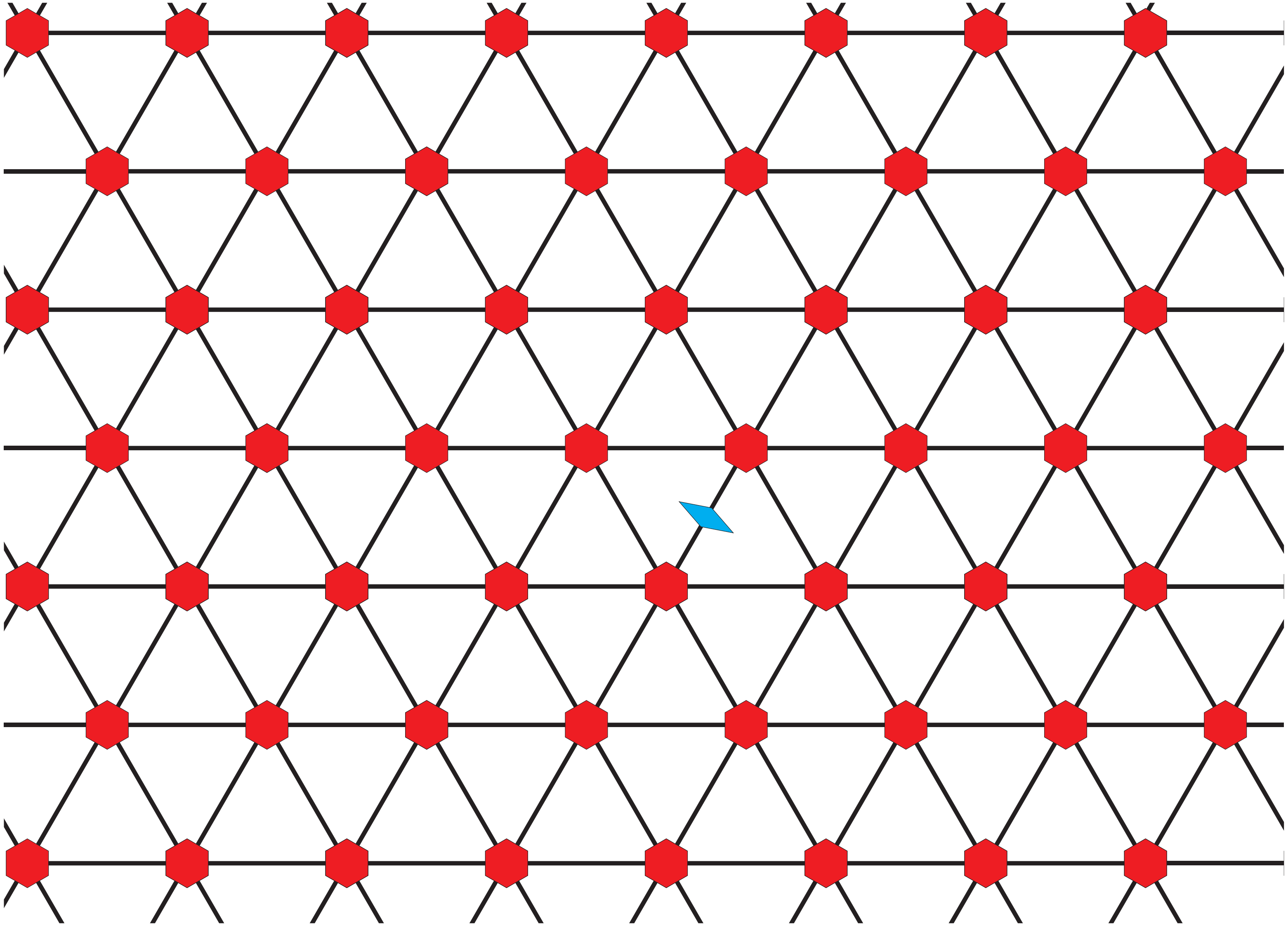
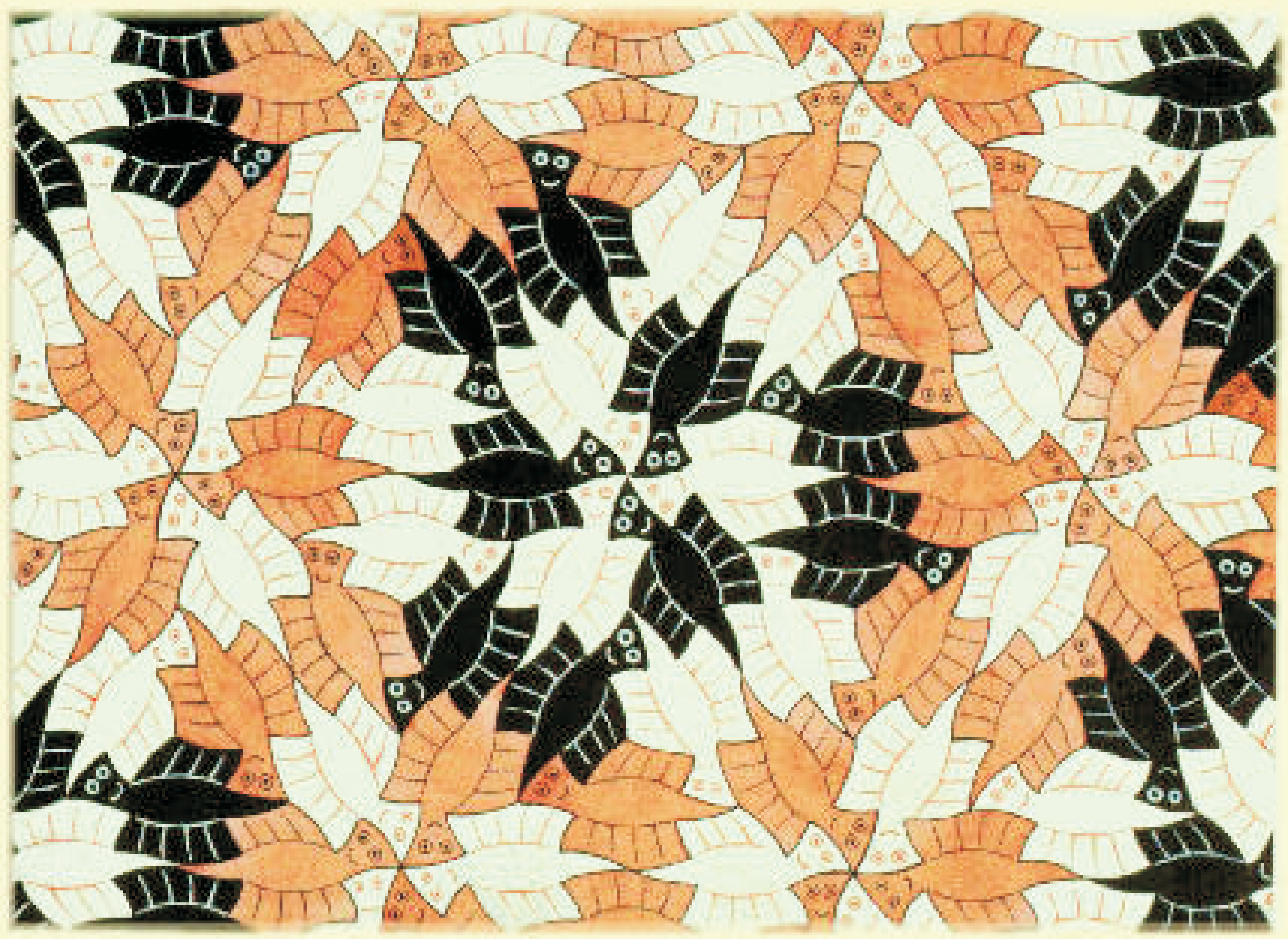


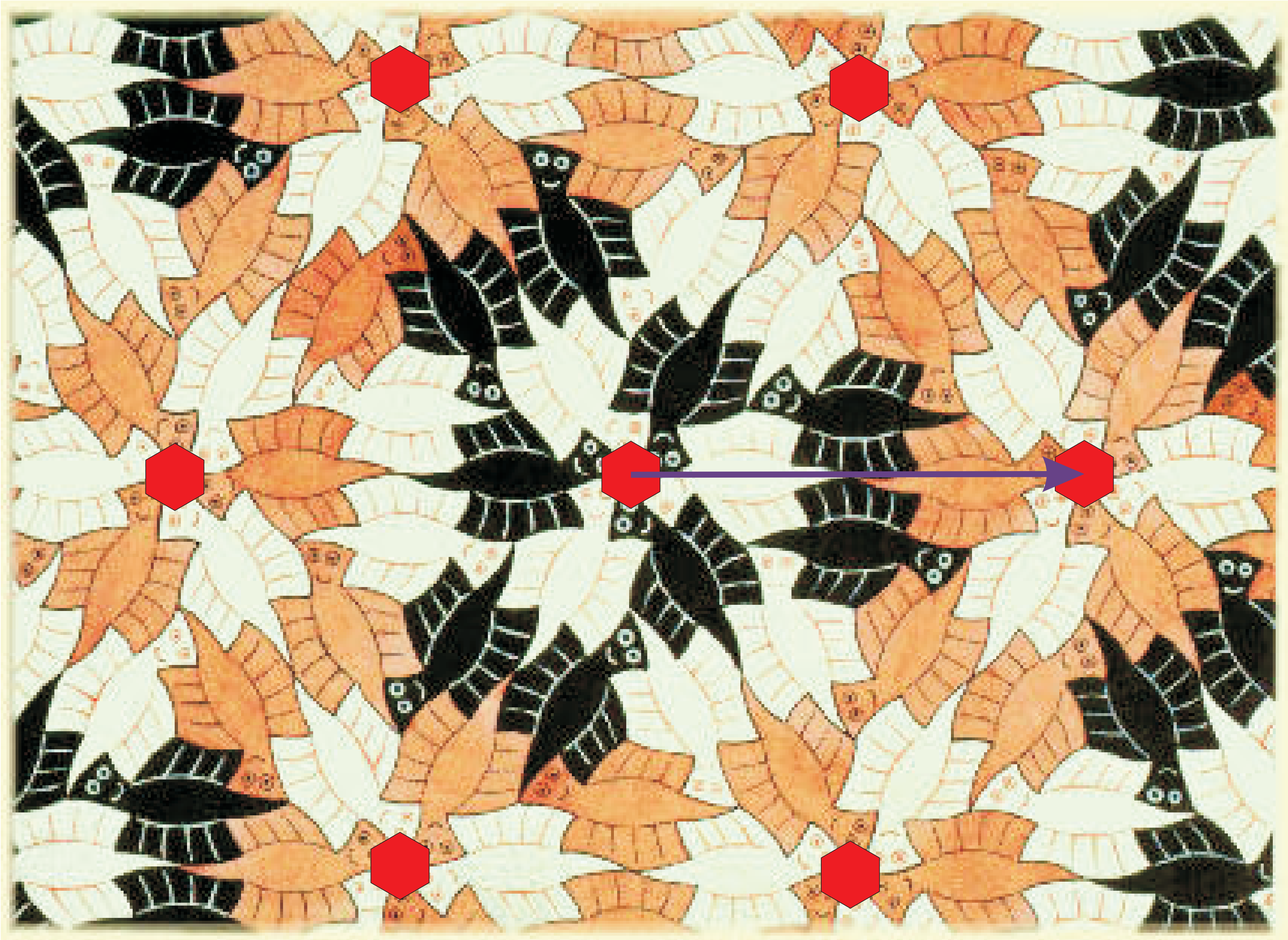
In addition to the sixfold rotations at the lattice points, the group also contains threefold rotational symmetries at centers of the triangles.



The group also contains half-turns around the midpoints of neighboring lattice points.



An example of a painting by Escher with symmetry group W_6 (ignoring colors).



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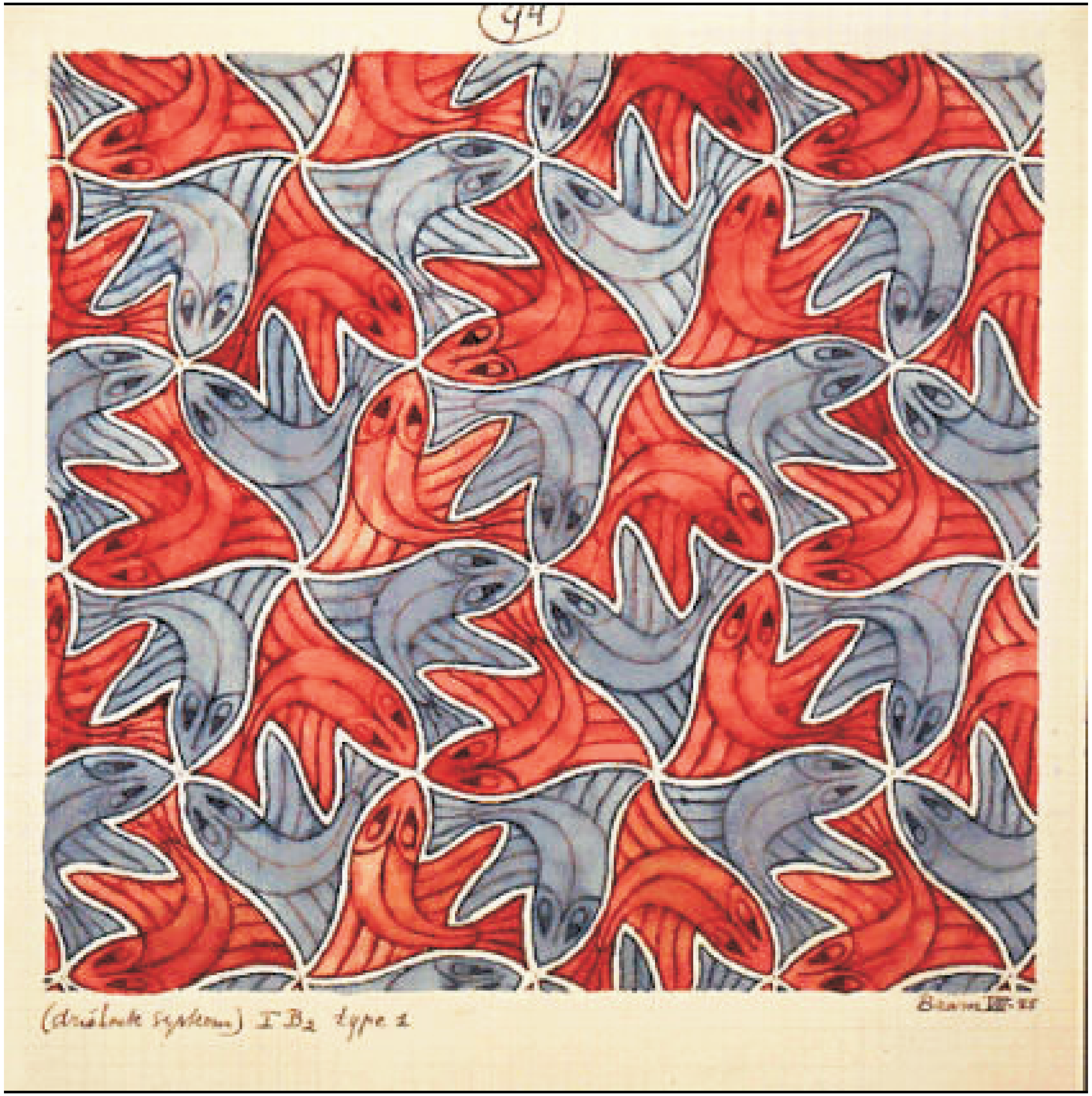
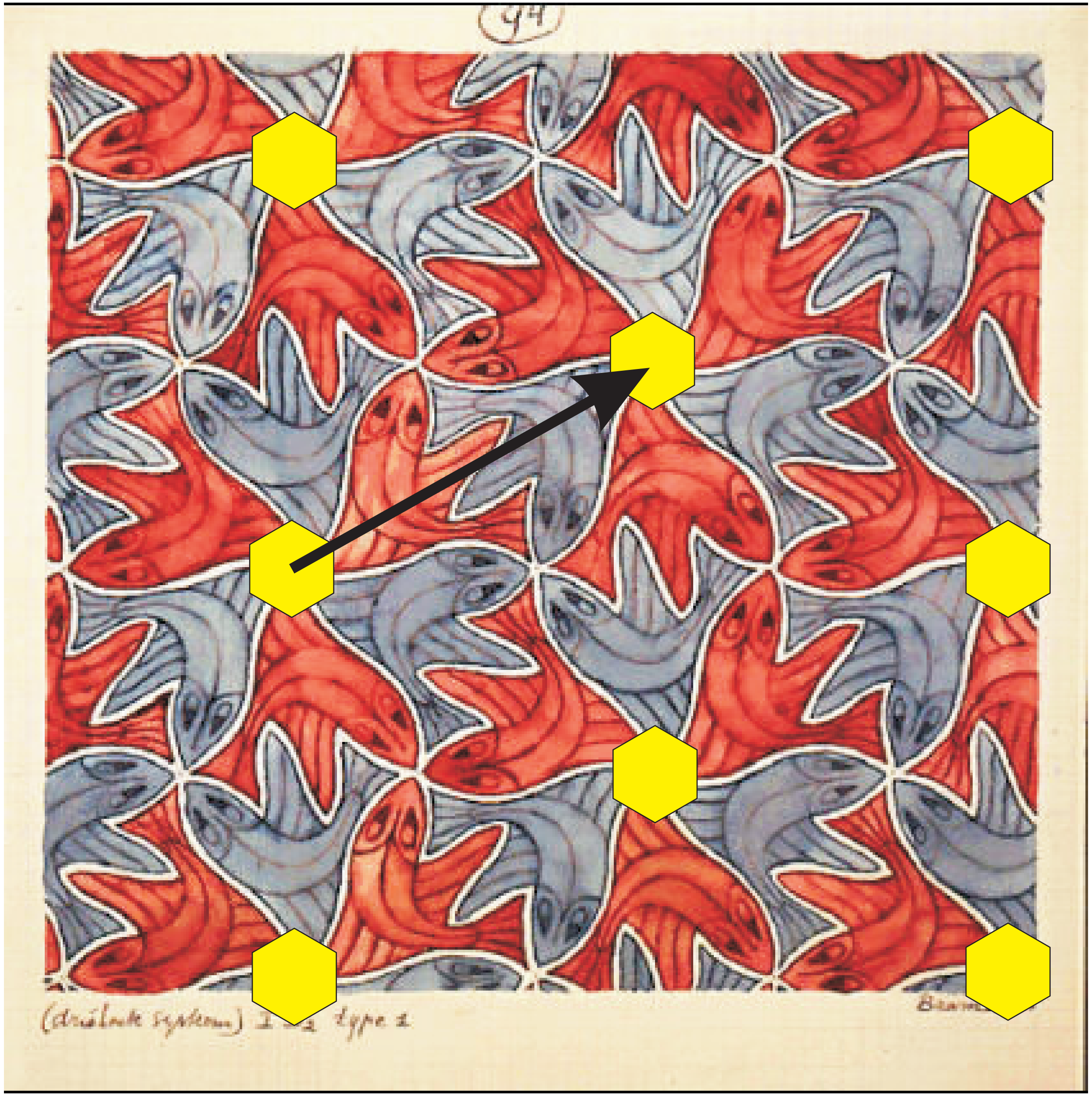
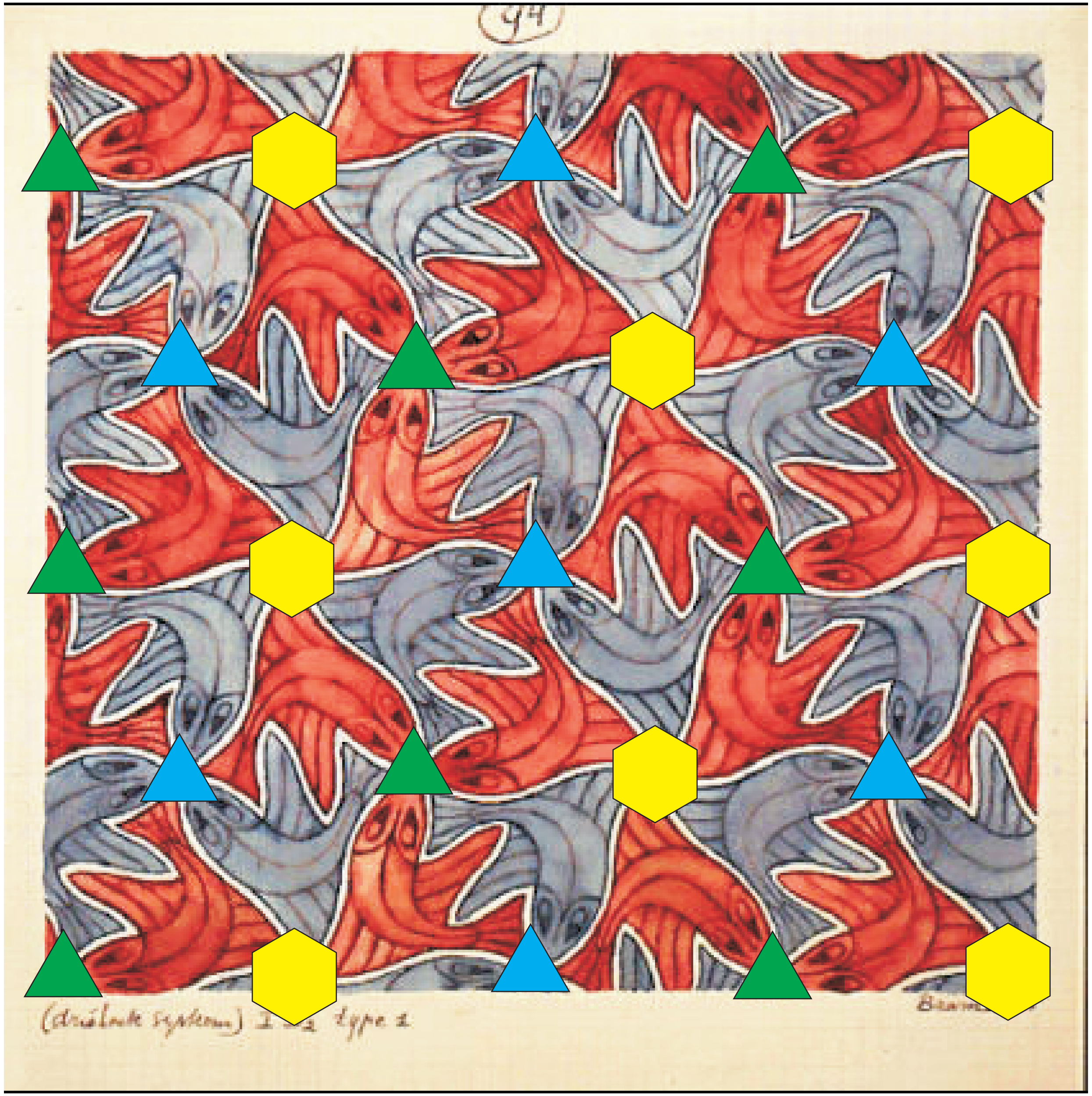


Another example of W_6 (ignoring coloring).

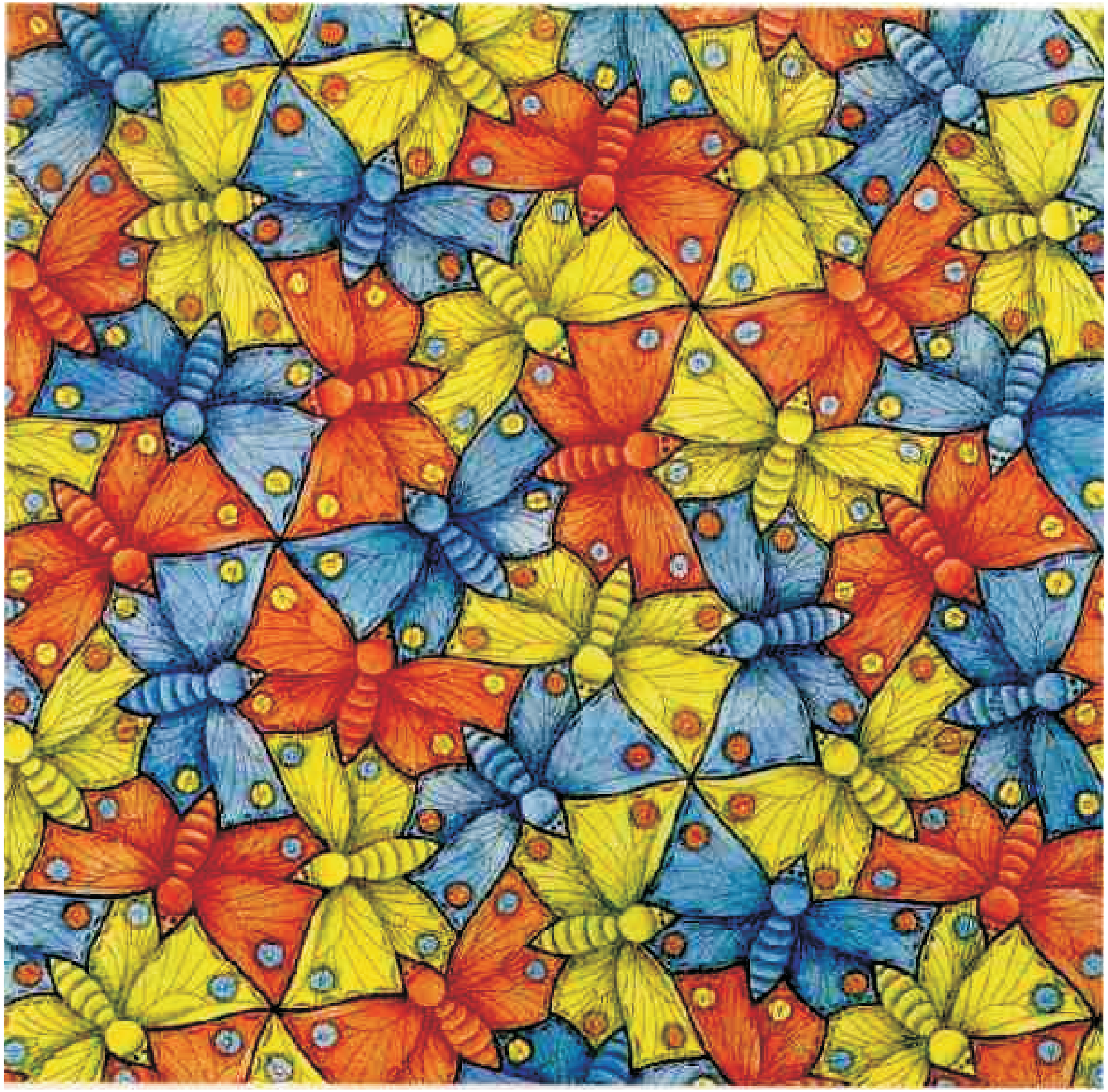


6-fold rotation centers and the shortest translation.

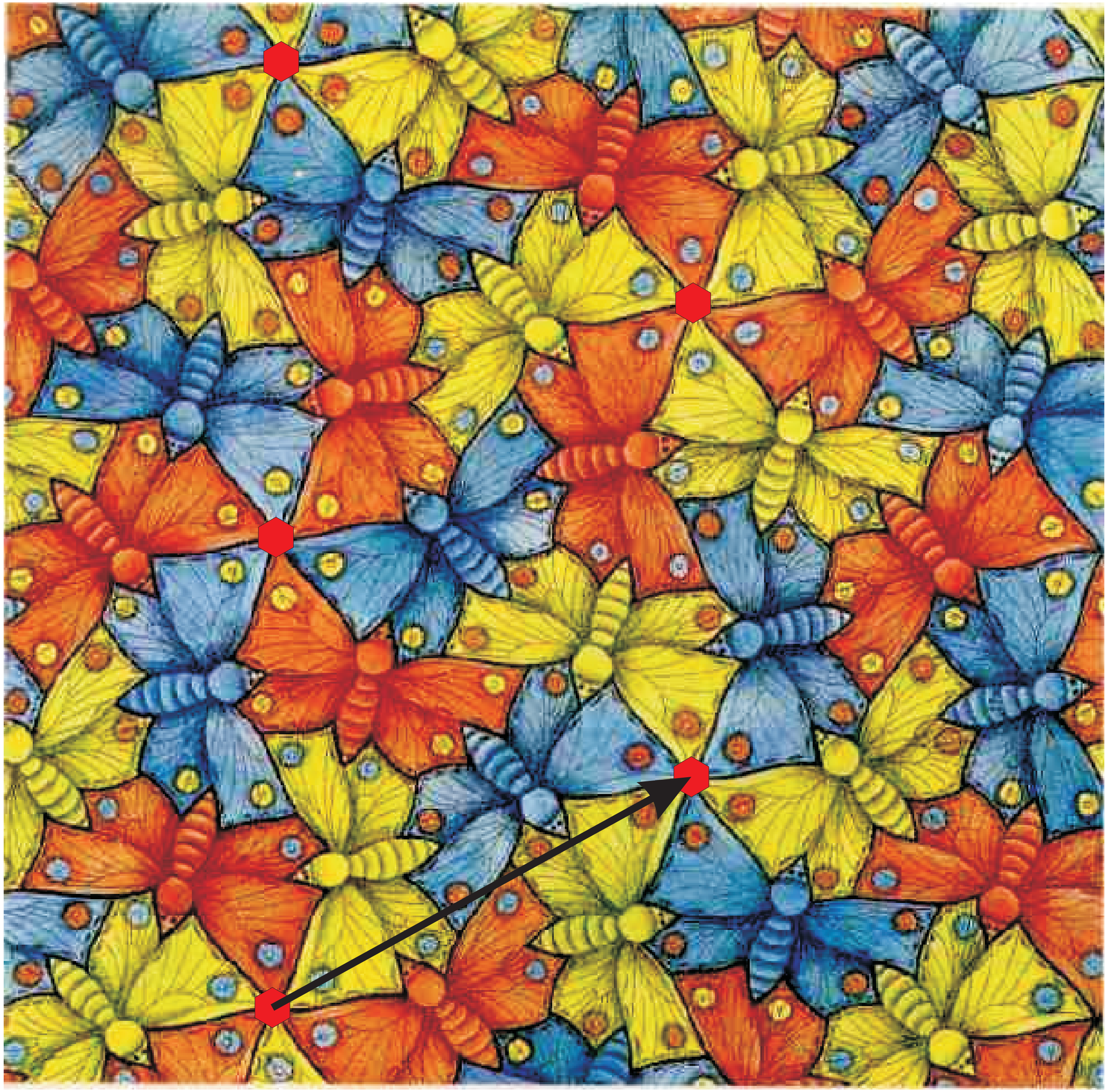


3-fold rotation centers at the centers of lattice triangles.

And the 2-fold rotation centers.



Yet another example of W_6, ignoring the coloring.



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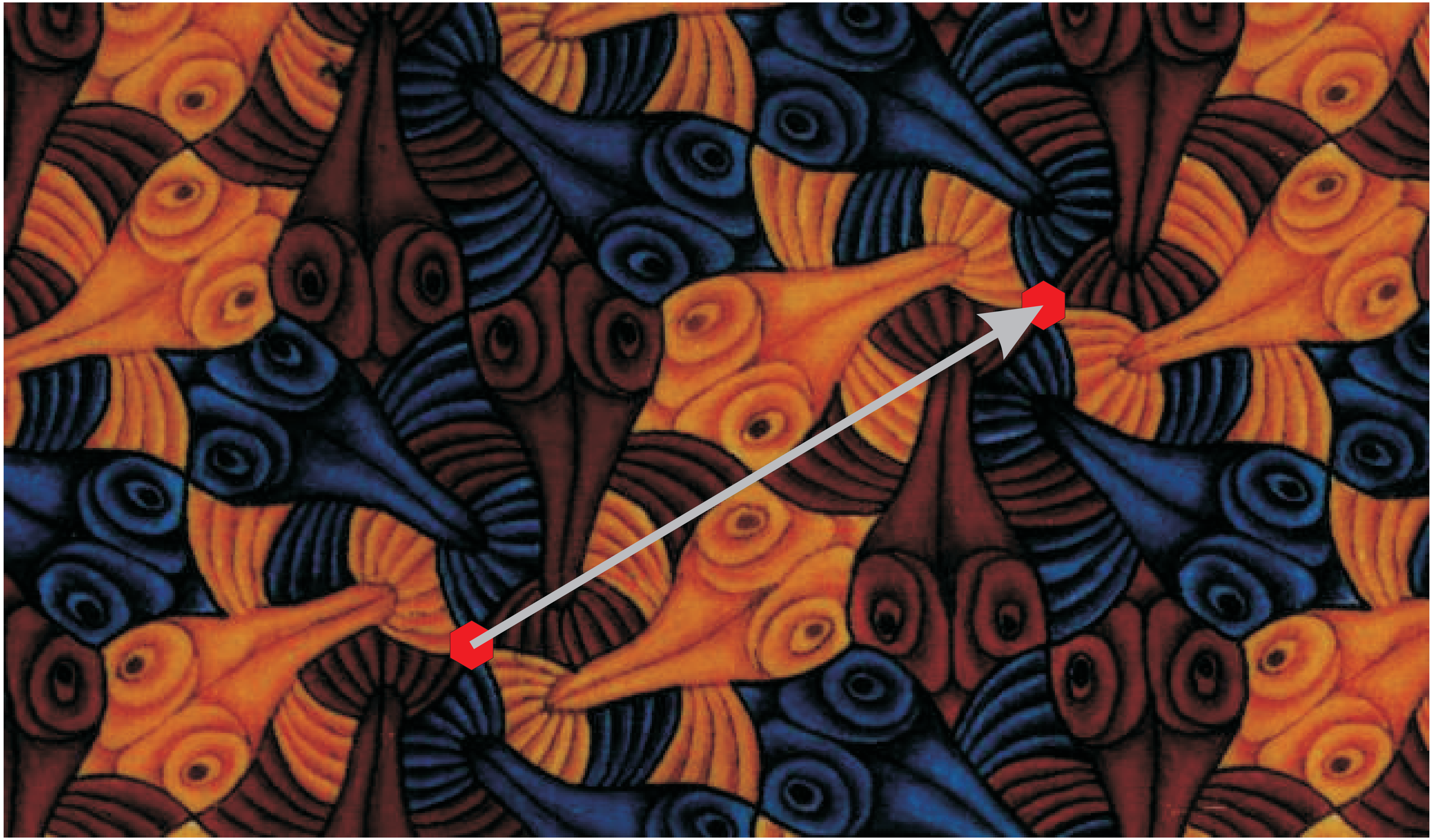

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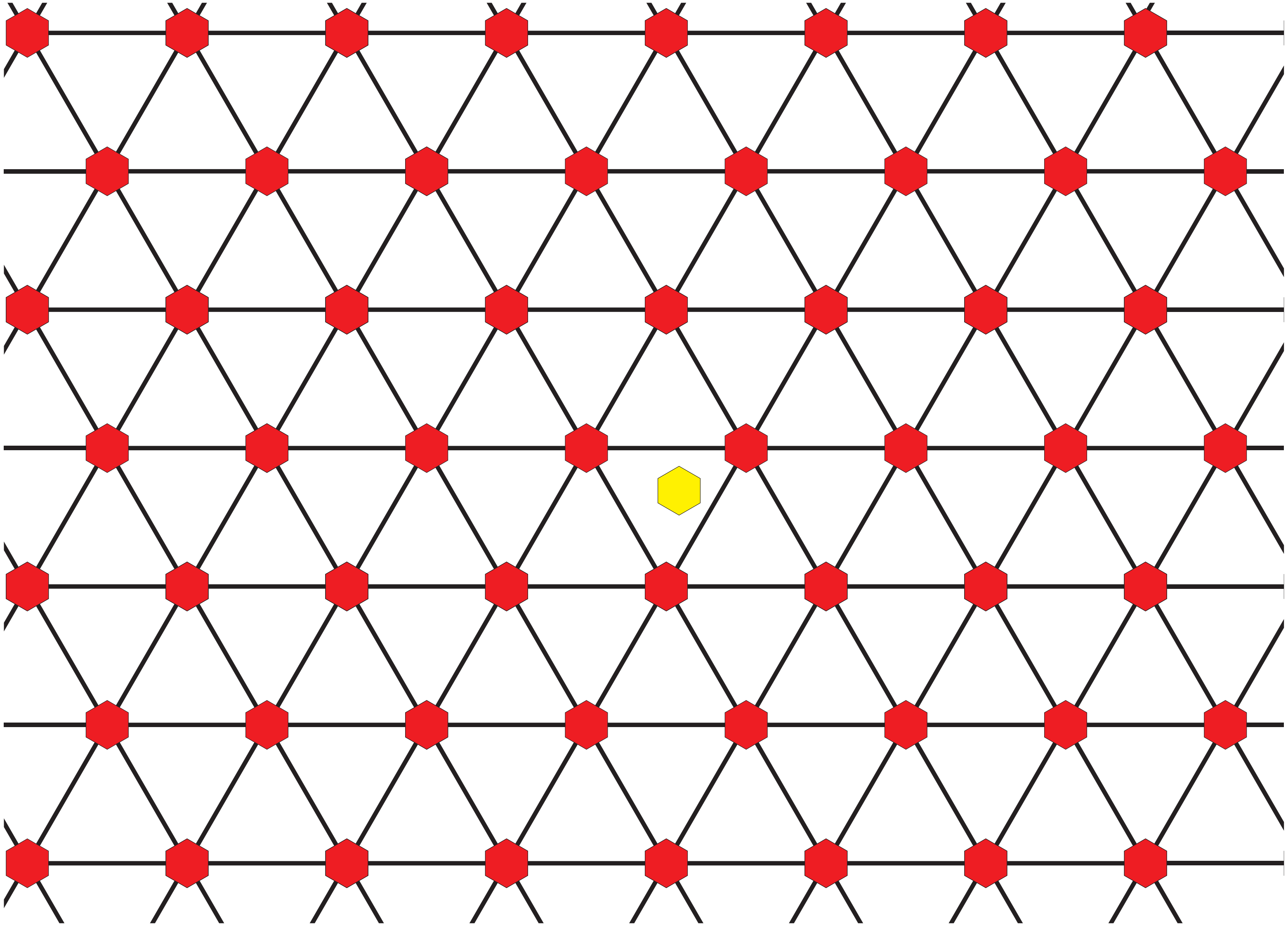
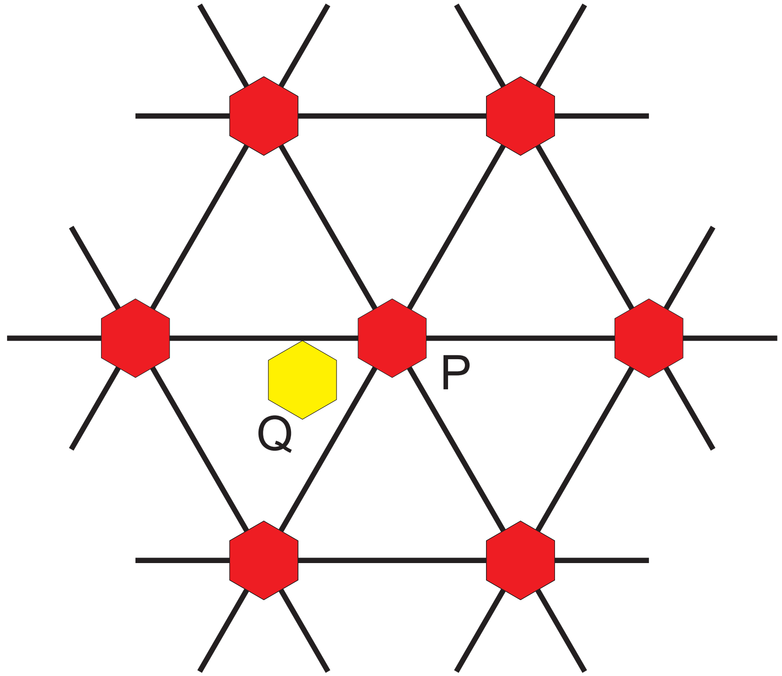
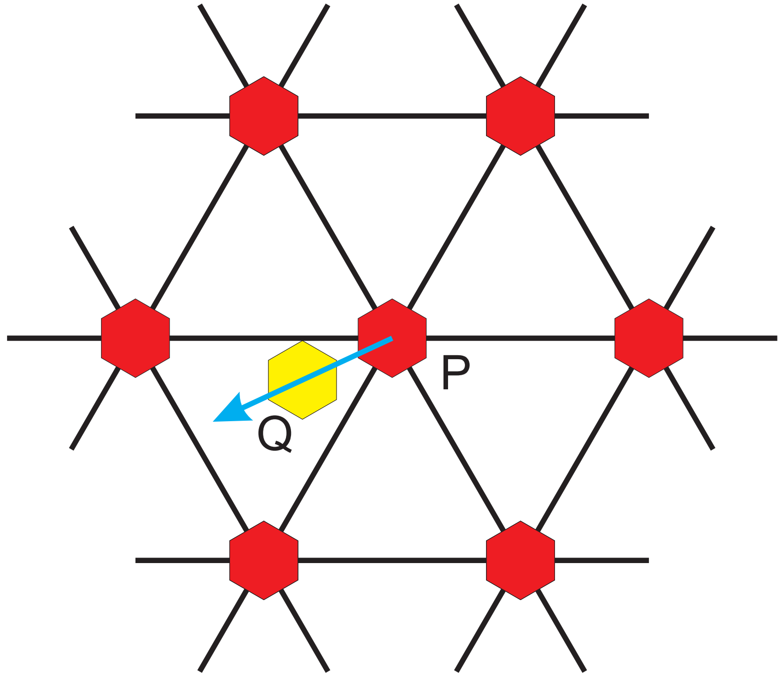


Let us prove that there are no sixfold rotations around any non-lattice points.

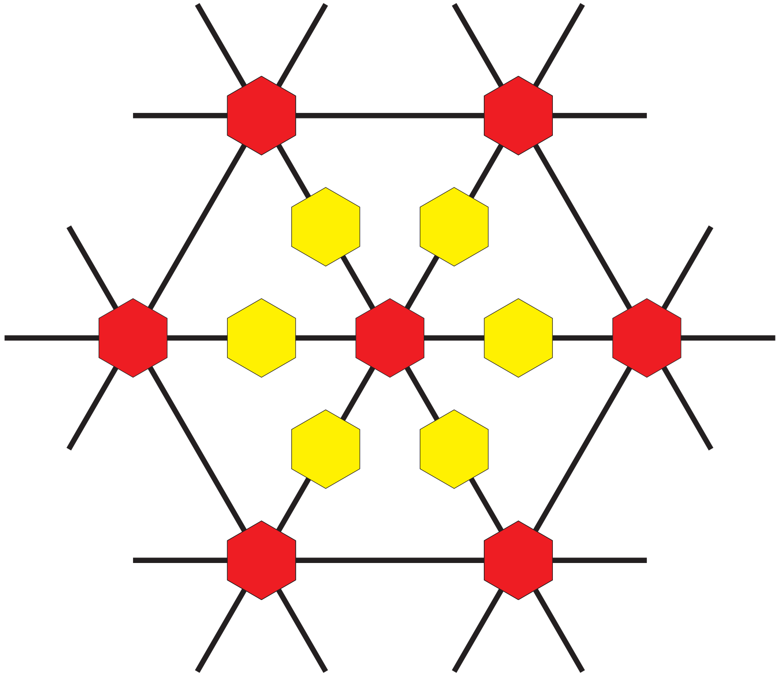


Let Q be a center of a 60° rotation in G. Then also the halfturn σ_Q around Q is in G.

Let P be the lattice point closest to Q. Also the halfturn σ_P around P is in G.



The composition $\sigma_Q \sigma_P$ is the translation by the vector $2\overrightarrow{PQ}$. Because $\sigma_Q \sigma_P$ is an even isometry in G, it maps lattice points to lattice points. In particular, the image of P must be one of the six lattice points surrounding P.



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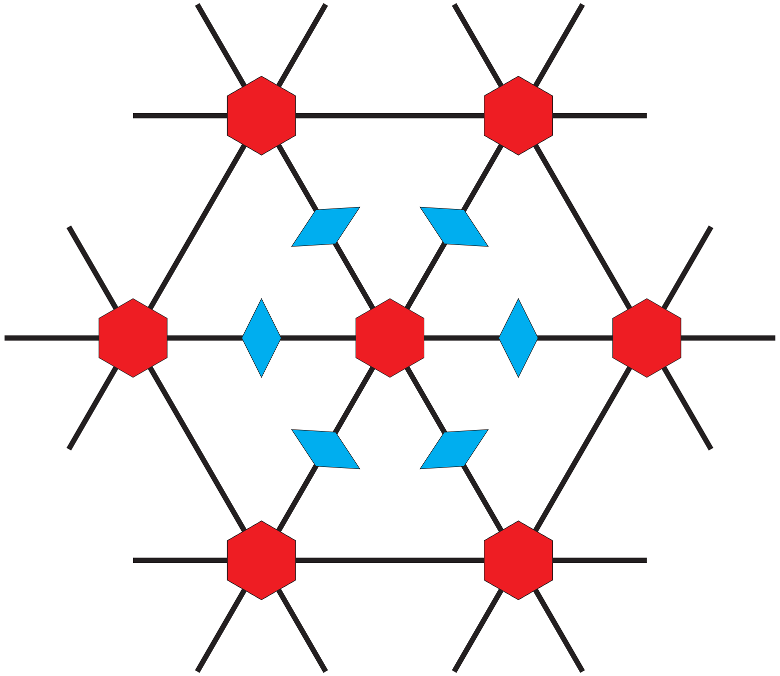
This is only possible if Q is the midpoint between P and an adjacent lattice point.



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This is only possible if Q is the midpoint between P and an adjacent lattice point.

But these six points are not centers of sixfold symmetry of the lattice. They are only centers of halfturns.

Suppose next that group G contains some odd isometry α.

If ρ is the 60° rotation around a lattice point P then the conjugate $\alpha\rho\alpha^{-1}$ is a sixfold rotation around point $\alpha(P)$.

All sixfold rotations are around lattice points, so $\alpha(P) \in S_6$, and α is an odd symmetry of the lattice S_6.

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Conversely, if β is an arbitrary odd symmetry of S_6 then $\alpha\beta$ is an even symmetry of S_6 and hence $\alpha\beta \in G$. Because $\alpha \in G$, we have $\beta \in G$.

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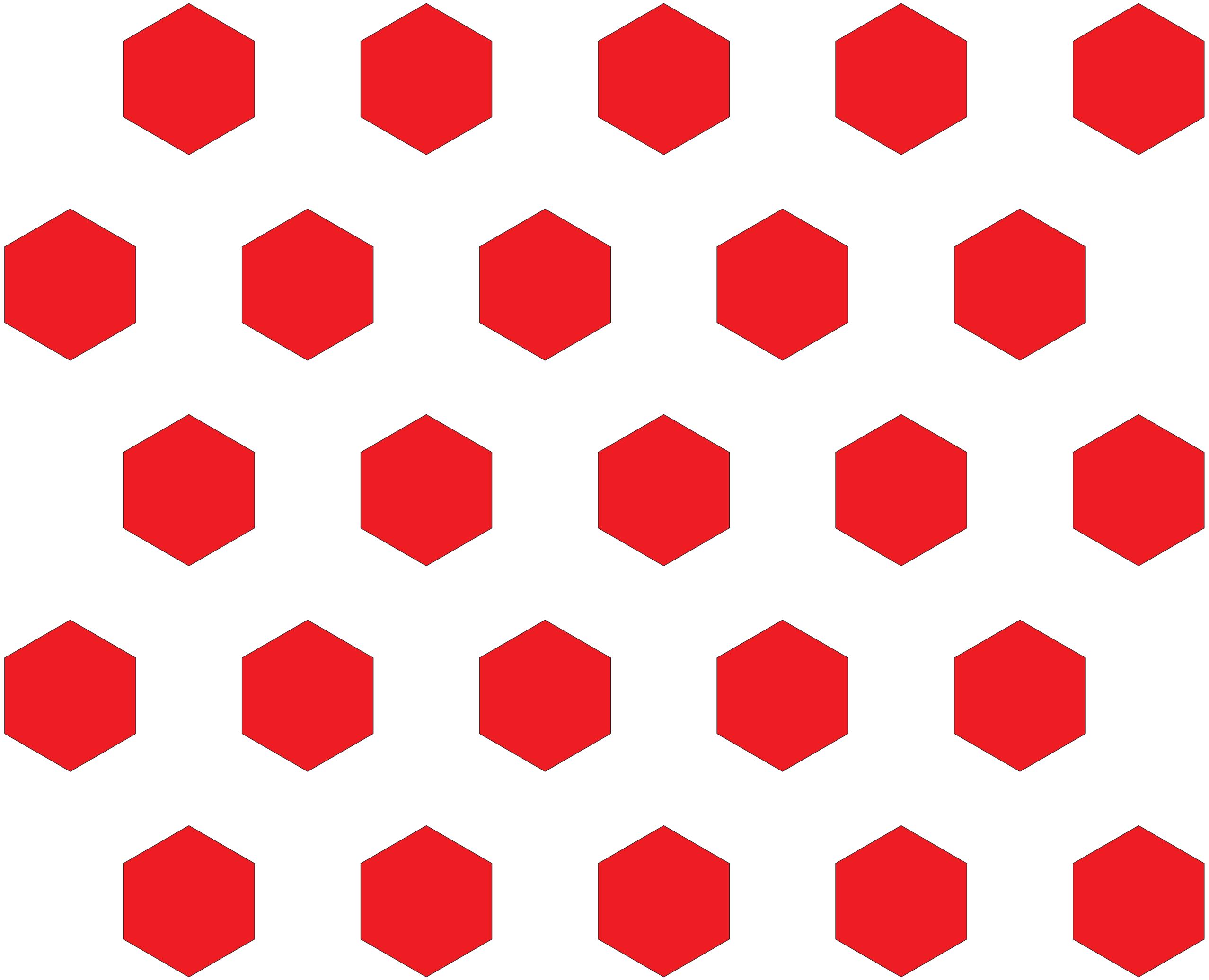
Conclusion: G is the symmetry group of the lattice S_6. This is our second wallpaper group

$$W_6^1 = \langle \rho, \tau, \alpha \rangle.$$



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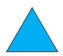
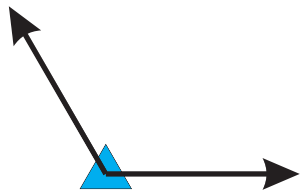


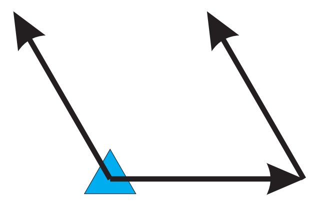
Case (2): Suppose G contains 120° rotations, but no 60° rotations.

Let P be the center of some threefold rotation $\rho \in G$.

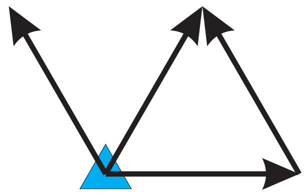
Let τ be a shortest translation in G.



The conjugate $\rho\tau\rho^{-1}$ of τ is the translation that moves point P to point $\rho\tau(P)$. The conjugate is also in group G.



The sum of the two translations is also a translation in G.



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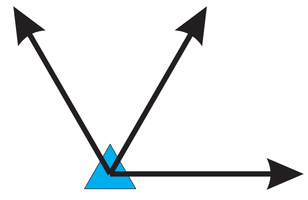

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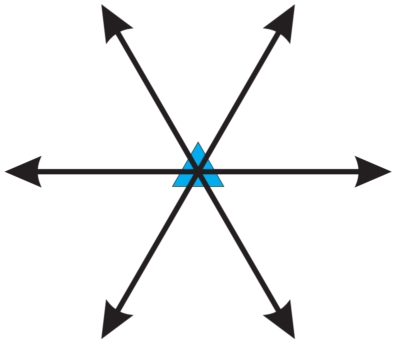
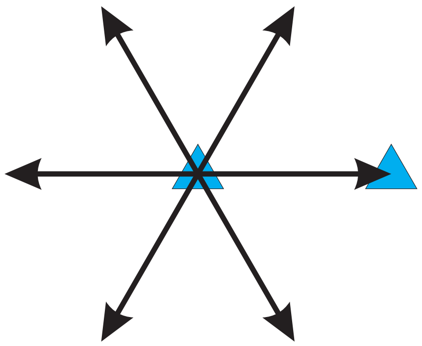
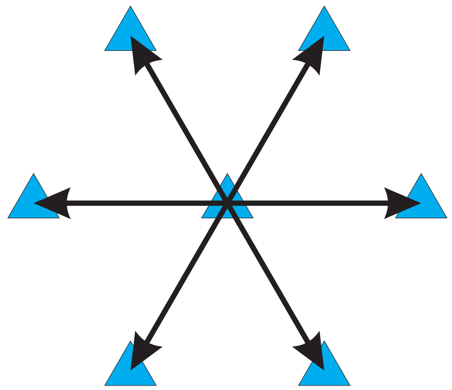


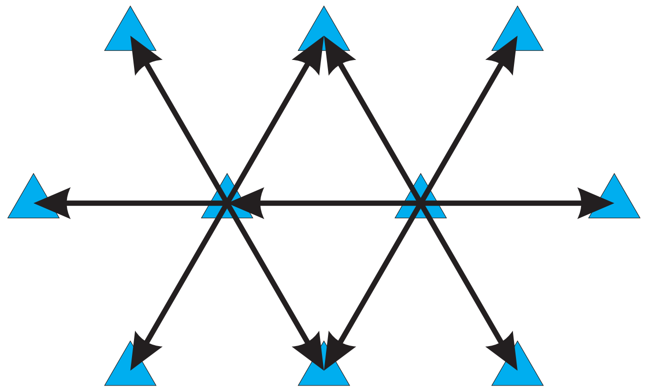
Also inverse translations are in G.



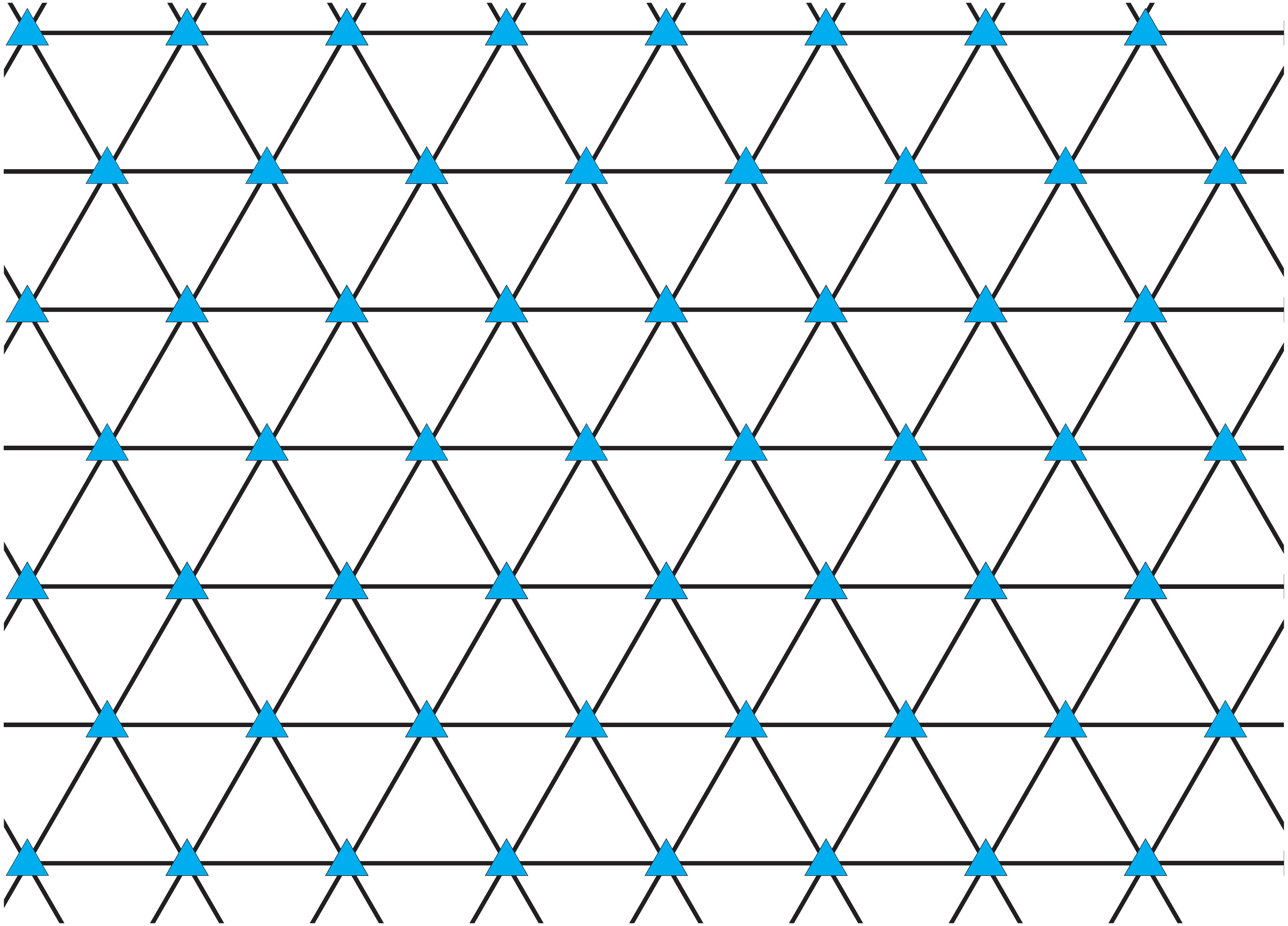
The conjugate $\tau\rho\tau^{-1}$ of ρ is a threefold rotation around point $\tau(P)$.



Analogously, conjugates of ρ by the six translations provide six centers of threefold rotations.

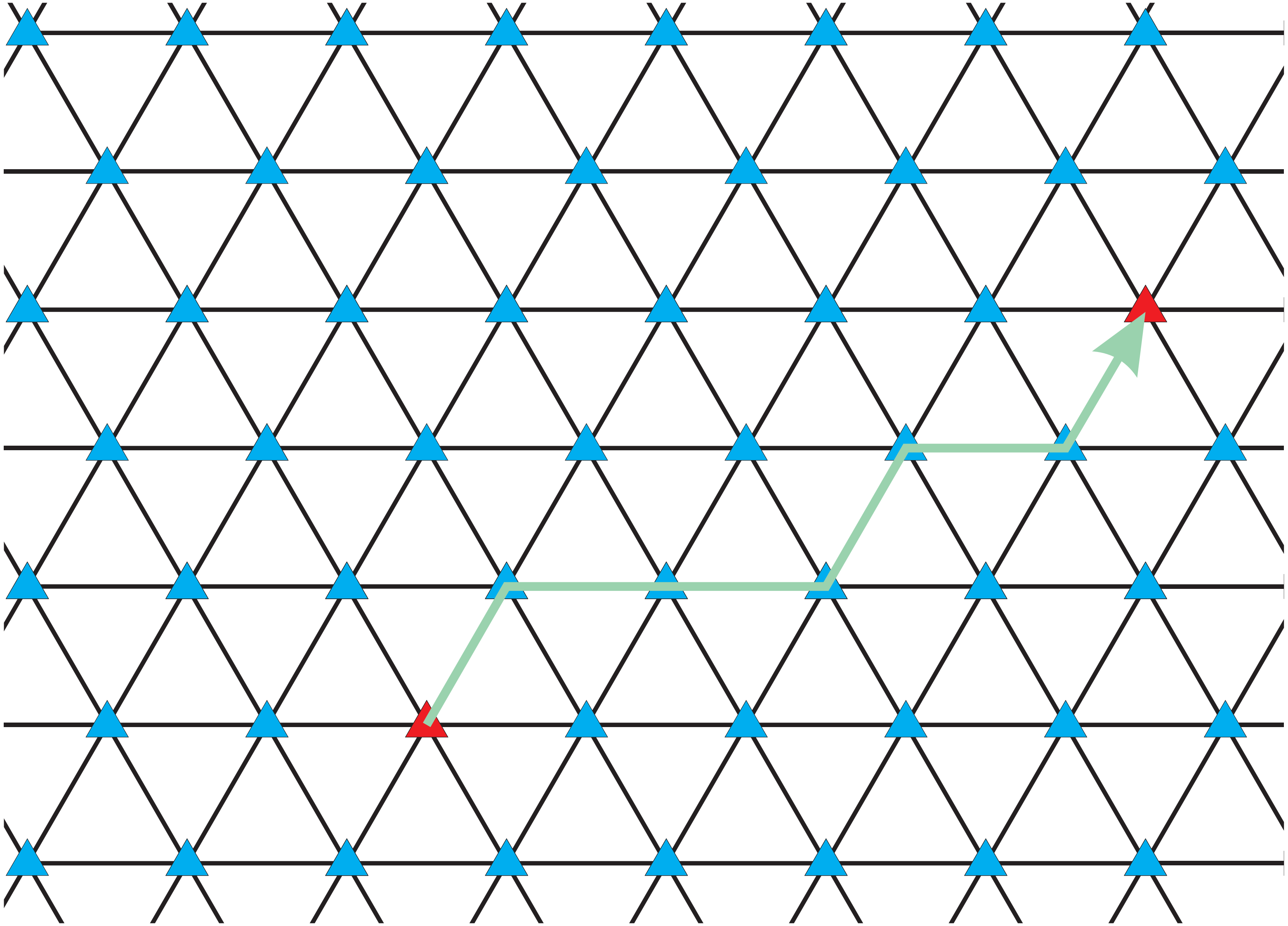


The same reasoning around the rotation center $\tau(P)$ provides new centers of rotation.



This reasoning can be repeated for all centers of rotation, which provides us with an infinite triangular lattice of rotation centers.

Let S_6 be the set of these rotation centers.



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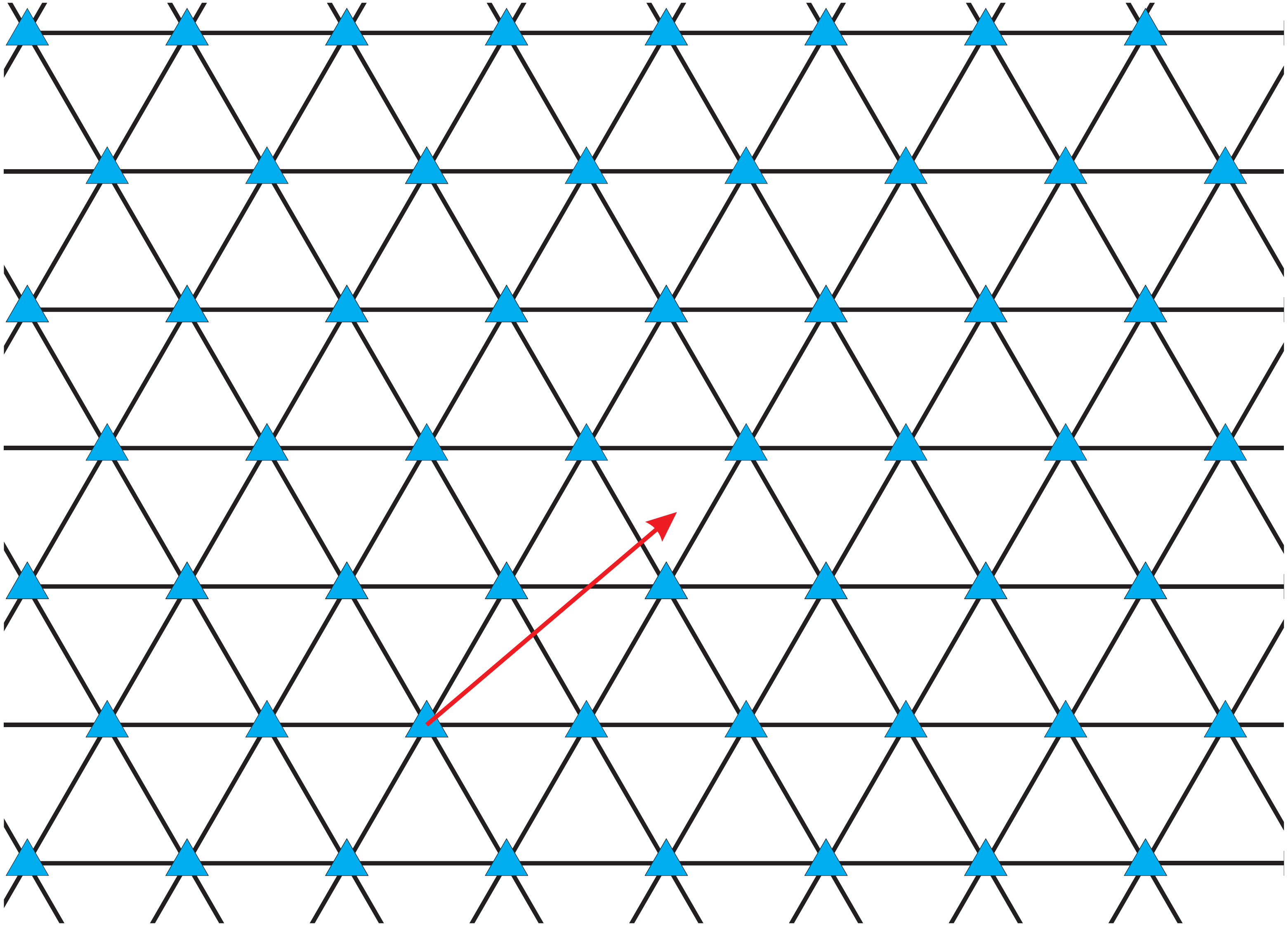

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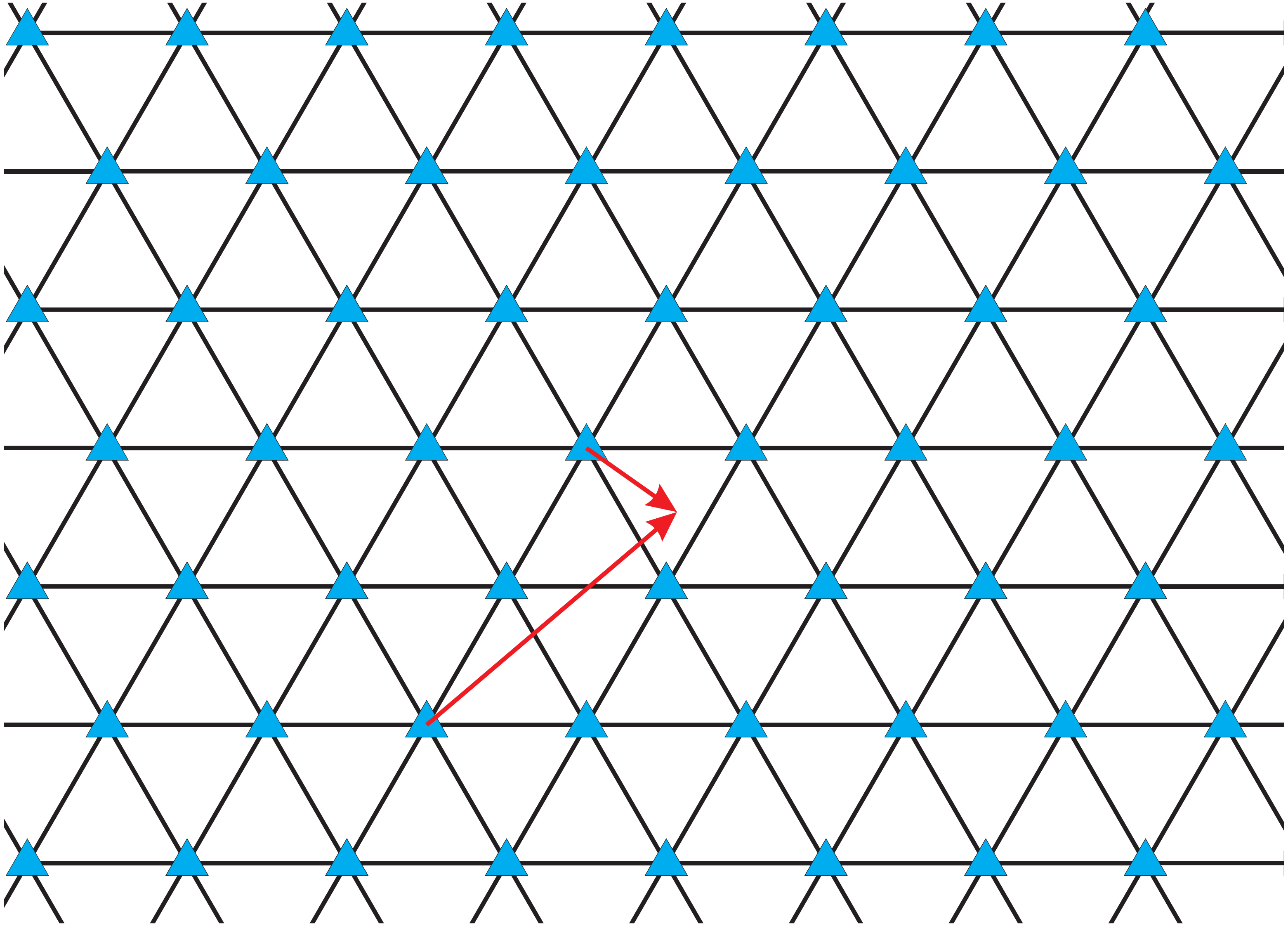
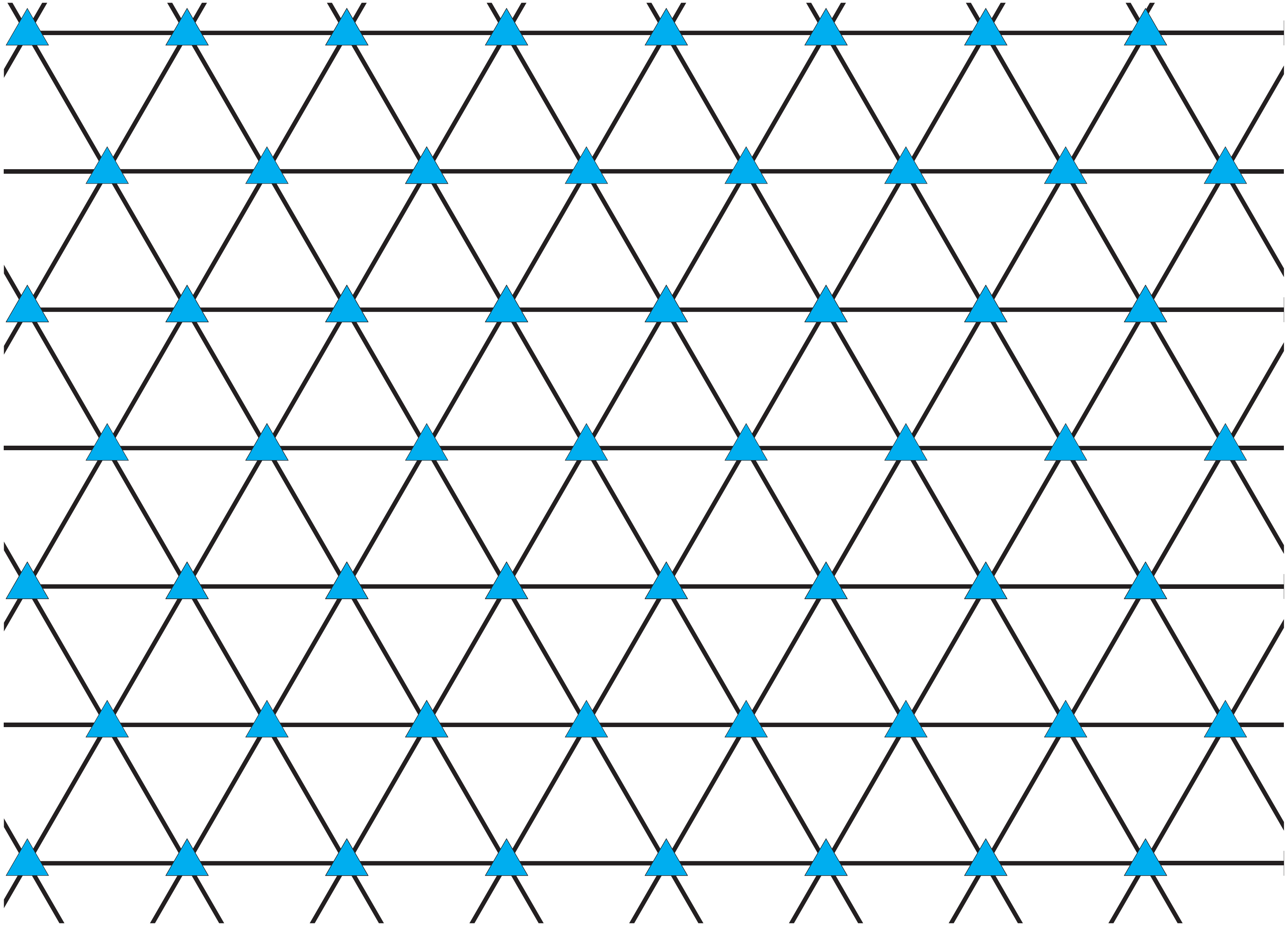


Then G also contains a translation taking a vertex of one of the equilateral triangles of the lattice into a non-vertex point of the triangle.

But this contradicts the minimality of translation τ.



We have proved that the translations in G are exactly the translations that map the lattice S_6 into itself.

Let us prove that rotations in G are exactly the 3-fold symmetries of the lattice.

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Then $\rho' = \tau' \rho^{\mp 1}$ takes lattice points to lattice points since both ρ and τ' are symmetries of the lattice.

If $\rho' \in G$ is any non-trivial rotation then – by the crystallographic restriction and by the assumption that there are no sixfold rotations in G – the rotation must be by $\pm 120°$.

It follows that $\rho' \rho^{\pm 1} = \tau'$ is a translation where ρ is our initial 3-fold rotation around a lattice point.

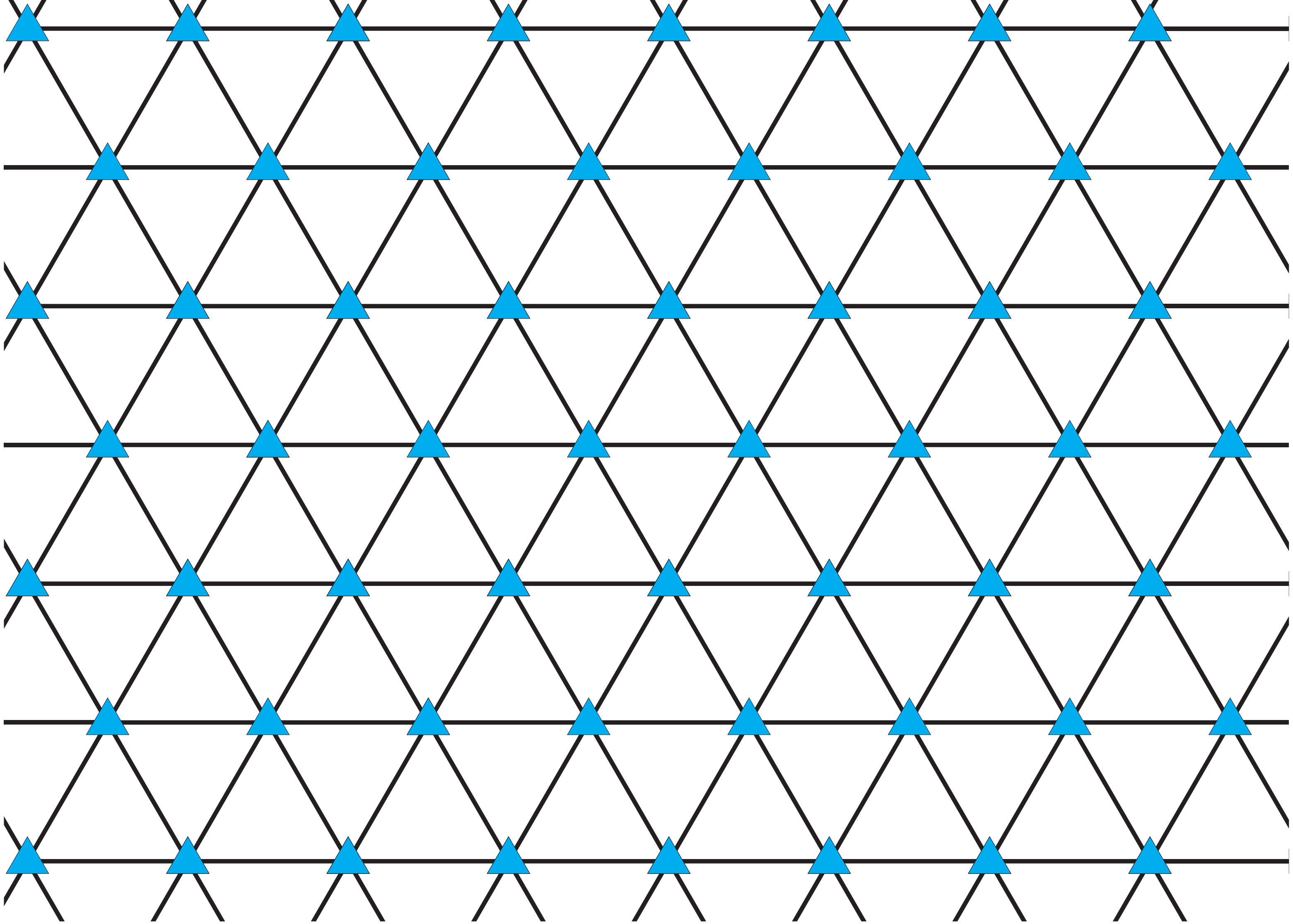
Then $\rho' = \tau' \rho^{\mp 1}$ takes lattice points to lattice points since both ρ and τ' are symmetries of the lattice.

We have proved that the rotations in G are also symmetries of the lattice S_6.

Conversely, let us show that all 3-fold rotational symmetries of S_6 are in G. So let ρ' be a rotation by 120° that is a symmetry of S_6.

Then, $\rho'\rho^{-1} = \tau'$ is a translation. This translation maps lattice points to lattice points and hence $\tau' \in G$.

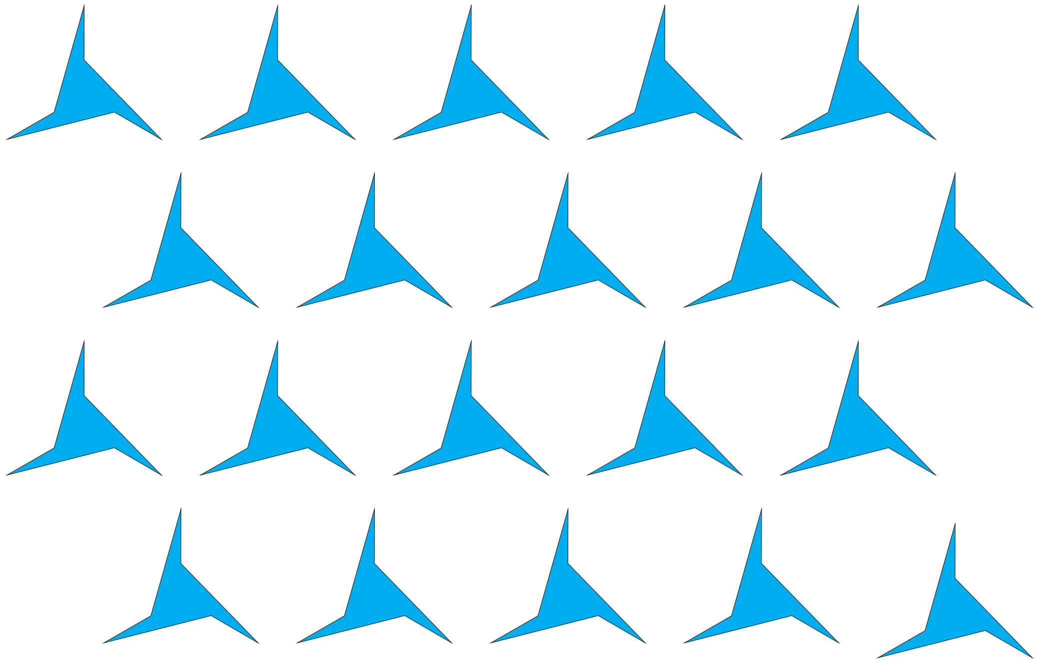
We have $\rho' = \tau'\rho \in G$.



We have proved that the even isometries in G are exactly the translational and 3-fold rotational symmetries of S_6.

If G contains no odd isometries then G is uniquely determined: It is the wallpaper group

$$W_3 = \langle \rho, \tau \rangle.$$



An example of a pattern whose symmetry group is

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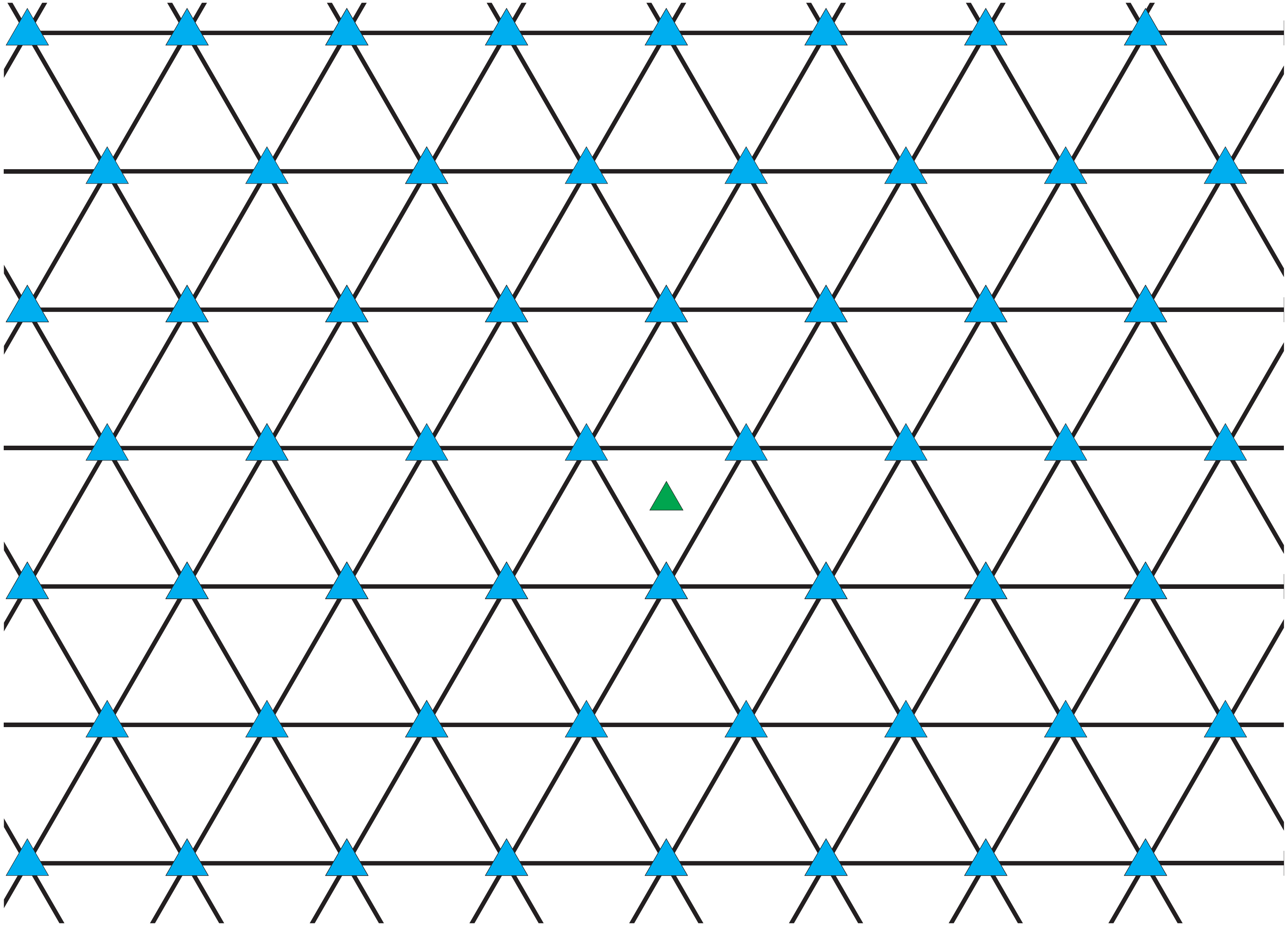
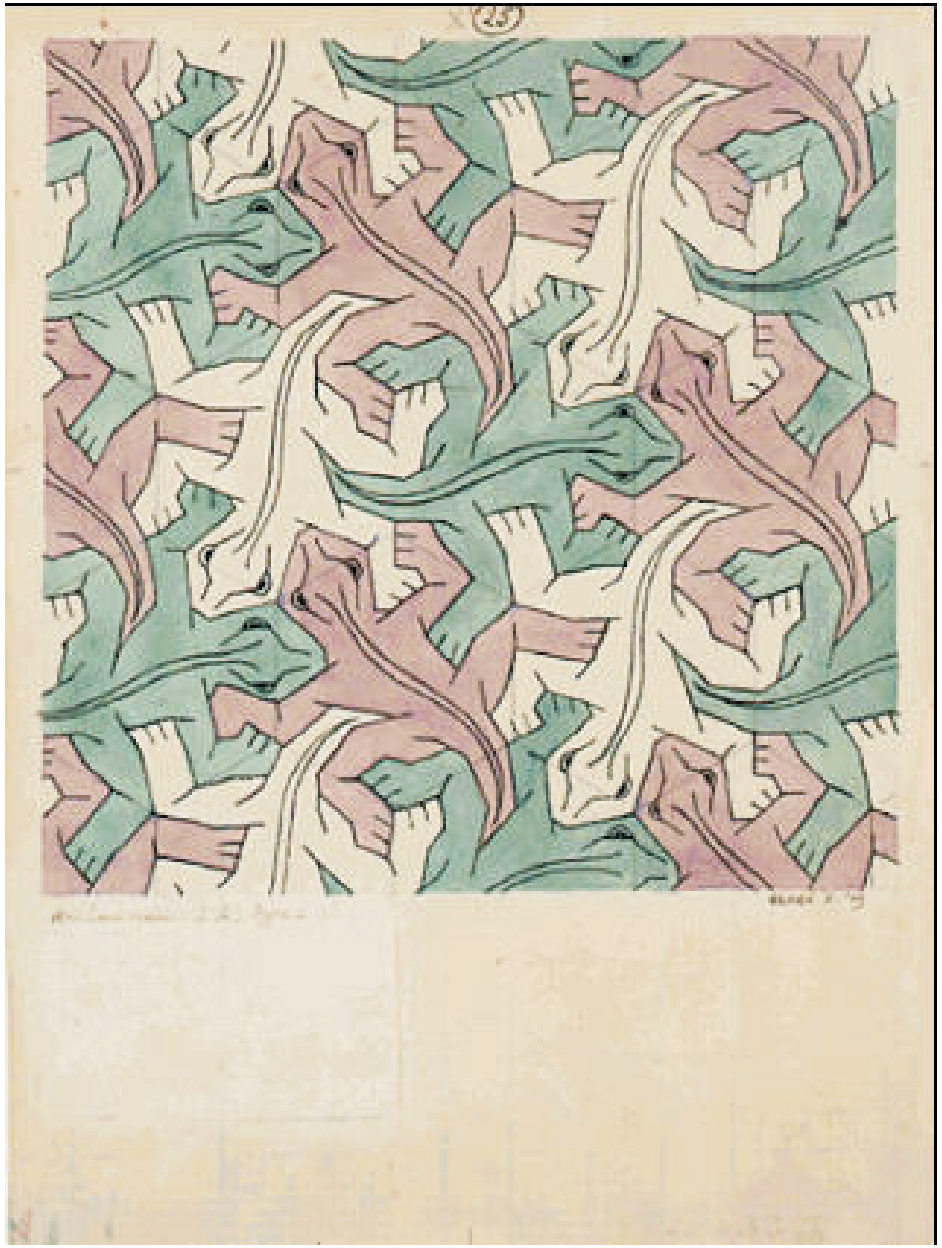
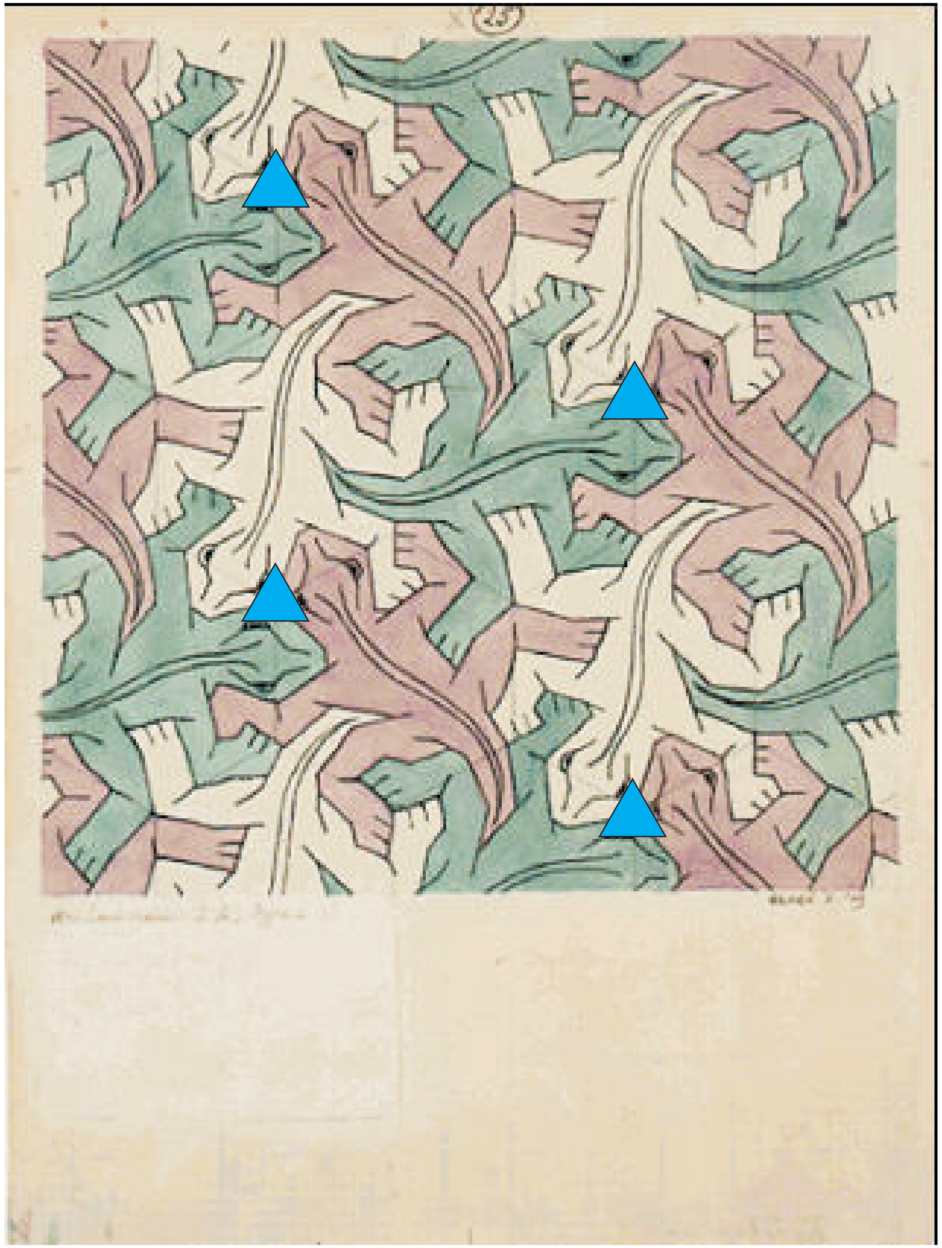


In addition to the 3-fold rotations at the lattice points, the group also contains 3-fold rotational symmetries at centers of the triangles.



An Escher painting with symmetry W_3.



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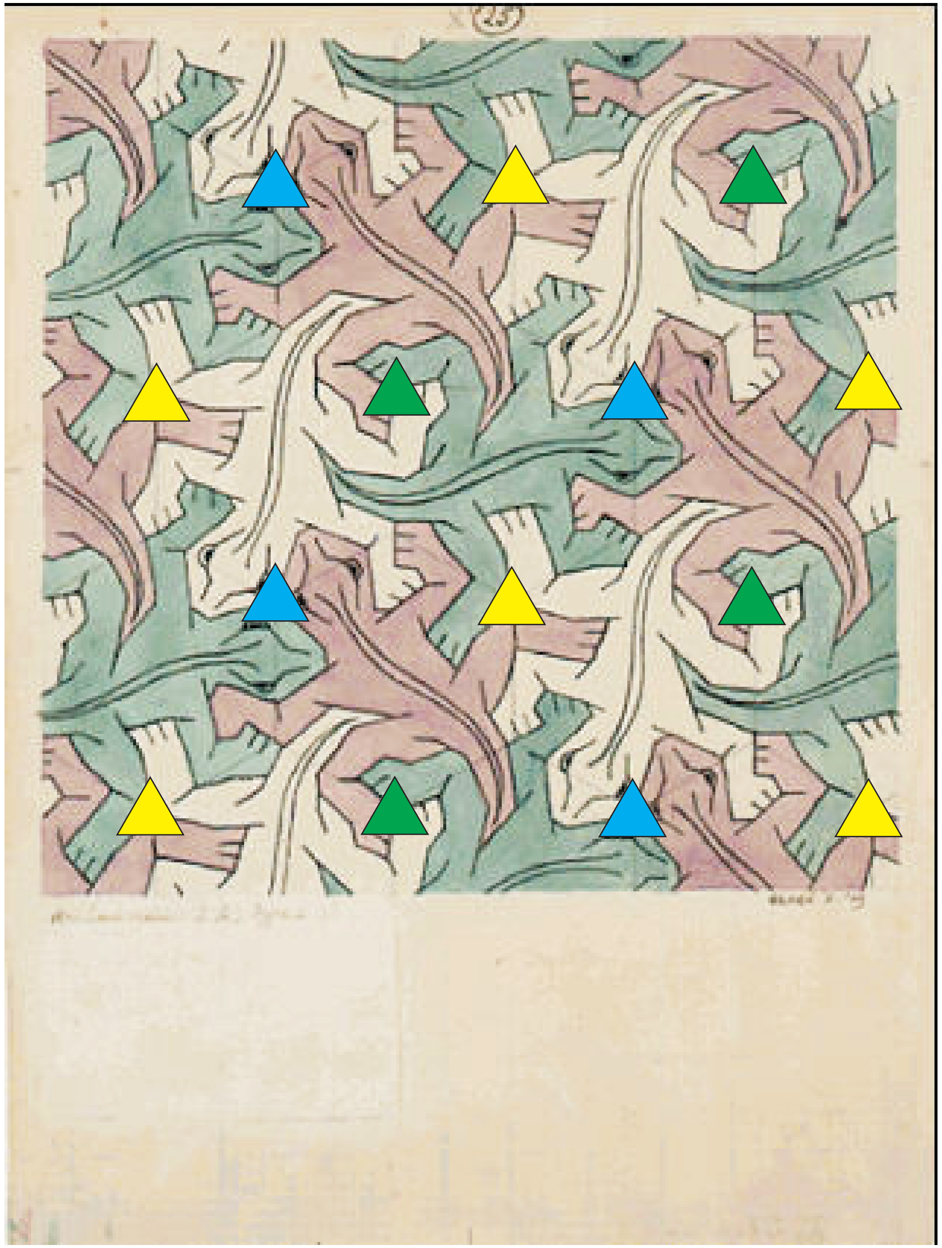

An Escher painting with symmetry W_3.

Suppose next that group G contains some odd isometry α.

A glide reflection γ and a 3-fold rotation ρ generate a reflection

$$\gamma\rho\gamma\rho^{-1}\gamma$$

so the group contains a reflection σ_m.

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A glide reflection γ and a 3-fold rotation ρ generate a reflection

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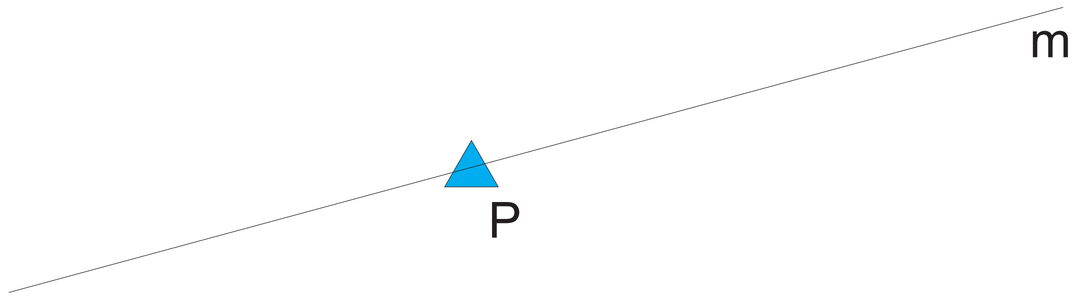
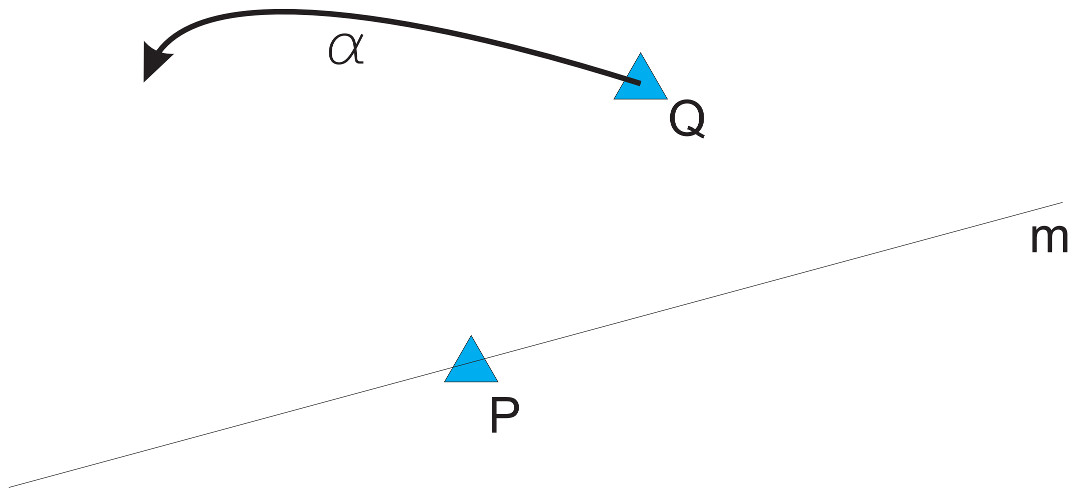
so the group contains a reflection σ_m.

As $\rho\sigma_m\rho^{-1}$ is a reflection on line $\rho(m)$ that intersects m, group G contains a non-trivial rotation around the intersection point.
Conclusion: every reflection line contains rotation centers.

As the initial rotation center was arbitrary, we can choose it so that its center P is on the reflection line m.

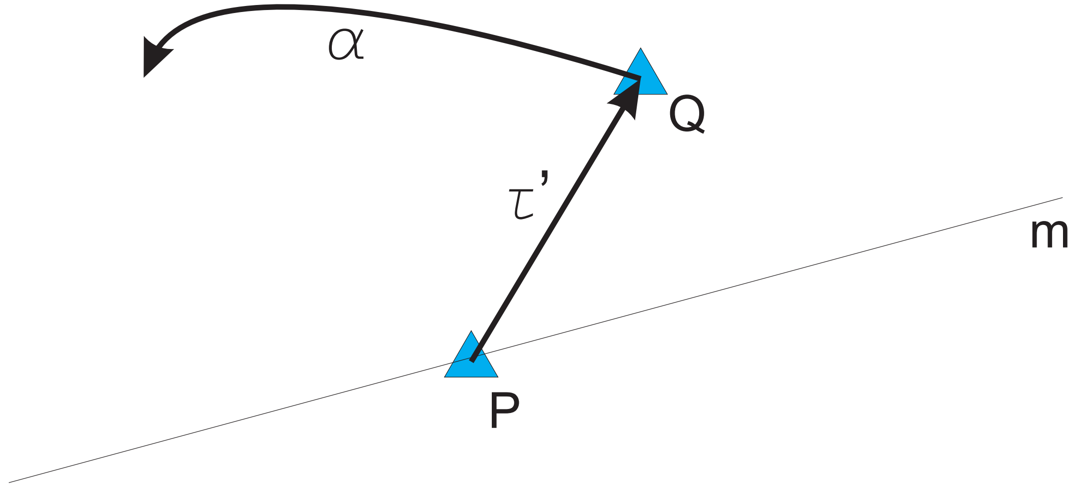


We have $\sigma_m \in G$ and $P \in m$.



We have $\sigma_m \in G$ and $P \in m$.

Let $\alpha \in G$ be odd. Let us prove that α is a symmetry of the lattice S_6. It is enough to show that an arbitrary lattice point Q is mapped into a lattice point.



We have $\sigma_m \in G$ and $P \in m$.

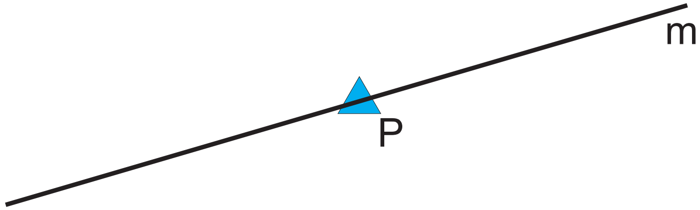
Let $\alpha \in G$ be odd. Let us prove that α is a symmetry of the lattice S_6. It is enough to show that an arbitrary lattice point Q is mapped into a lattice point.

If $\tau' \in G$ is the translation that takes P to Q then

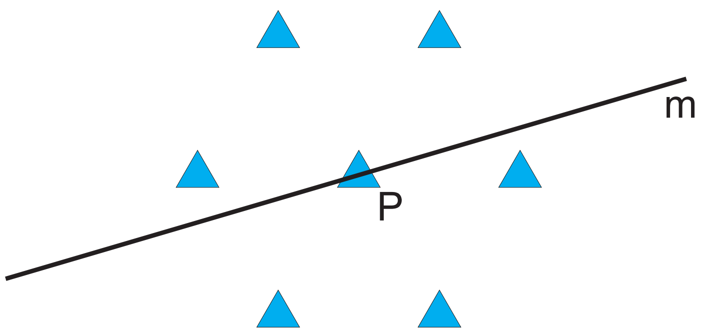
$$\alpha\tau'\sigma_m$$

is an even element of G, and hence a symmetry of the lattice. It maps $P \mapsto \alpha(Q)$, so $\alpha(Q)$ is a lattice point.

Now we know that all elements of G are symmetries of lattice S_6.

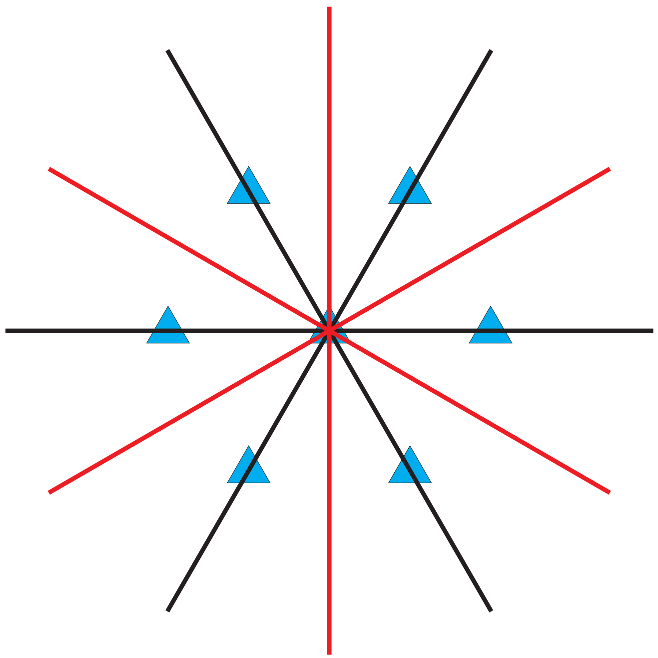


Let P be again a rotation center through which a reflection line m passes.



Let P be again a rotation center through which a reflection line m passes.

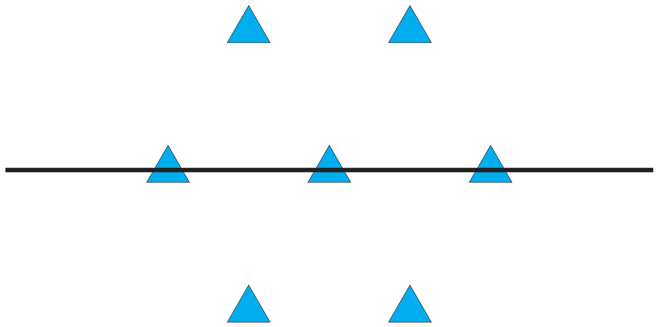
Because σ_m keeps the lattice invariant, it has to be also a symmetry for the six closest lattice points around P.



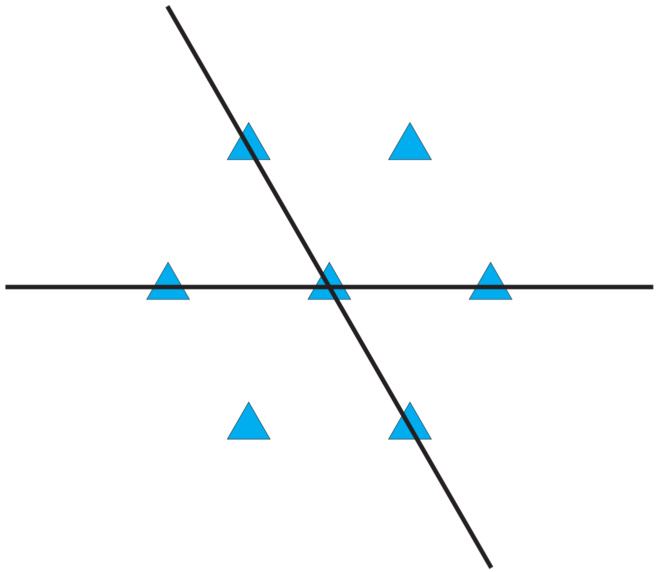
Let P be again a rotation center through which a reflection line m passes.

Because σ_m keeps the lattice invariant, it has to be also a symmetry for the six closest lattice points around P.

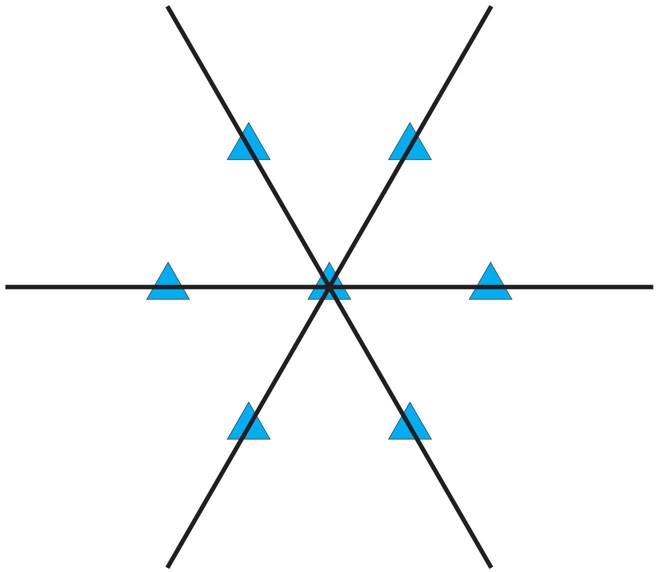
The symmetries for these six points form the dihedral group D_6, which contains six reflections: three on lines through opposite points, and three between them.



The three reflections on the black lines are conjugate to each other through the $120°$ rotation around point P.



The three reflections on the black lines are conjugate to each other through the $120°$ rotation around point P.



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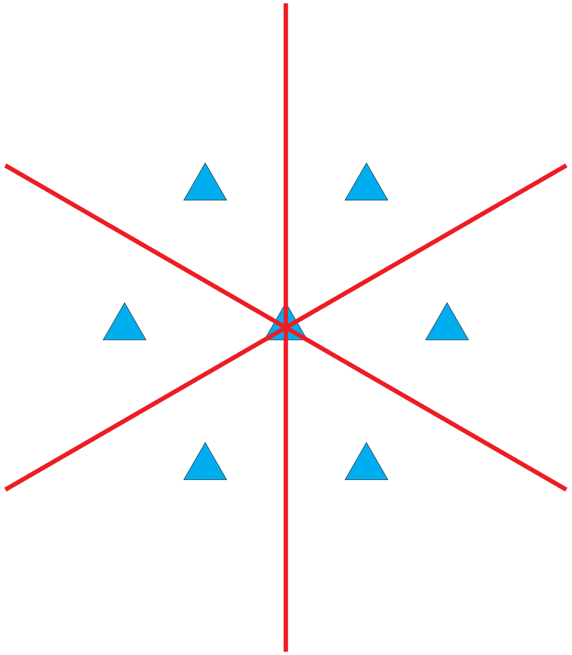
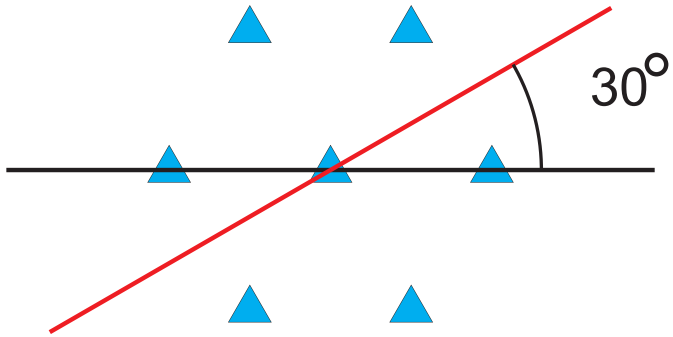
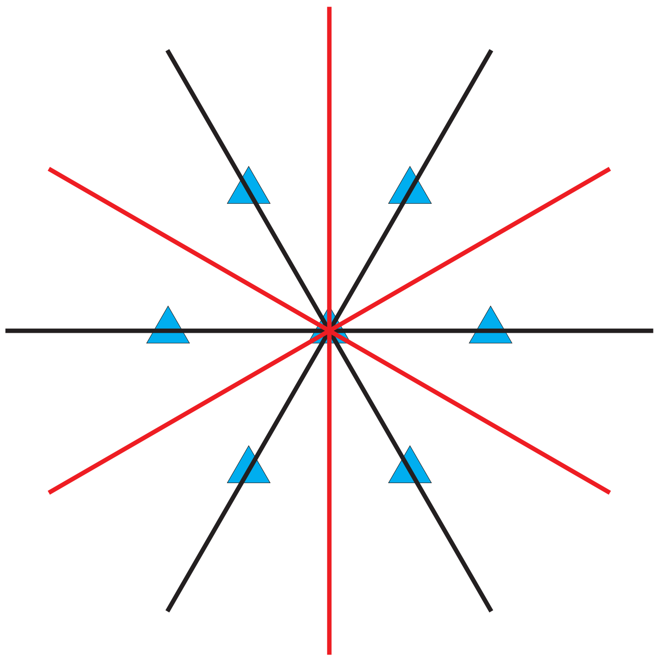


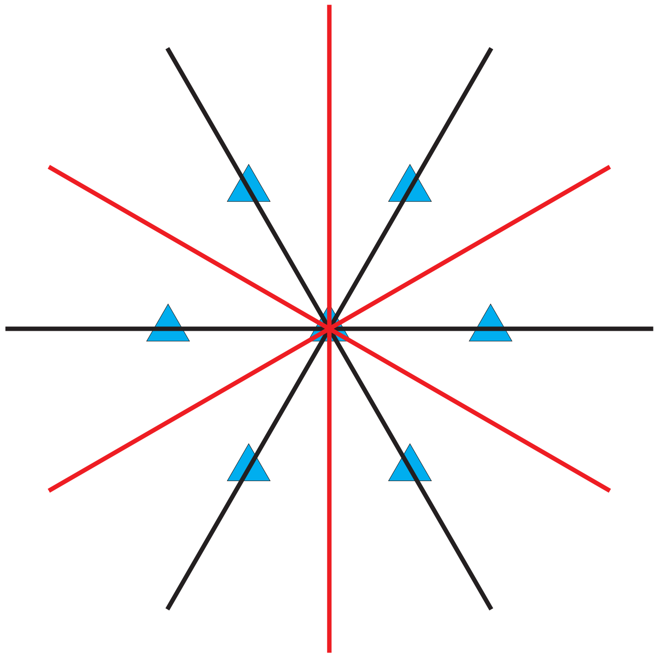
Analogously, the three reflections on the red lines are conjugate to each other in G.



But G cannot contain both black and red reflection lines because the product of two such reflections is a rotation by $60°$.



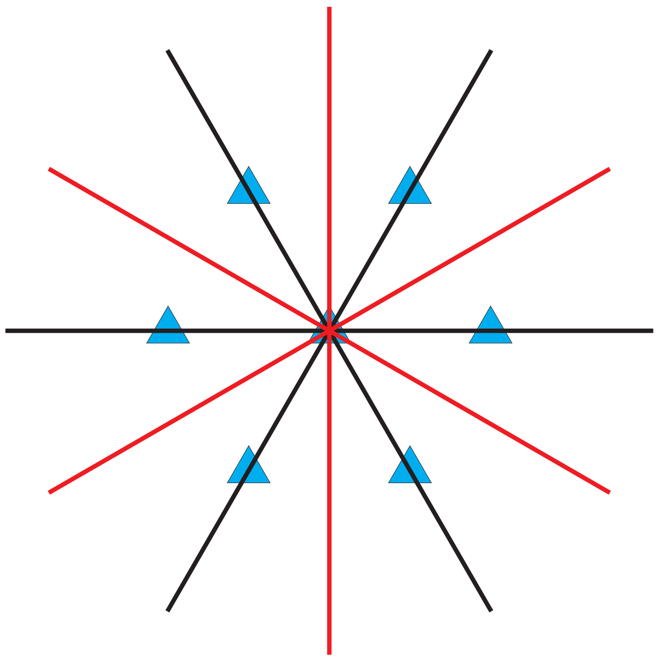
Conclusion: G either contains all three black reflection lines or all three red reflection lines, but not both.



Conclusion: G either contains all three black reflection lines or all three red reflection lines, but not both.

Knowing one odd element α of G determines uniquely all of them: they are exactly $\alpha\beta$ where β goes through all even elements of G.

Hence we have just two possible wallpaper groups with odd isometries.

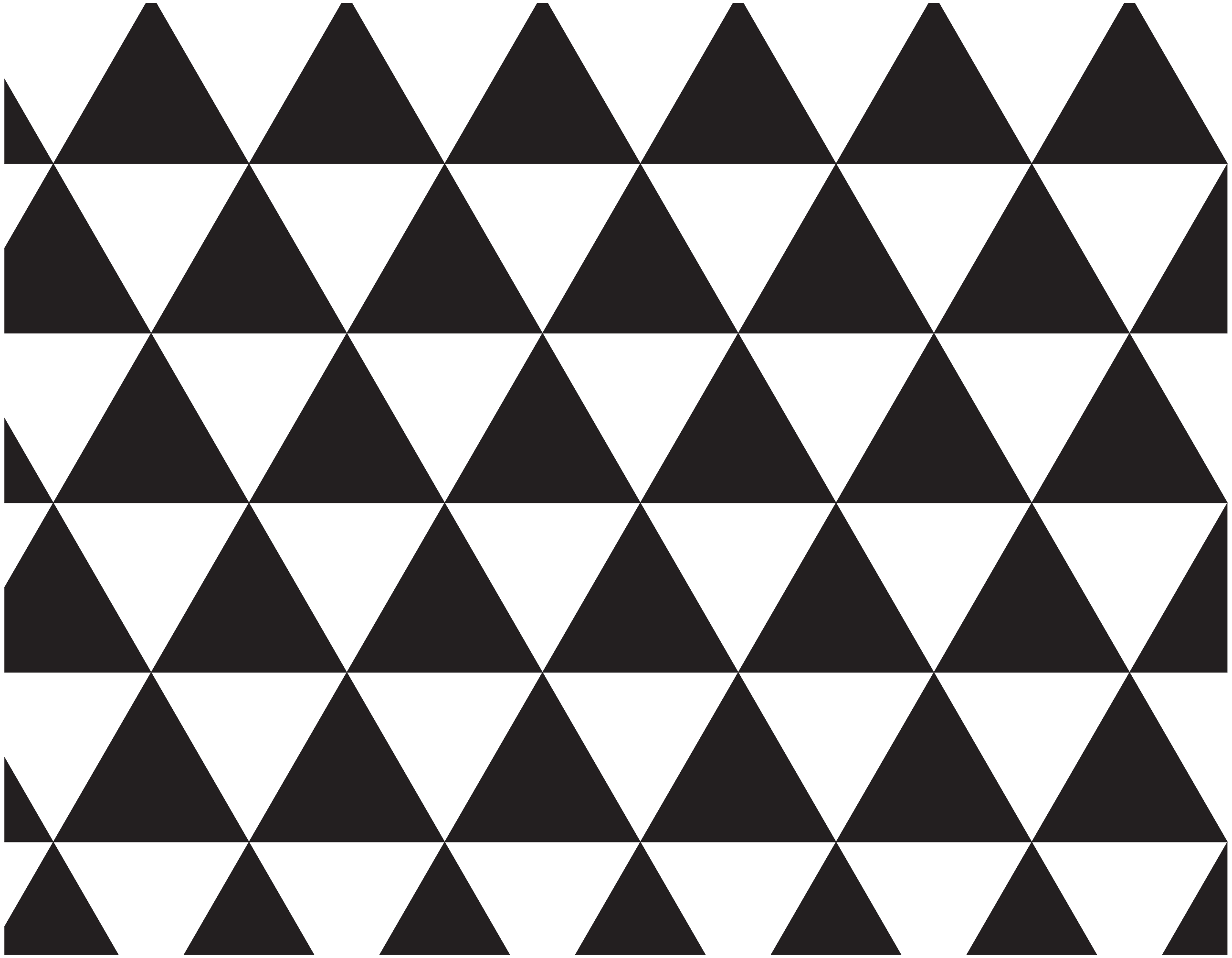


If we take the red reflection lines through point P we obtain the wallpaper group

$$W_3^1 = \langle \tau, \rho, \sigma_r \rangle,$$

and if we take the black reflection lines we obtain the wallpaper group

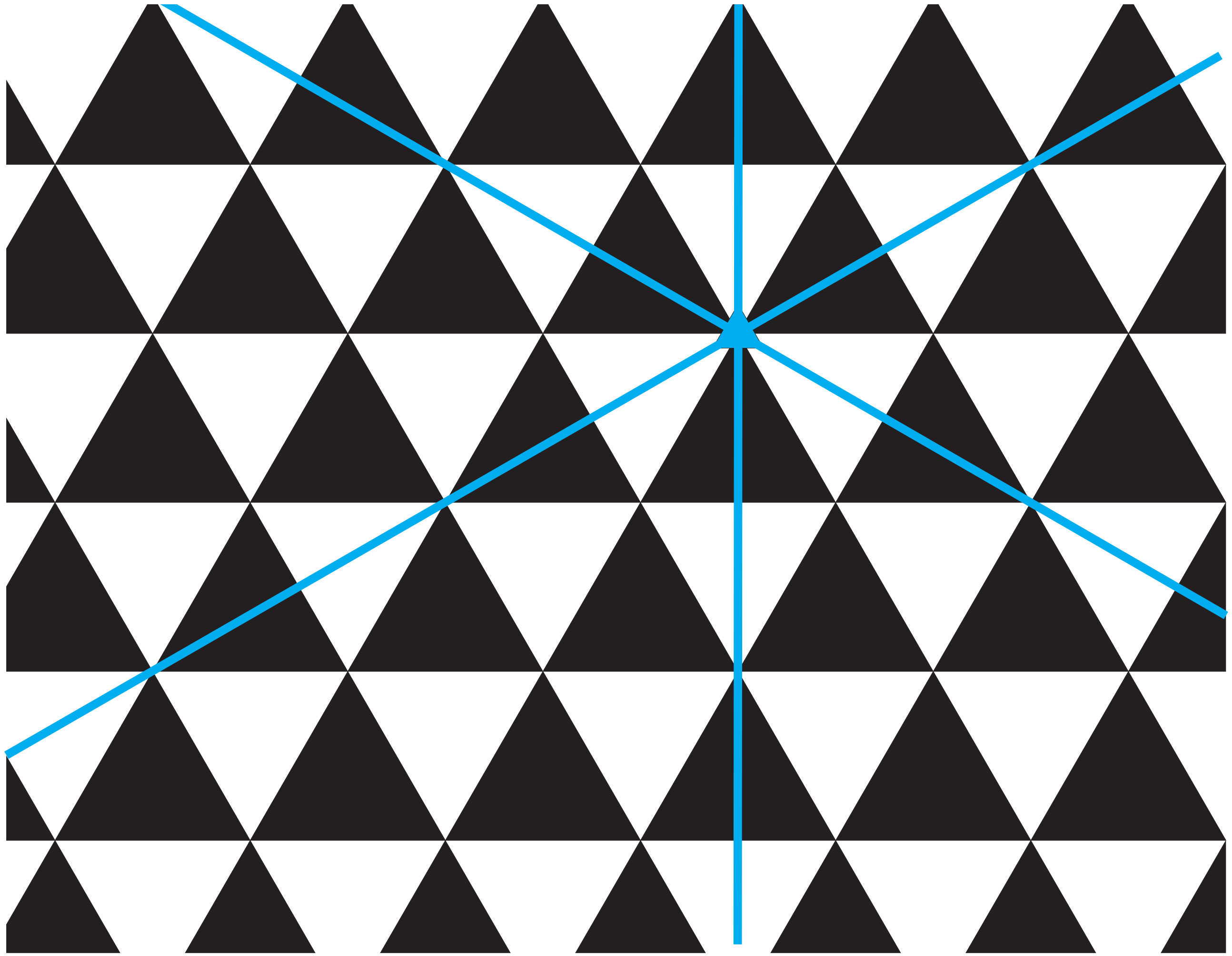
$$W_3^2 = \langle \tau, \rho, \sigma_b \rangle.$$



An example of a pattern whose symmetry group is

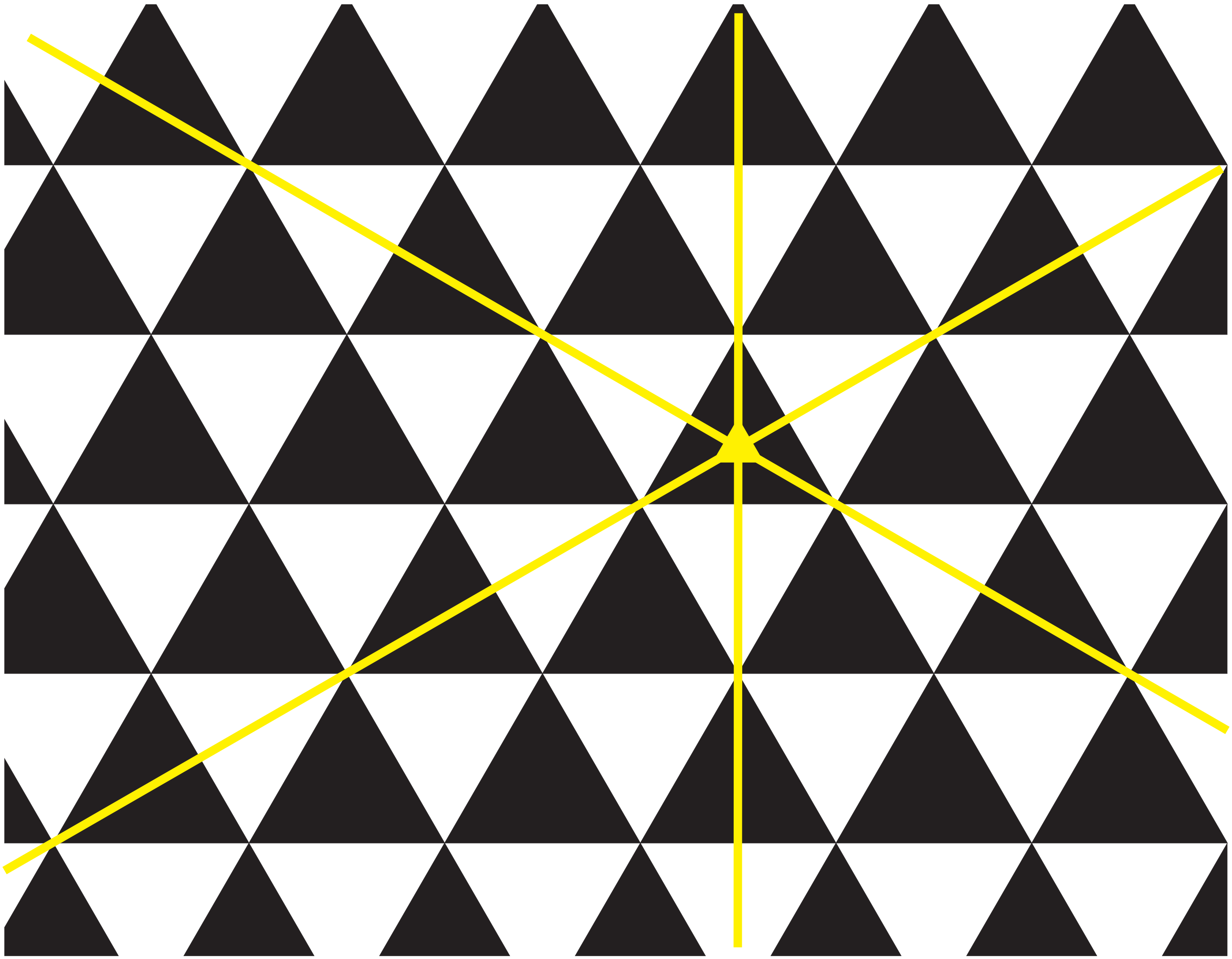
$$W_3^1 = \langle \tau, \rho, \sigma_r \rangle.$$

There are no reflection lines through neighboring lattice points.



But note the reflection lines through all rotation centers:

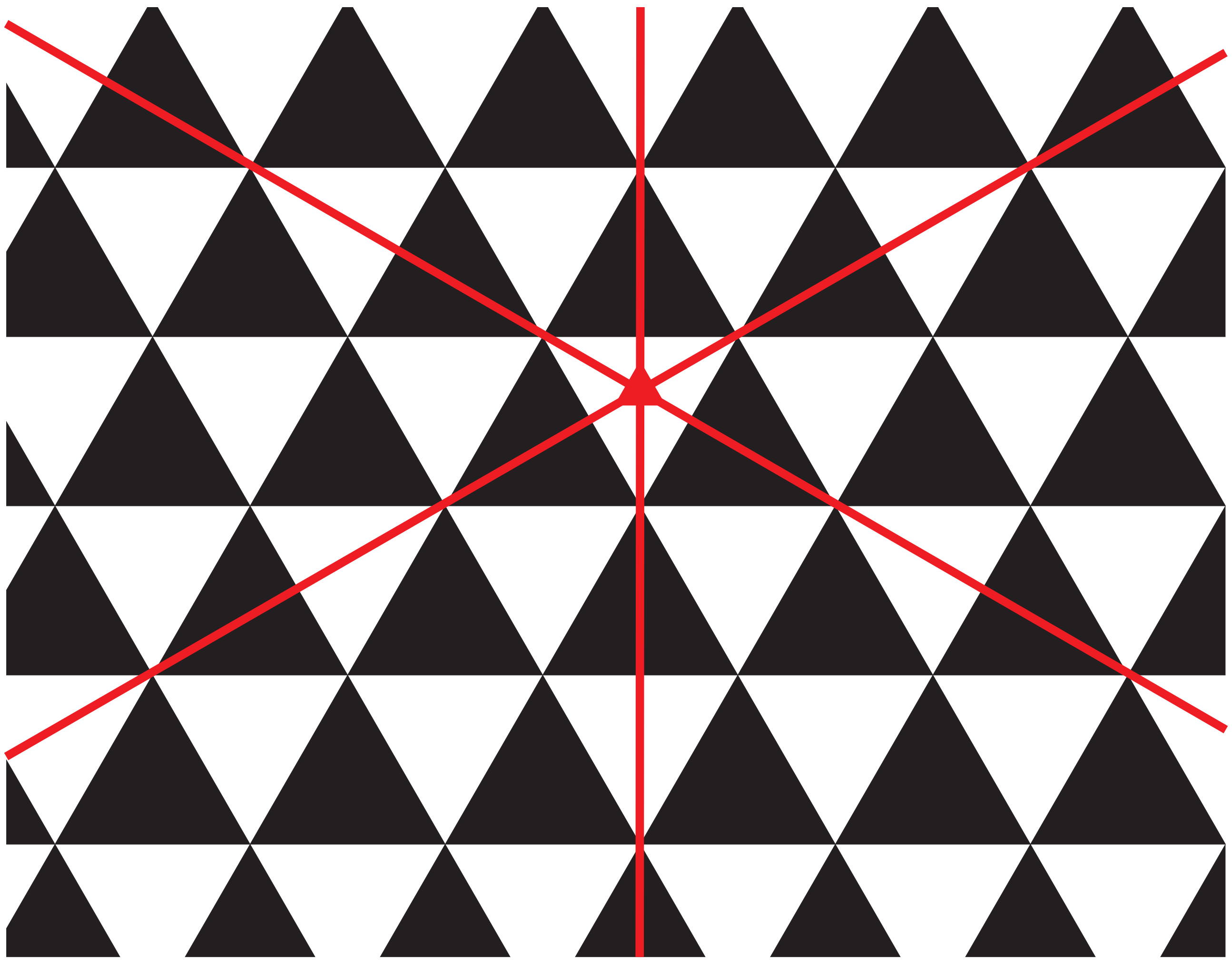
(1) the corners of the triangles,



But note the reflection lines through all rotation centers:

(1) the corners of the triangles,

(2) the centers of the black triangles,

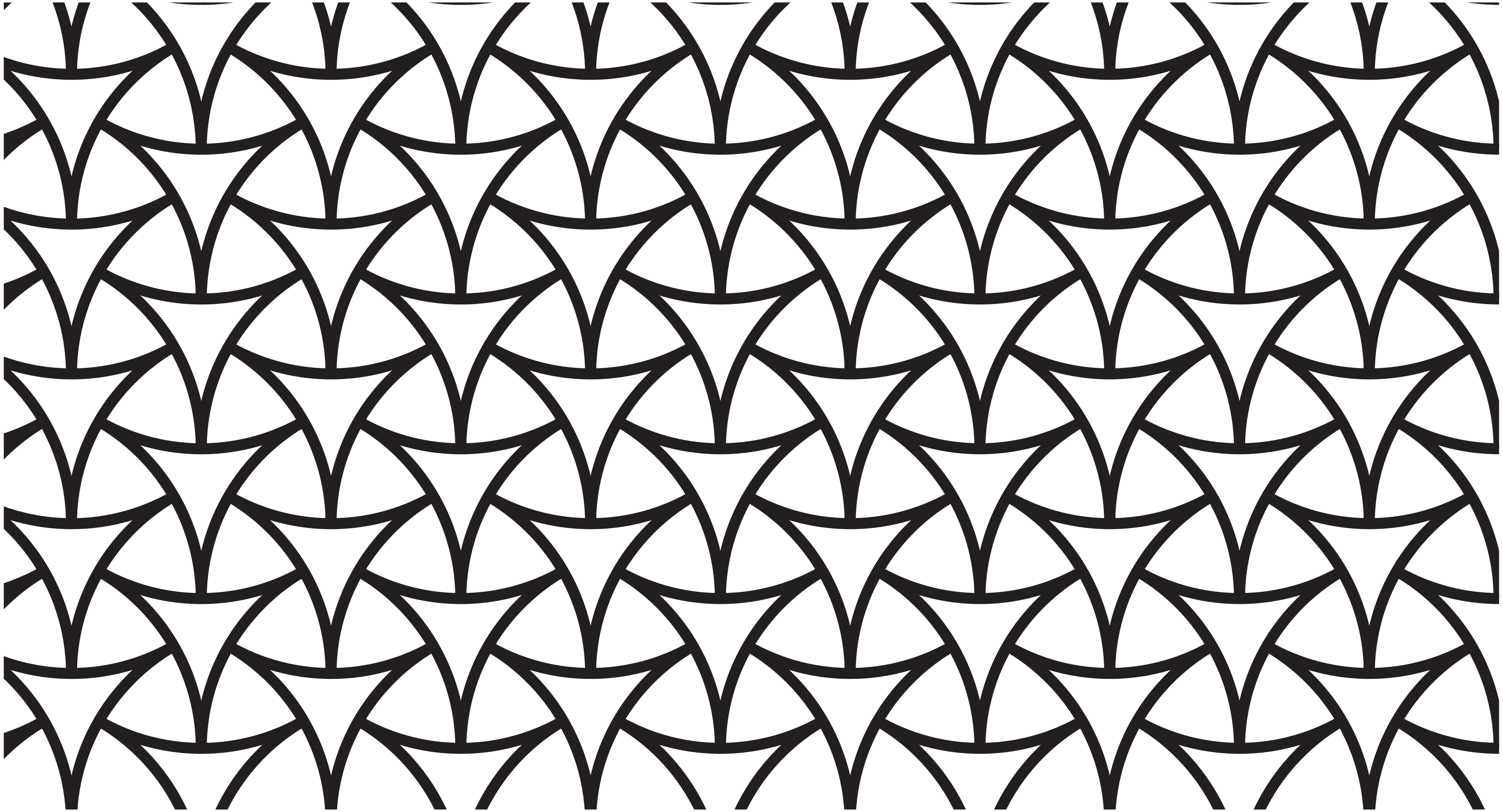


But note the reflection lines through all rotation centers:

(1) the corners of the triangles,

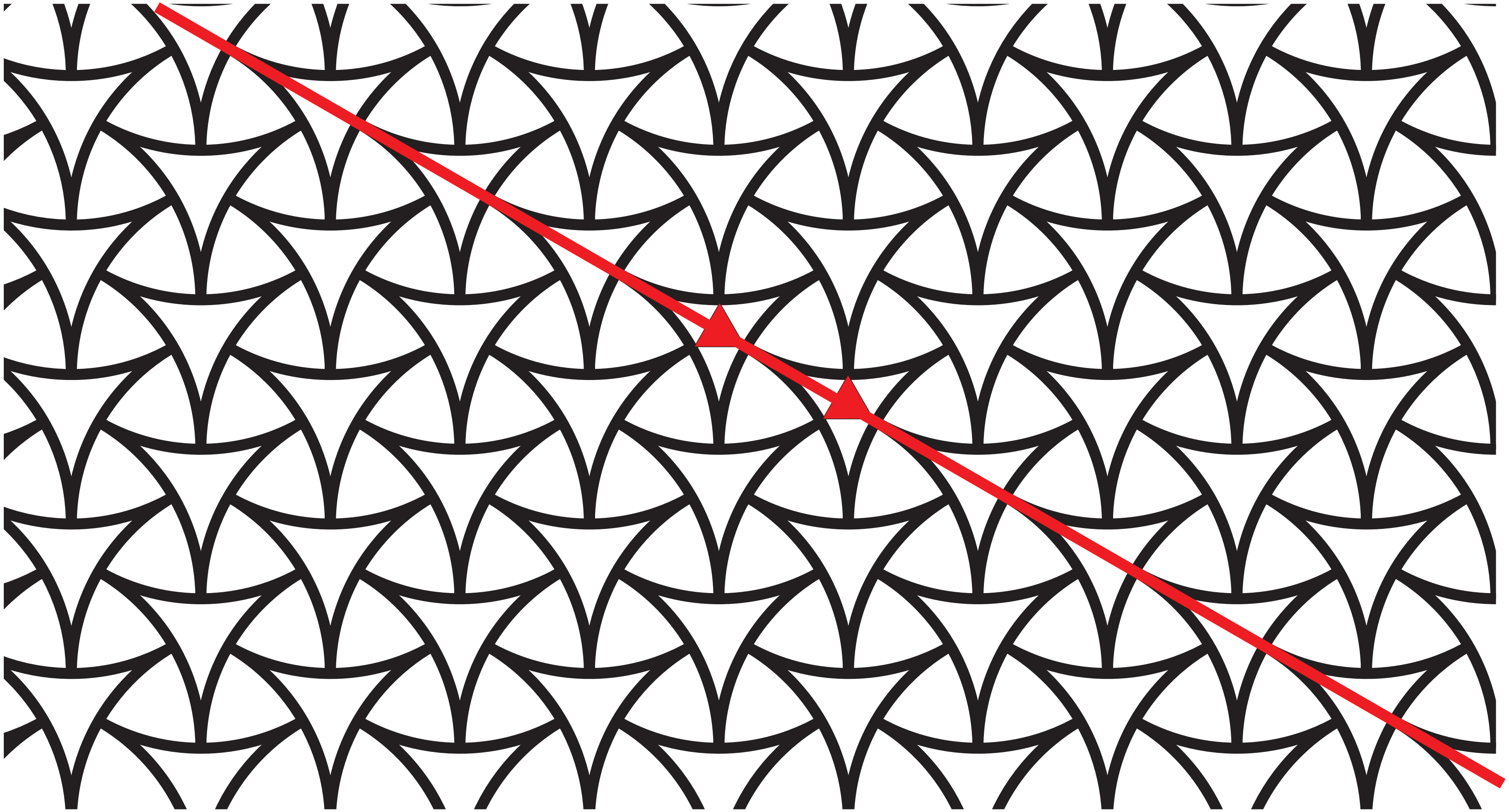
(2) the centers of the black triangles,

(3) the centers of the white triangles.

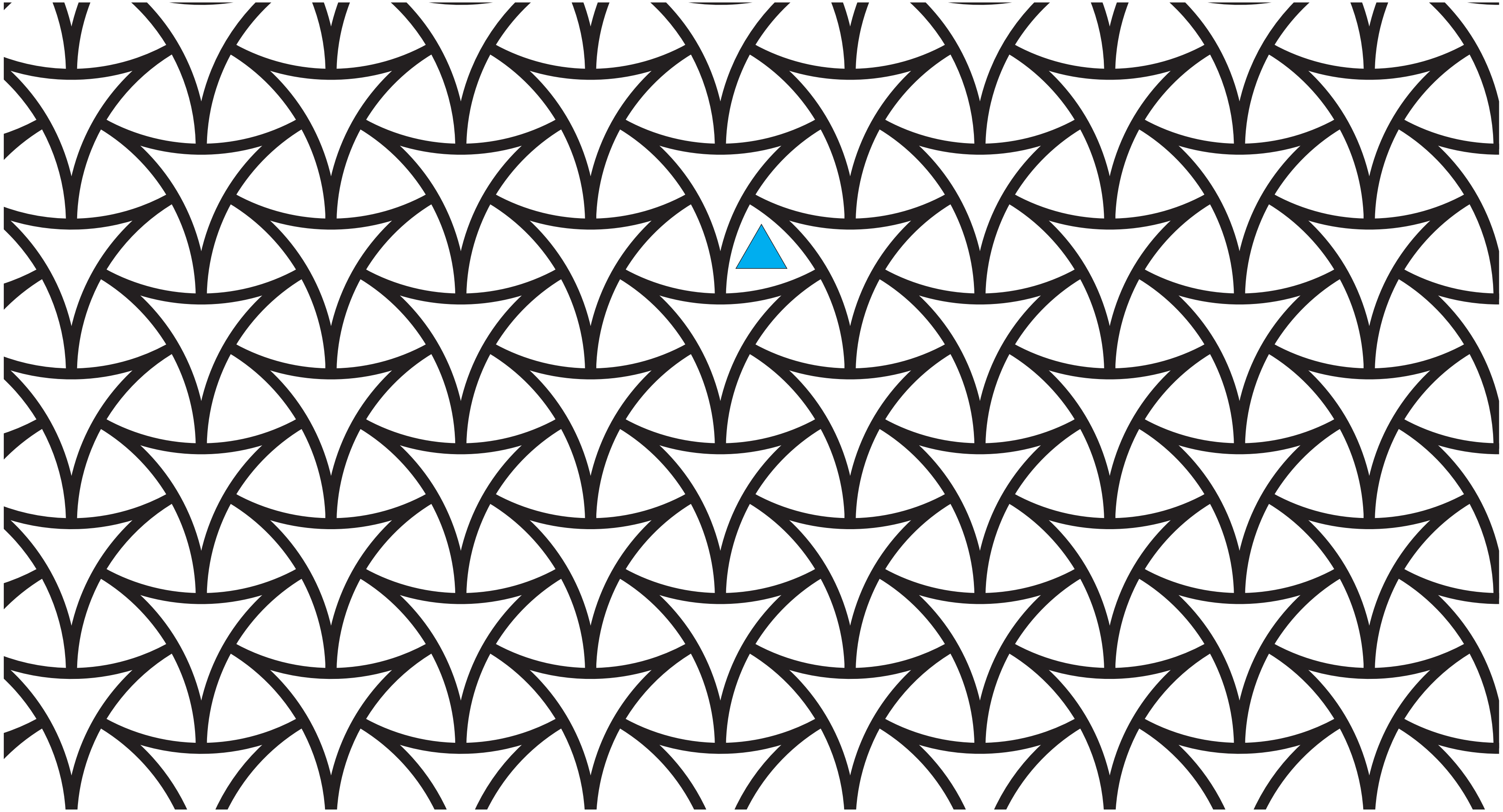


An example of a pattern whose symmetry group is

$$W_3^2 = \langle \tau, \rho, \sigma_b \rangle.$$



There are reflection lines through some neighboring lattice points, i.e., through rotation centers that are separated by the shortest translation.



Another difference to W_3^1 is that there are rotation centers that are not on any reflection line.

Here's an Escher painting with the symmetry group W_3^1.

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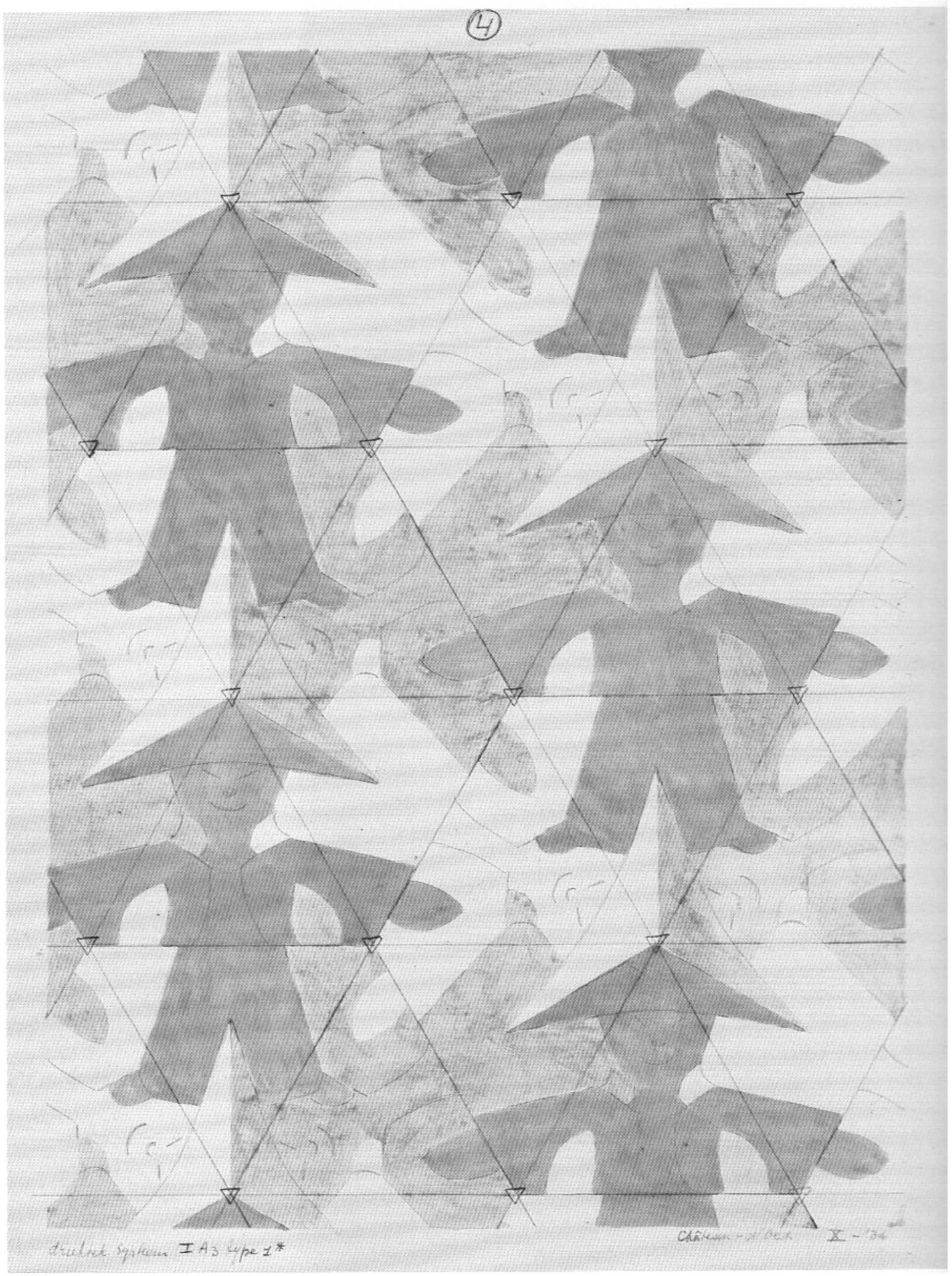
Three interlaced lattices of rotation centers

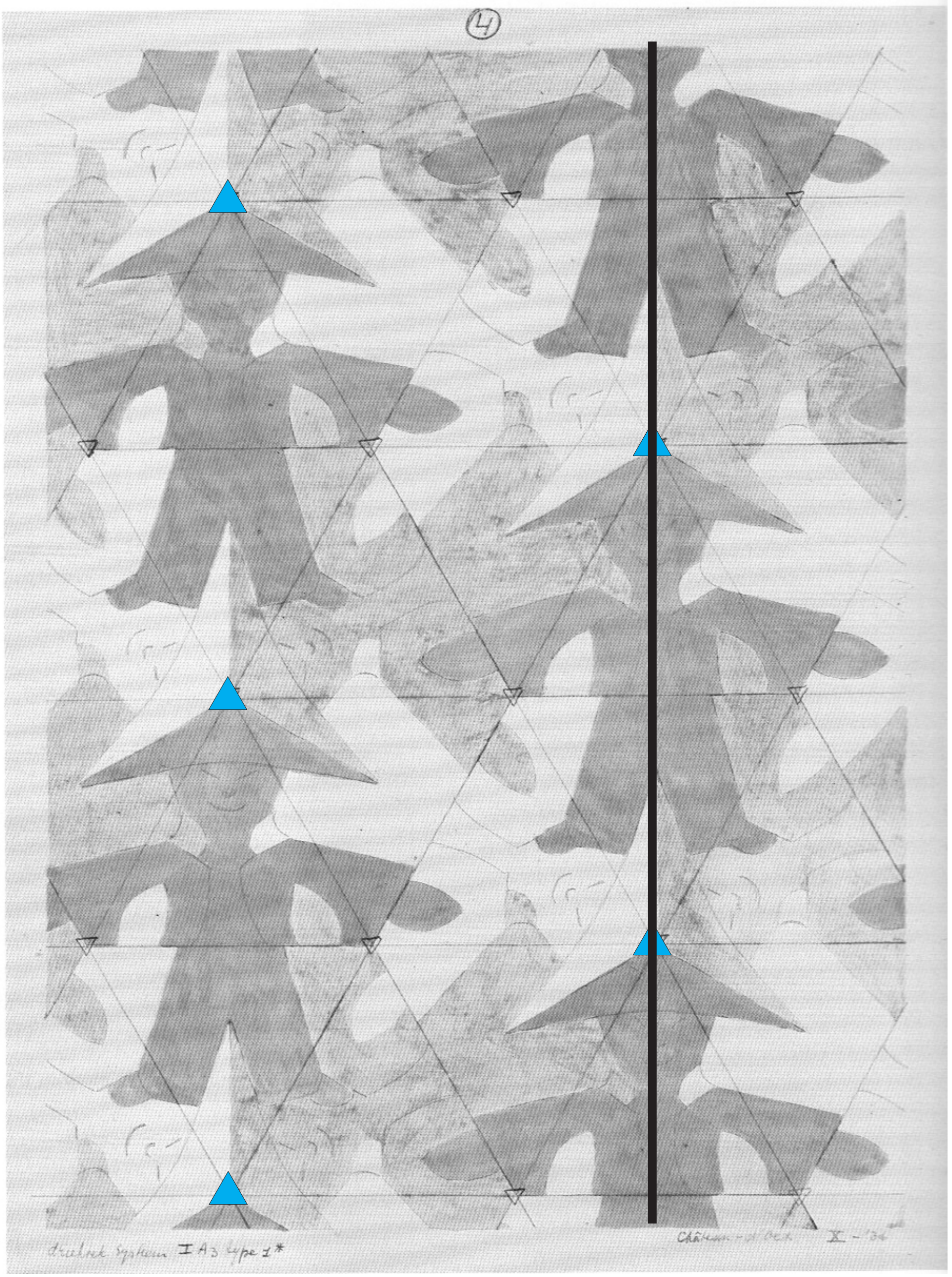
Here's an Escher painting with the symmetry group W_3^1.

Three interlaced lattices of rotation centers

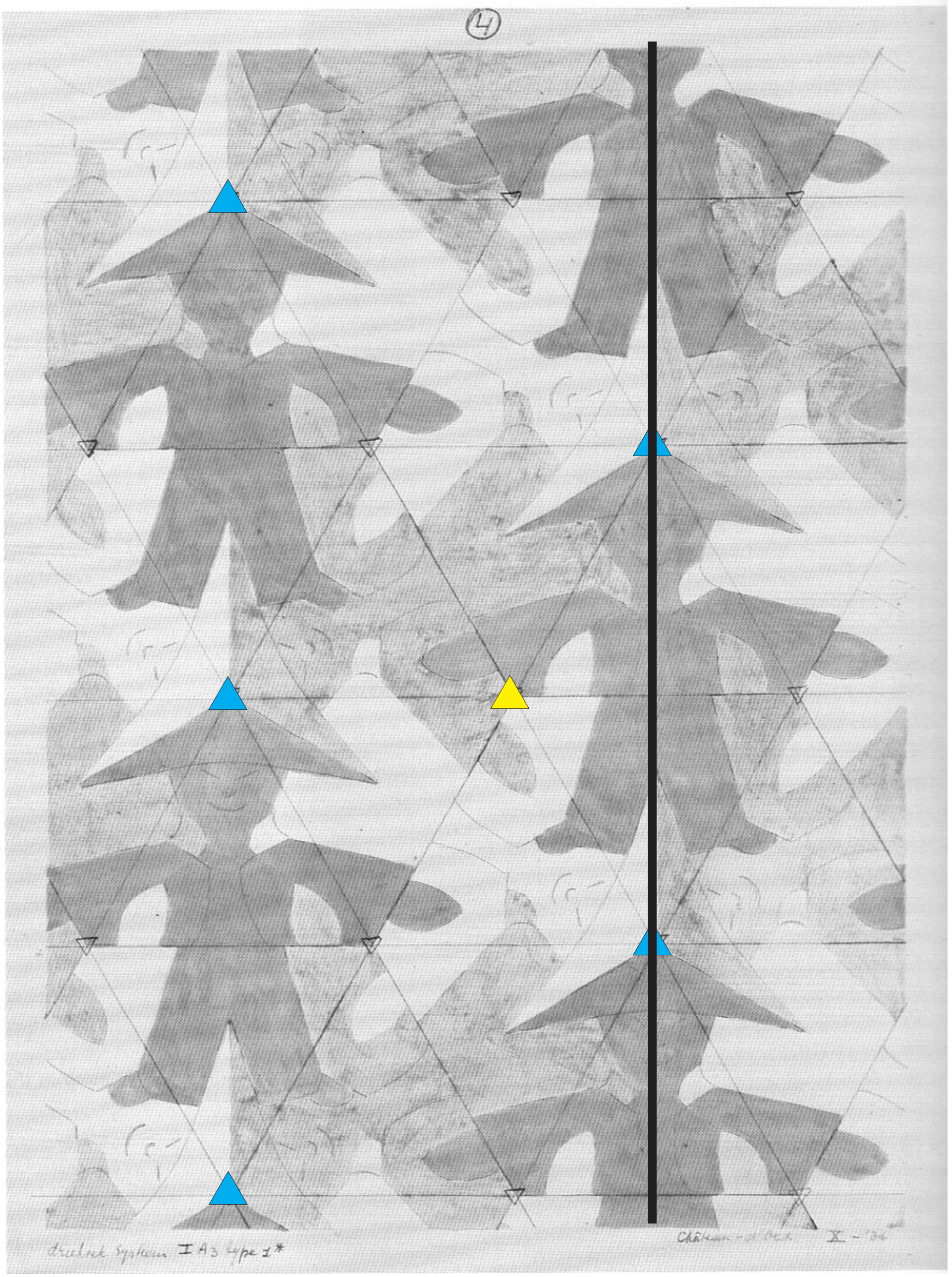
with lines of symmetry through all rotation centers.



This painting has the symmetry group W_3^2.



There are lines of symmetry through adjacent lattice points.

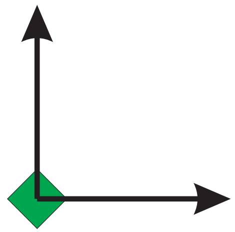


But there are also rotations centers that are not on any line of symmetry.

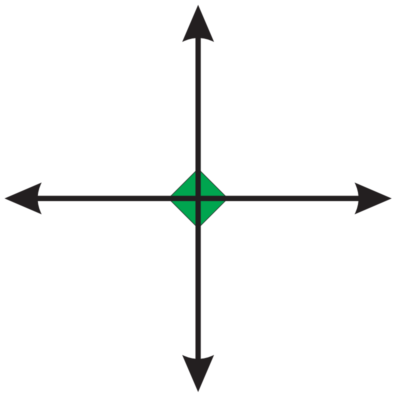
Case (3): Suppose G contains 90° rotations.

Let P be the center of some 4-fold rotation $\rho \in G$.

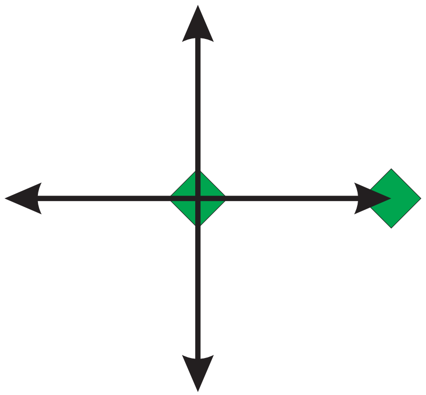
Let τ be a shortest translation in G.



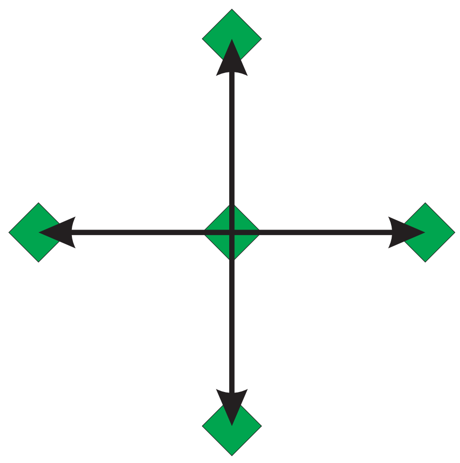
The conjugate $\rho\tau\rho^{-1}$ of τ is the translation that moves point P to point $\rho\tau(P)$. The conjugate is also in group G.



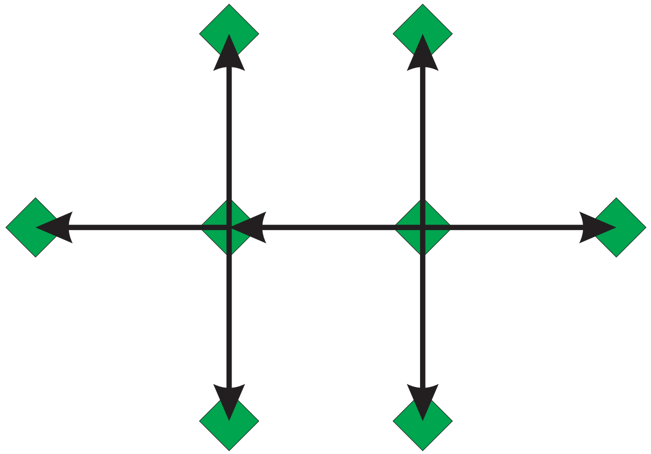
Conjugating τ by rotations around P gives more translations.



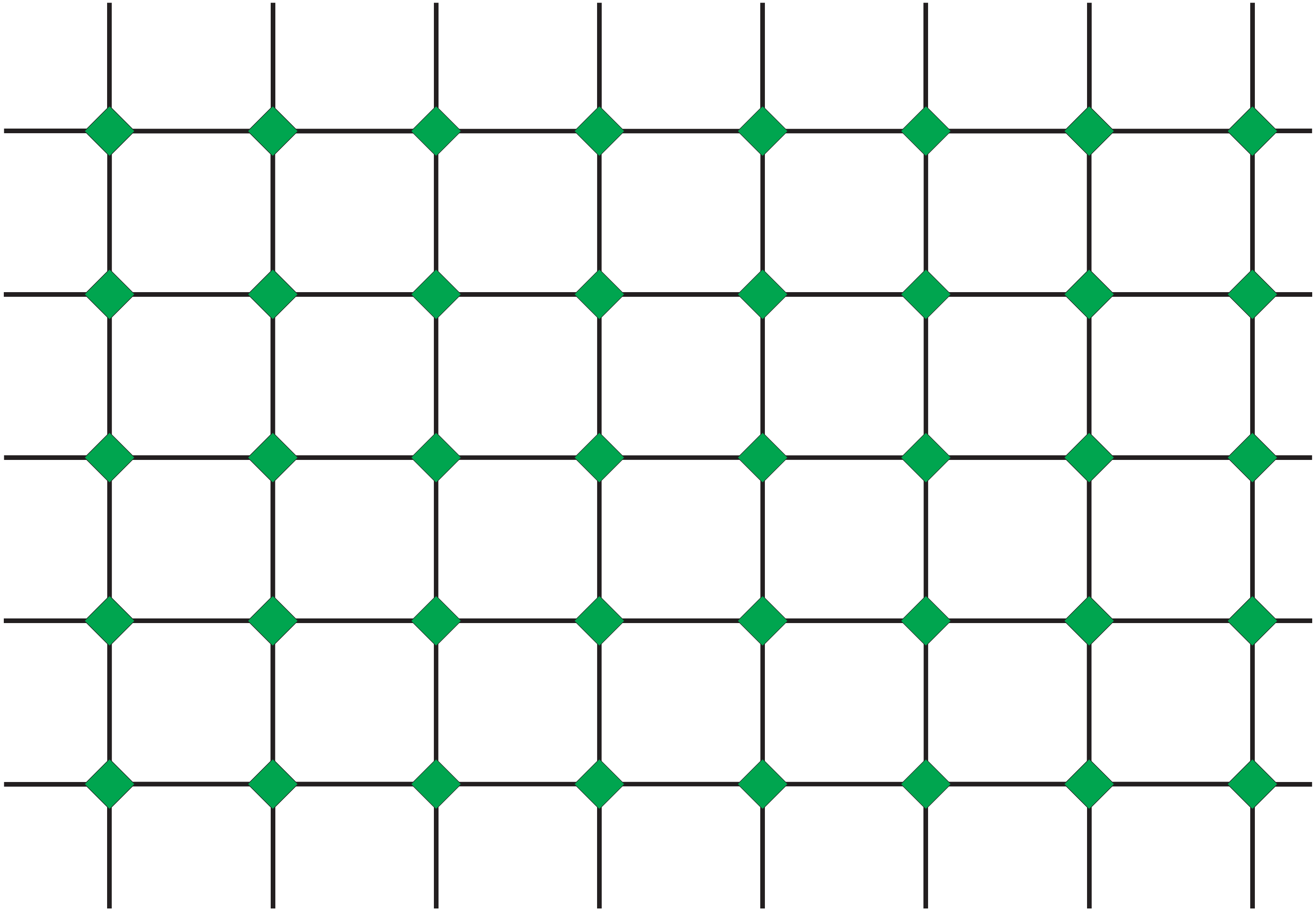
The conjugate $\tau\rho\tau^{-1}$ of ρ is a 90° rotation around point $\tau(P)$.



Analogously, conjugates of ρ by all four translations provide four centers of 4-fold rotation.

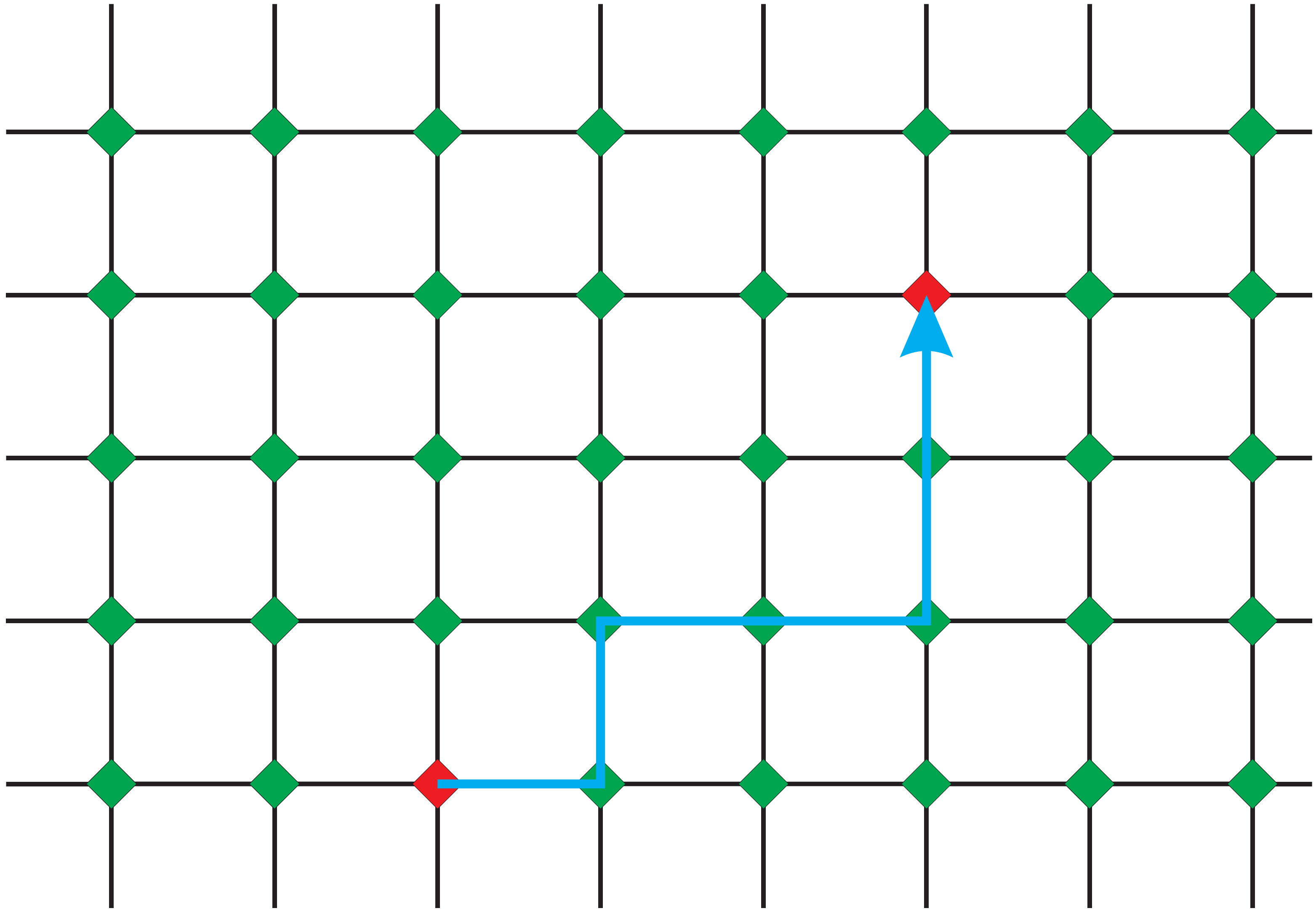


The same reasoning around the rotation center $\tau(P)$ provides new centers of rotation.

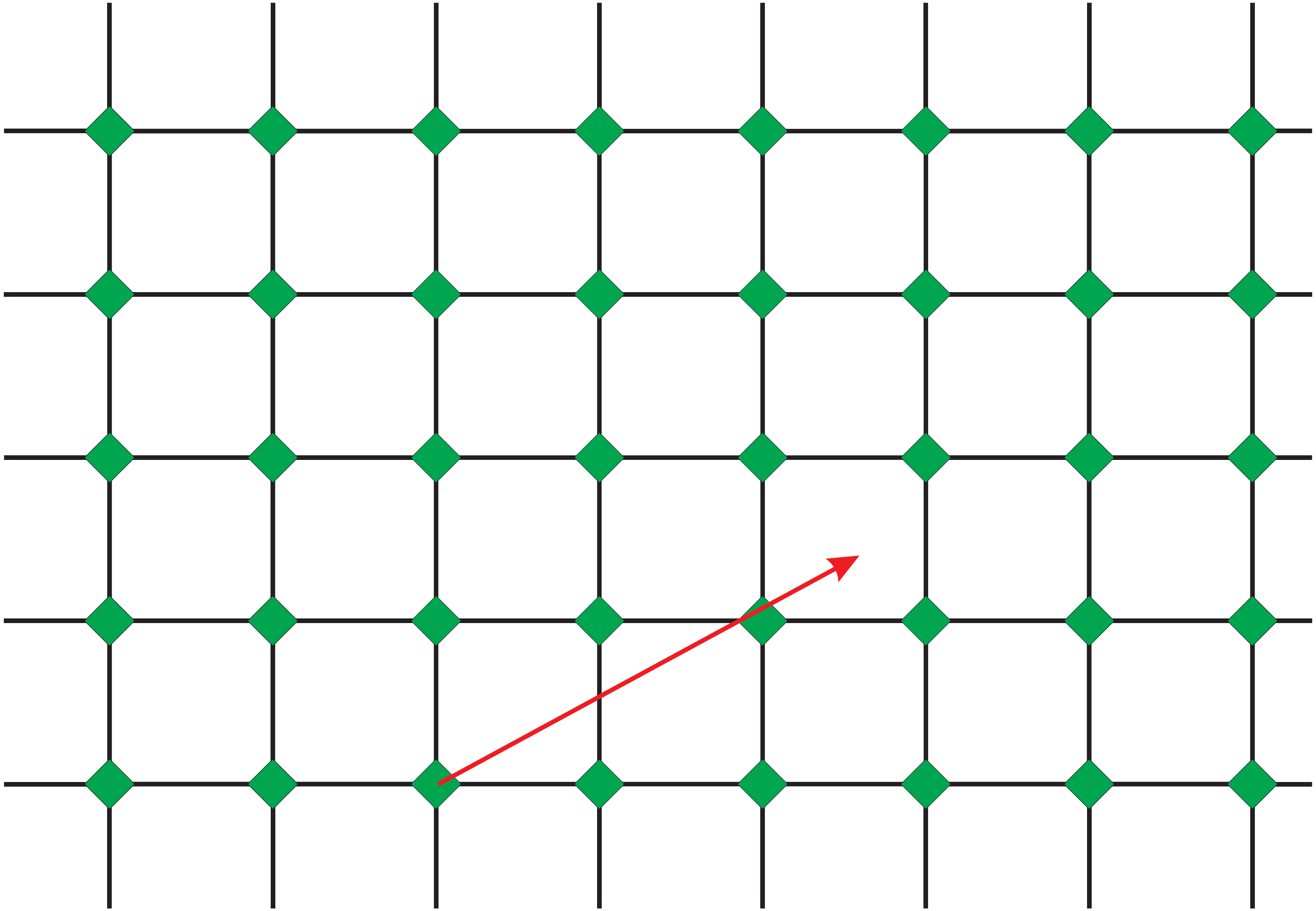


This reasoning can be repeated for all centers of rotation, which provides us with an infinite square lattice of rotation centers.

Let S_4 be the set of these rotation centers.

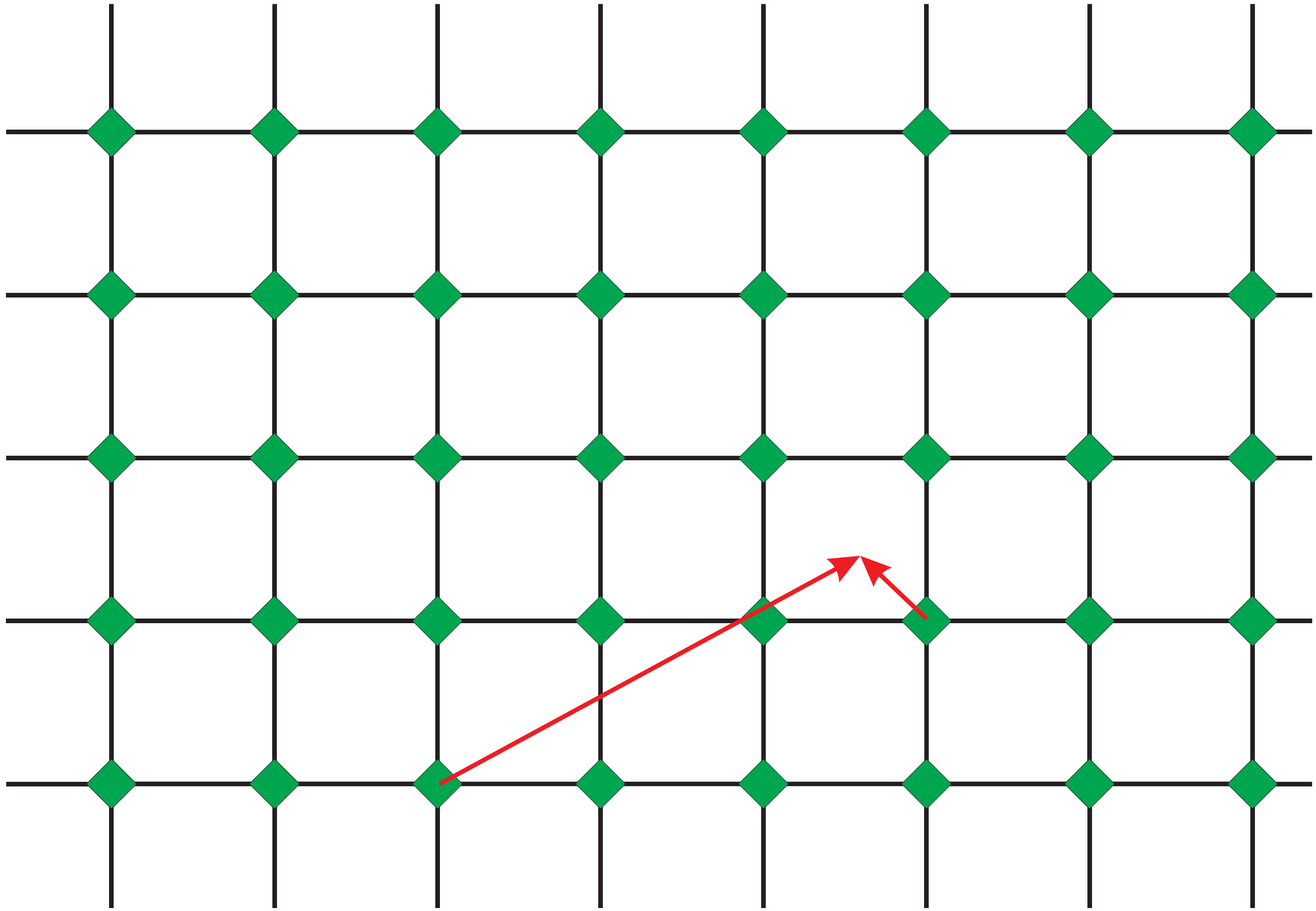


Any translation moving a lattice point into another lattice point is in group G, as it is a composition of the one step translations in the lattice.



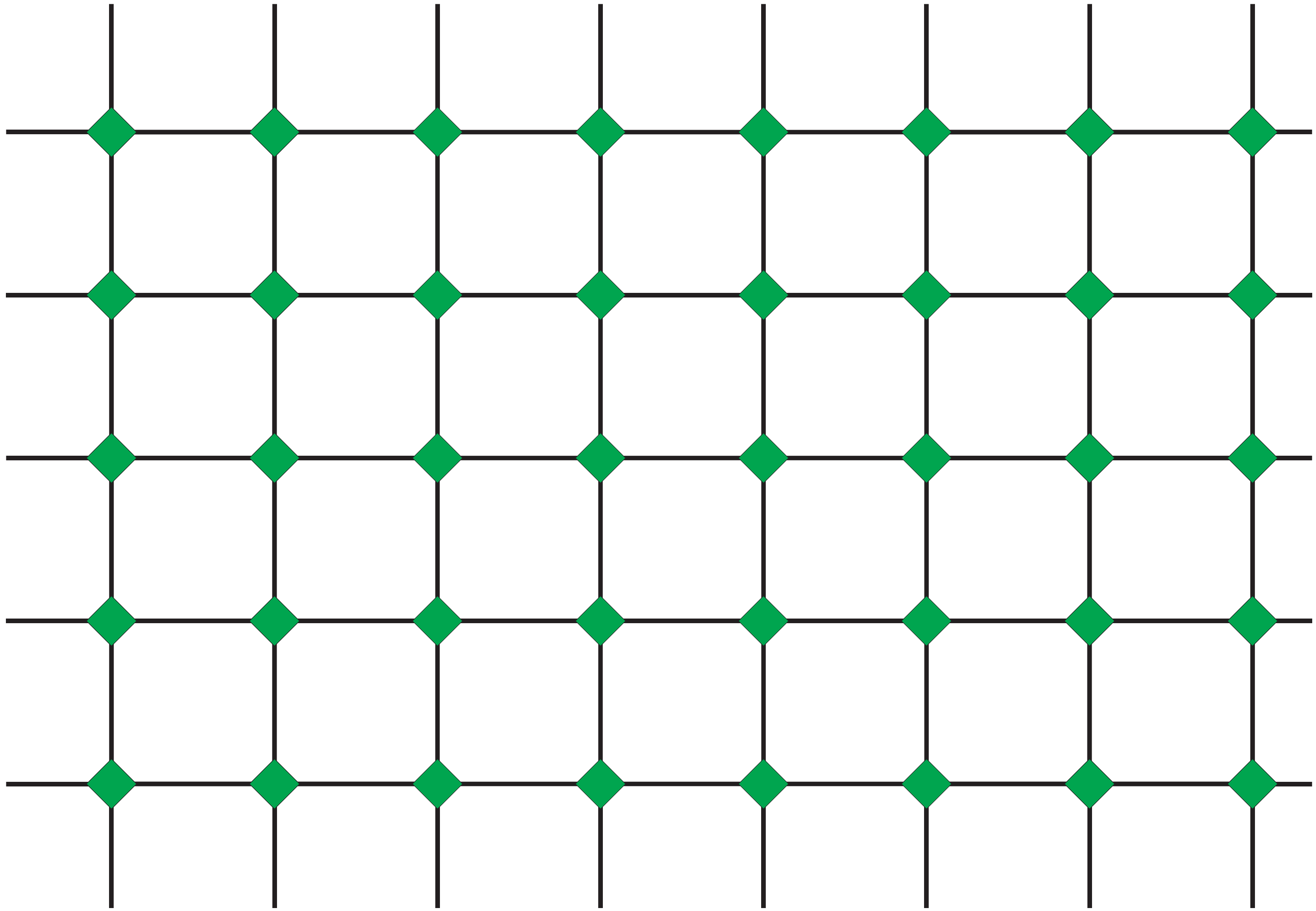
Conversely, all translations in G must take lattice points to lattice points.

Suppose the contrary: some translation takes a lattice point into a non-lattice point.



Then G also contains a translation taking a corner of one of the squares of the lattice into a non-vertex point of the square.

But this contradicts the minimality of translation τ.



We have proved that the translations in G are exactly the translations that map the lattice S_4 into itself.

Let us prove the same for rotations.

If $\rho' \in G$ is any rotation then – by the crystallographic restriction – the rotation angle is a multiple of 90°.

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It follows that $\rho'\rho^i = \tau'$ is a translation, for some i, where ρ is our initial rotation around a lattice point.

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It follows that $\rho'\rho^i = \tau'$ is a translation, for some i, where ρ is our initial rotation around a lattice point.

Then $\rho' = \tau'\rho^{-i}$ takes lattice points to lattice points since both ρ and τ' are symmetries of the lattice.

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It follows that $\rho'\rho^i = \tau'$ is a translation, for some i, where ρ is our initial rotation around a lattice point.

Then $\rho' = \tau'\rho^{-i}$ takes lattice points to lattice points since both ρ and τ' are symmetries of the lattice.

We have proved that the rotations in G are also symmetries of the lattice S_4.

Conversely, let us show that all rotational symmetries of S_4 are in G. So let ρ' be a rotation that is a symmetry of S_4.

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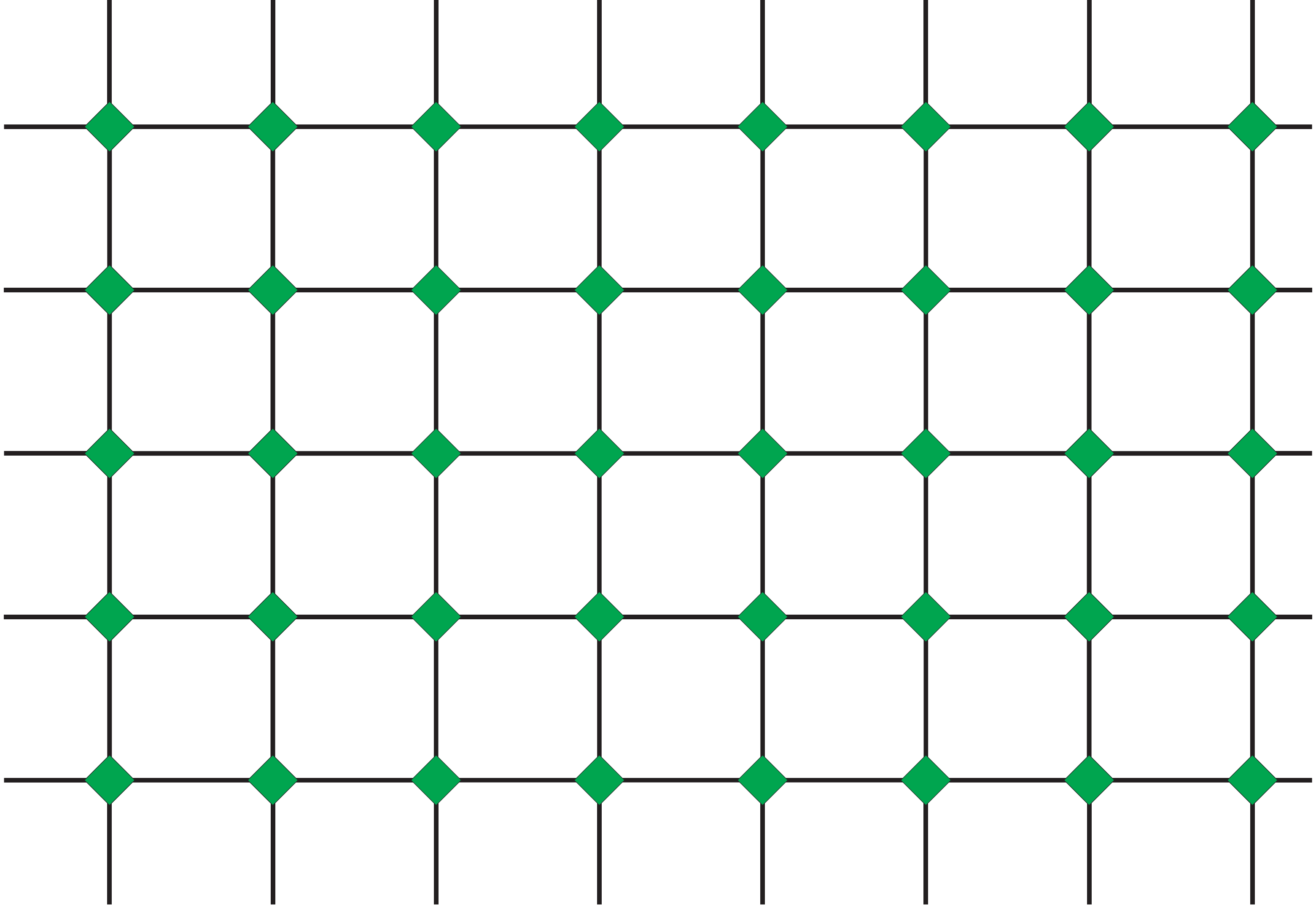
The symmetry group of S_4 is a wallpaper group that contains a 4-fold rotation, so by the crystallographic restriction, the rotation angle of ρ' is a multiple of $90°$.

Conversely, let us show that all rotational symmetries of S_4 are in G. So let ρ' be a rotation that is a symmetry of S_4.

The symmetry group of S_4 is a wallpaper group that contains a 4-fold rotation, so by the crystallographic restriction, the rotation angle of ρ' is a multiple of $90°$.

Then, $\rho'\rho^i = \tau'$ is a translation, for some i. This translation maps lattice points to lattice points and hence $\tau' \in G$.

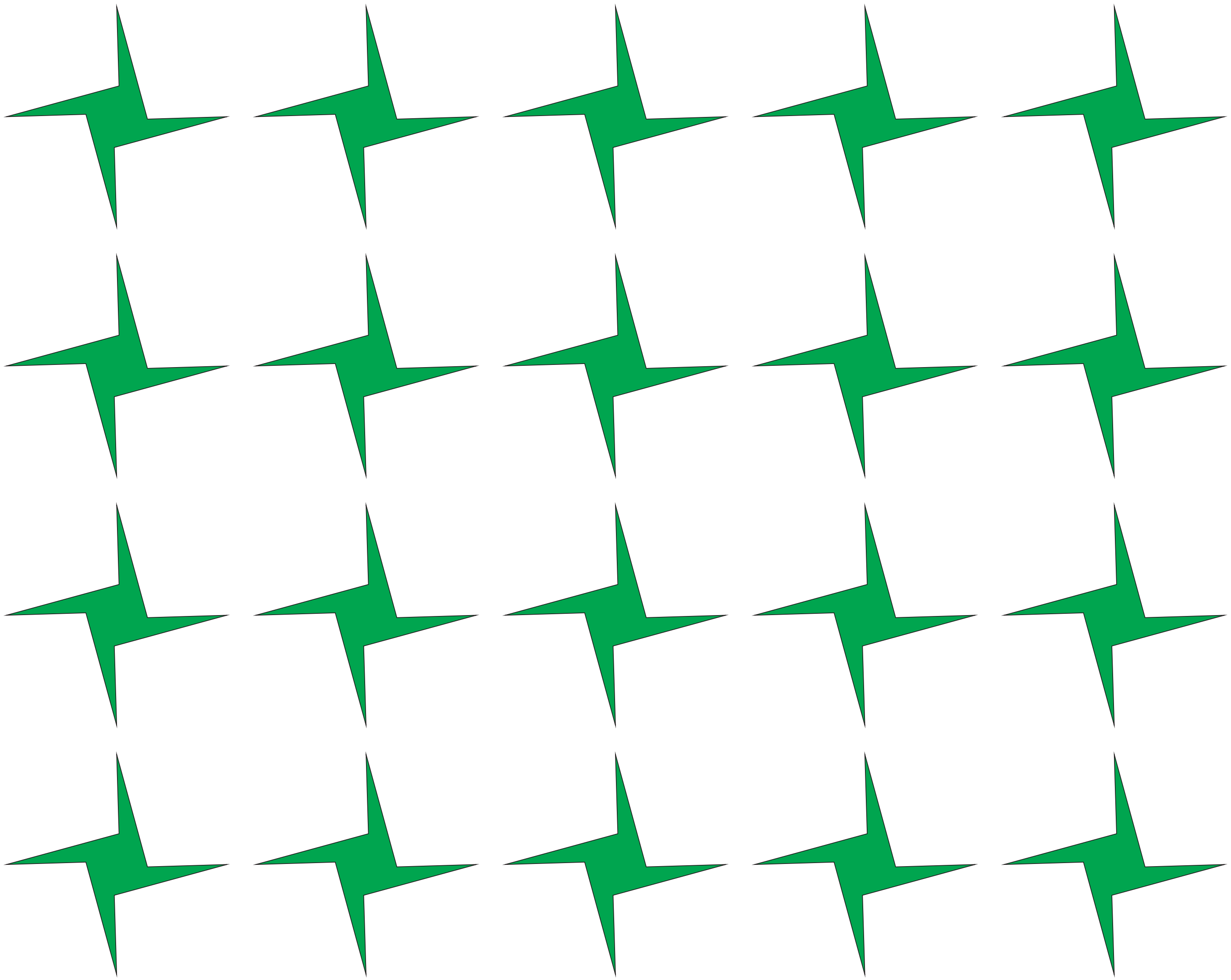
We have $\rho' = \tau'\rho^{-i} \in G$.



We have proved that the even isometries in G are exactly the even symmetries of the square lattice S_4.

If G contains no odd isometries then G is uniquely determined: It is the wallpaper group

$$W_4 = \langle \rho, \tau \rangle.$$



An example of a pattern whose symmetry group is

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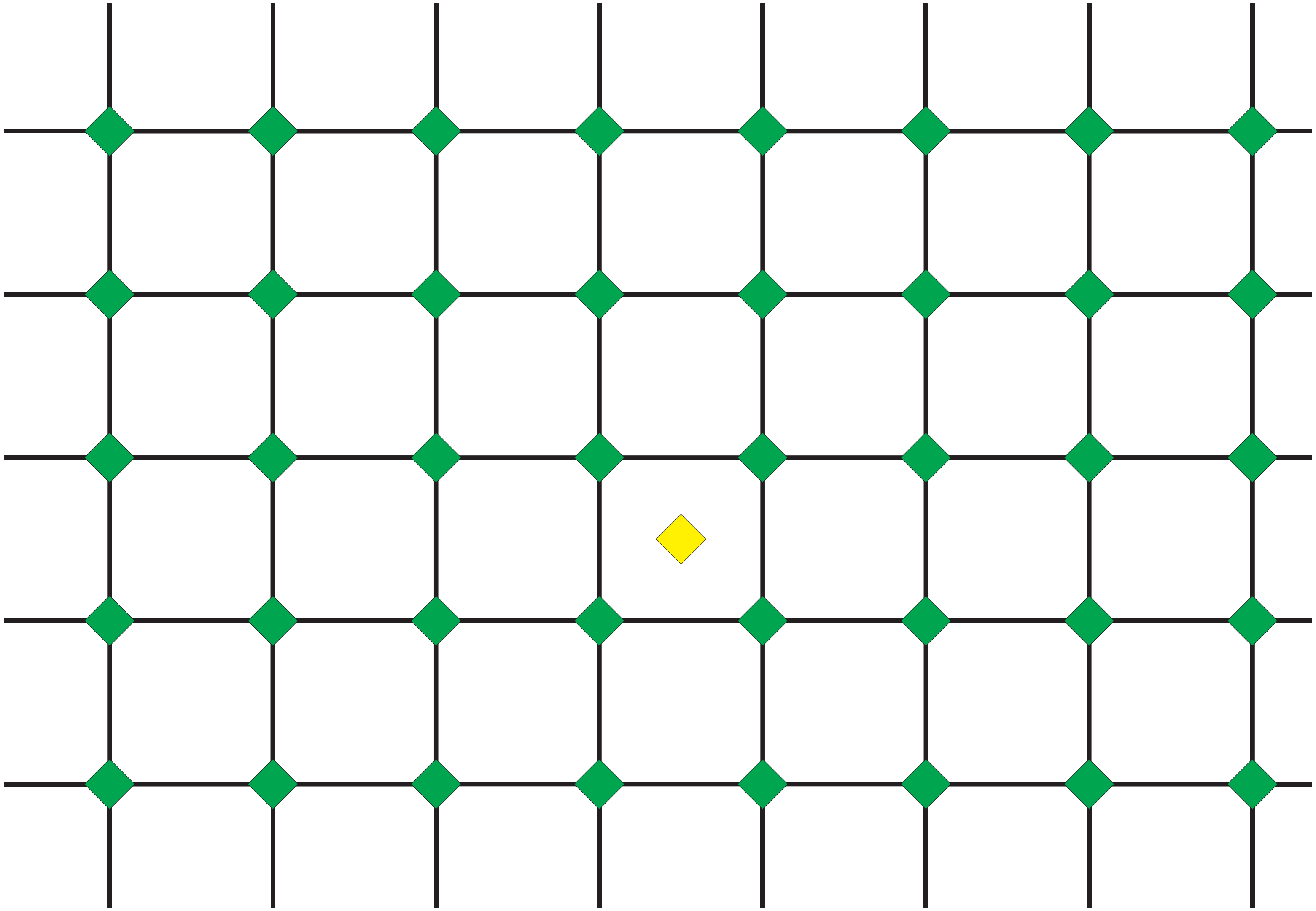
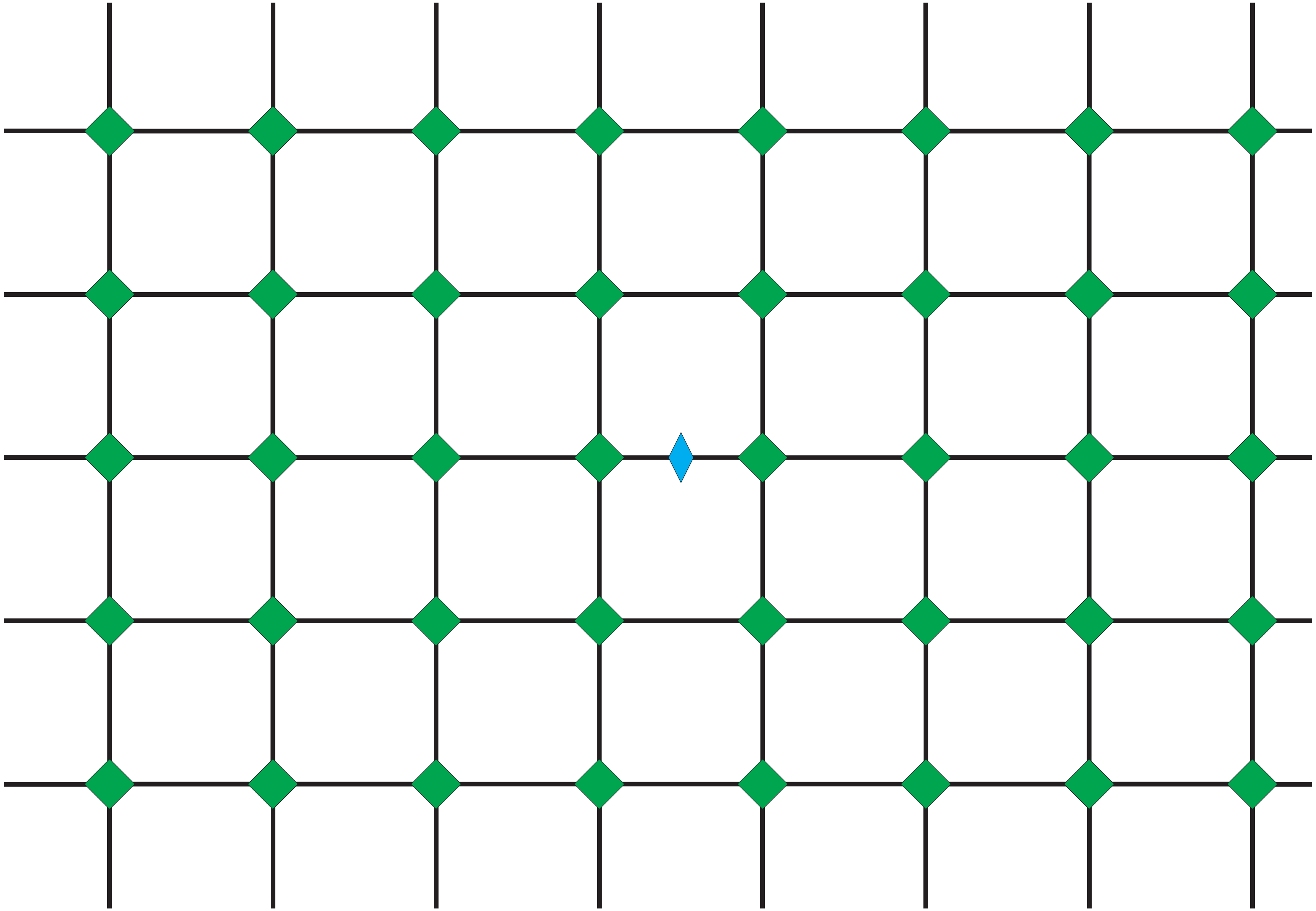
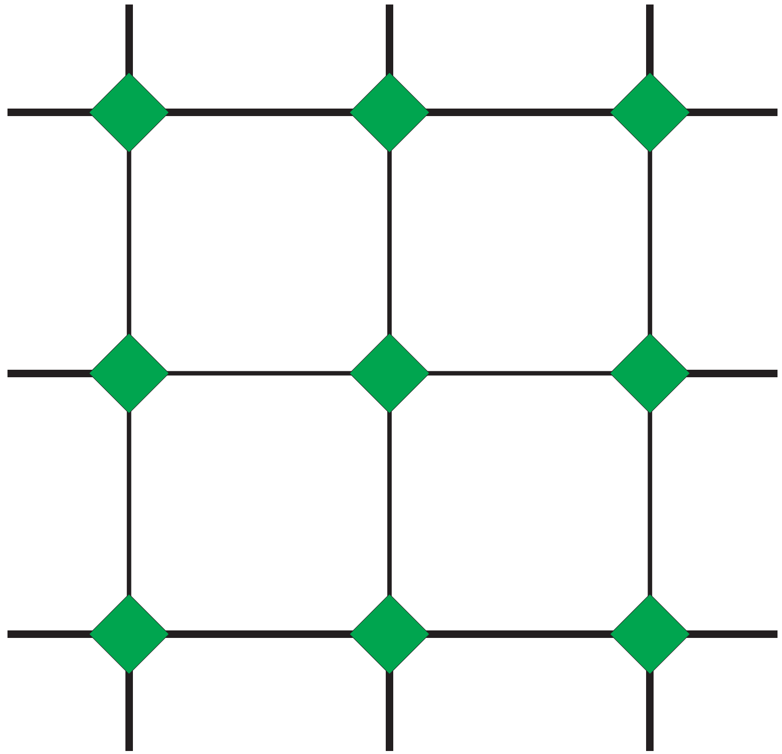


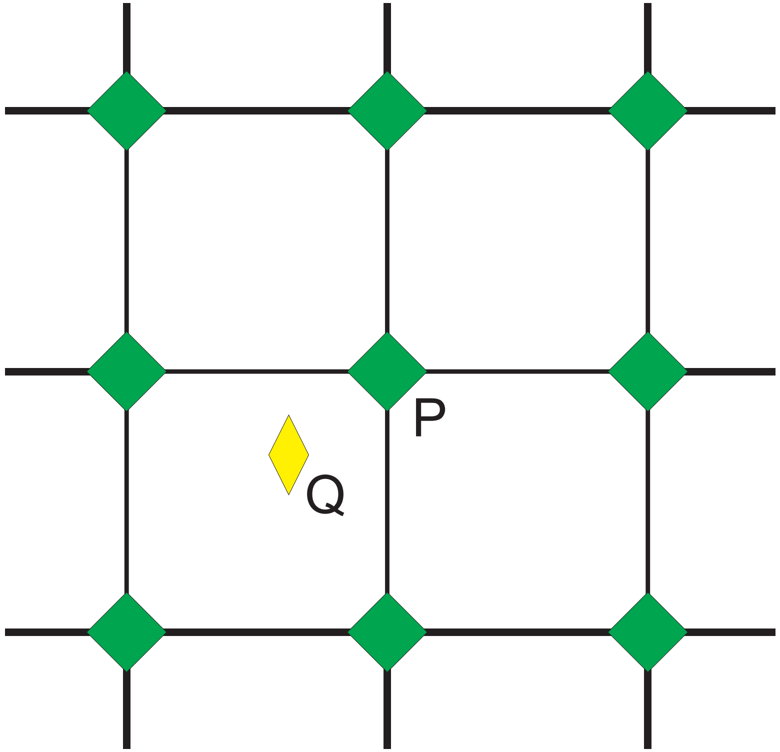
In addition to the 4-fold rotations at the lattice points, the group also contains 4-fold rotational symmetries at the centers of the squares.



The group also contains half-turns around the midpoints between neighboring lattice points.



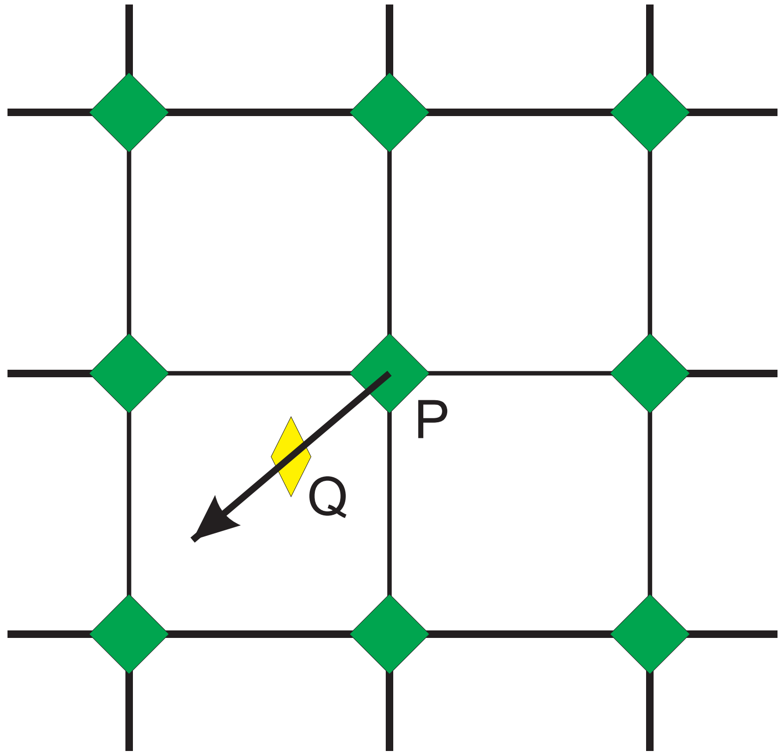
Let us show that these are the only rotations in G.



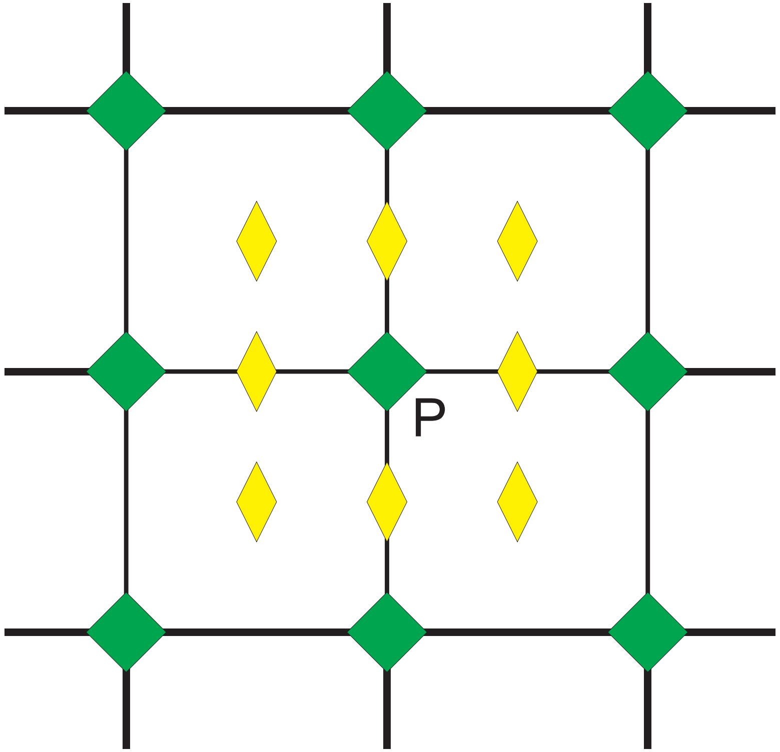
Let us show that these are the only rotations in G.

Let Q be a center of a rotation in G. By the crystallographic restriction, the halfturn σ_Q is in G.

Let P be the lattice point closest to Q. Also the halfturn σ_P around P is in G.



The composition $\sigma_Q\sigma_P$ is the translation by the vector $2\overrightarrow{PQ}$. Because $\sigma_Q\sigma_P$ is an even isometry in G, it maps lattice points to lattice points. In particular, the image of P must be one of the eight lattice points surrounding P.



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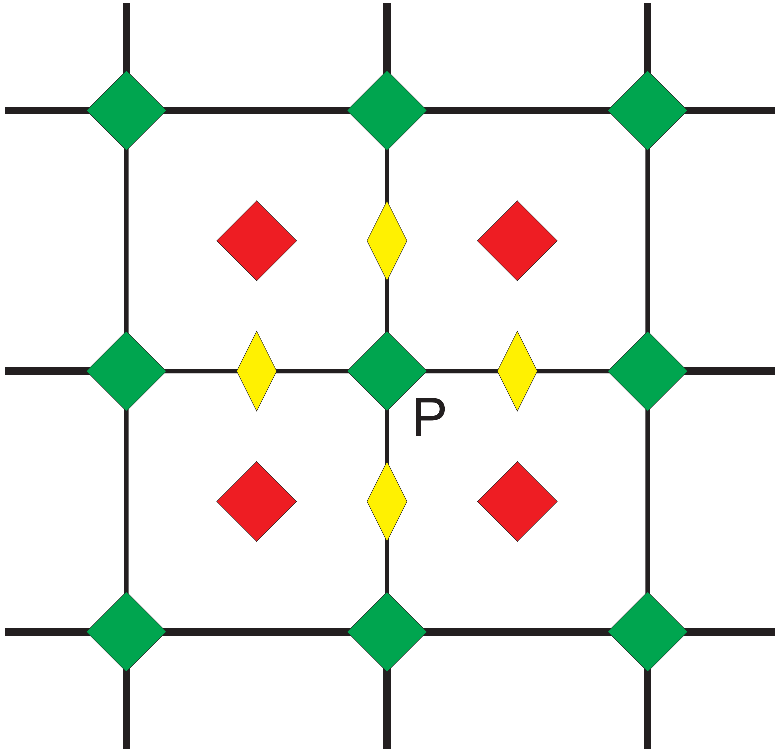
This is only possible if Q is the center of a lattice square or a midpoint between two adjacent lattice points.



The composition $\sigma_Q \sigma_P$ is the translation by the vector $2\overrightarrow{PQ}$. Because $\sigma_Q \sigma_P$ is an even isometry in G, it maps lattice points to lattice points. In particular, the image of P must be one of the eight lattice points surrounding P.

This is only possible if Q is the center of a lattice square or a midpoint between two adjacent lattice points.

Among these, the centers of squares have 4-fold symmetry.

An example of a painting by Escher with symmetry group W_4 (ignoring colors).

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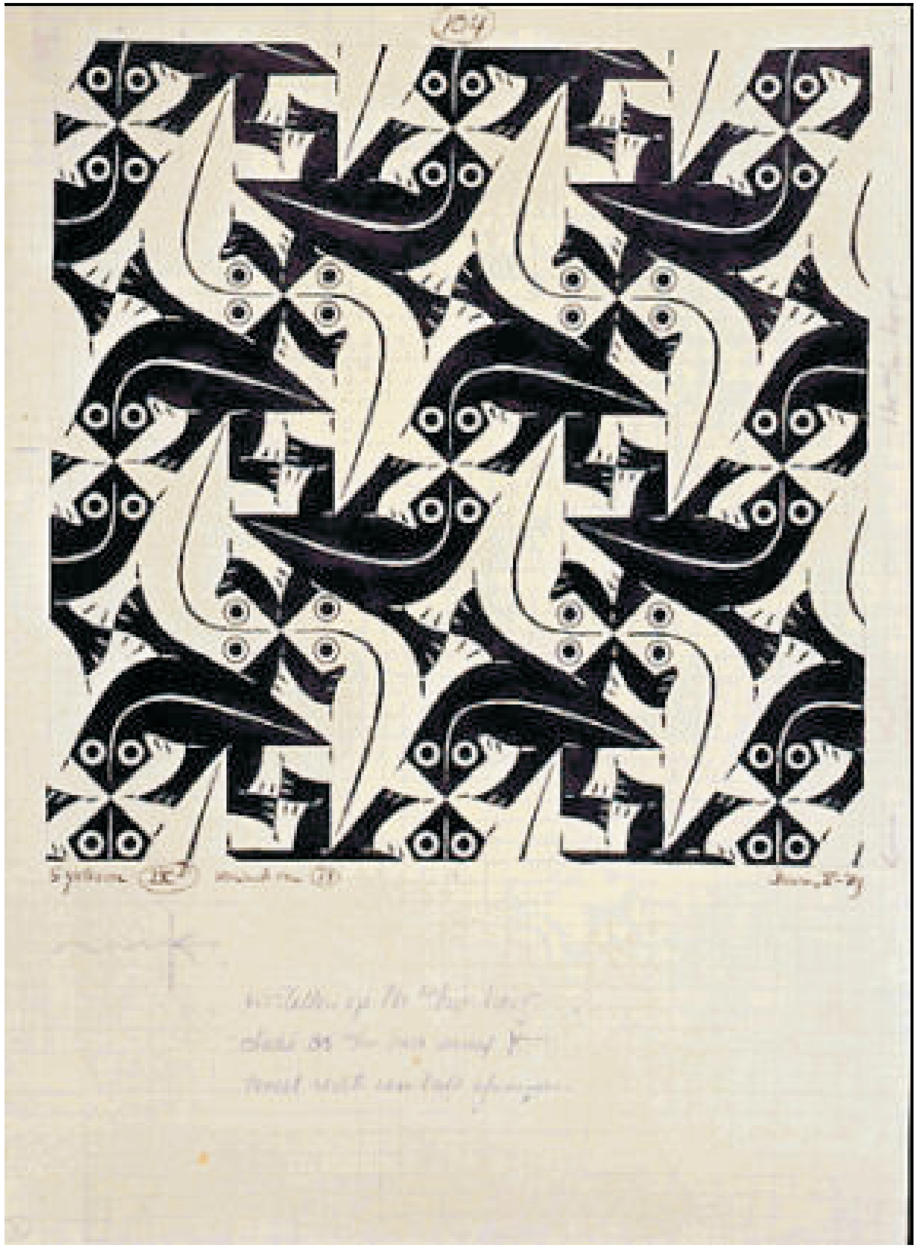
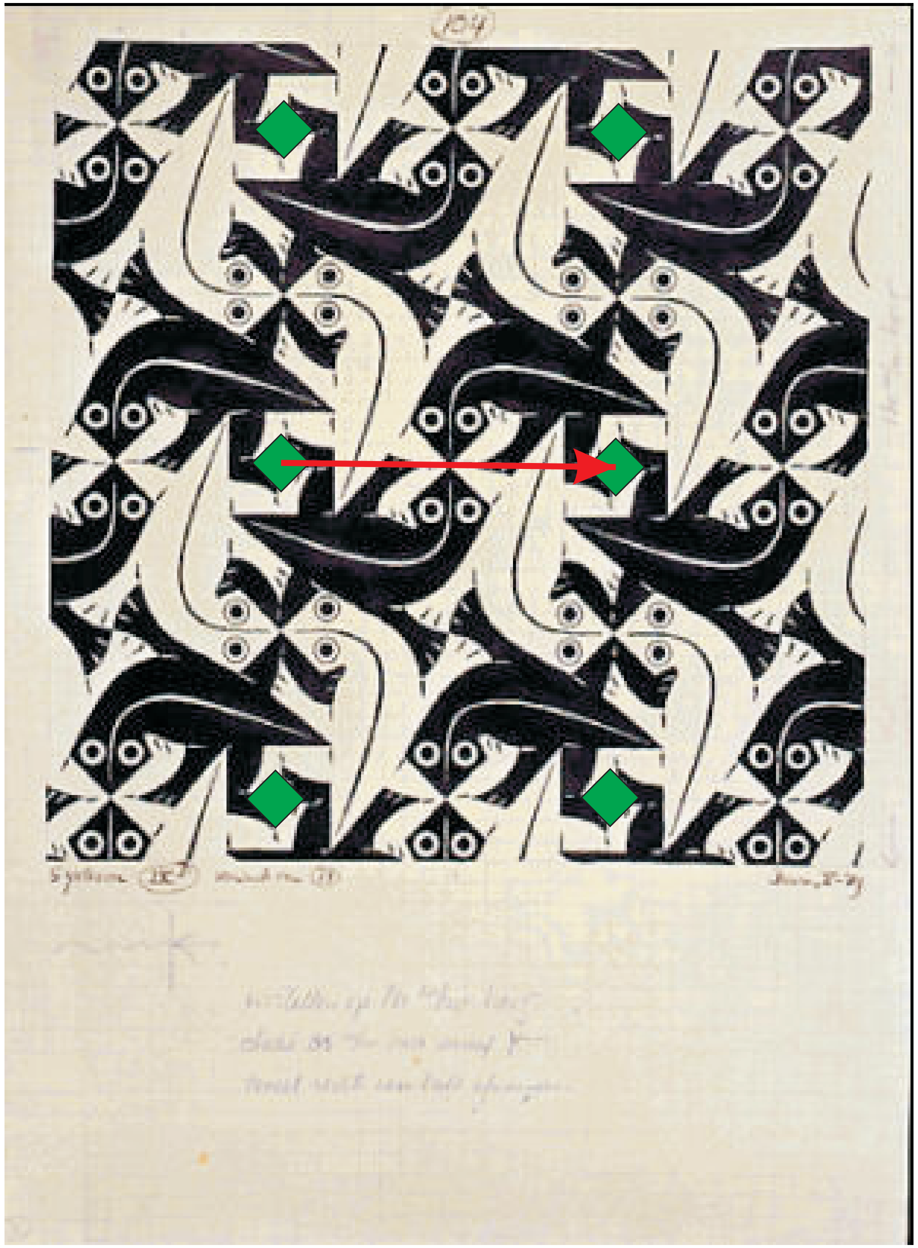
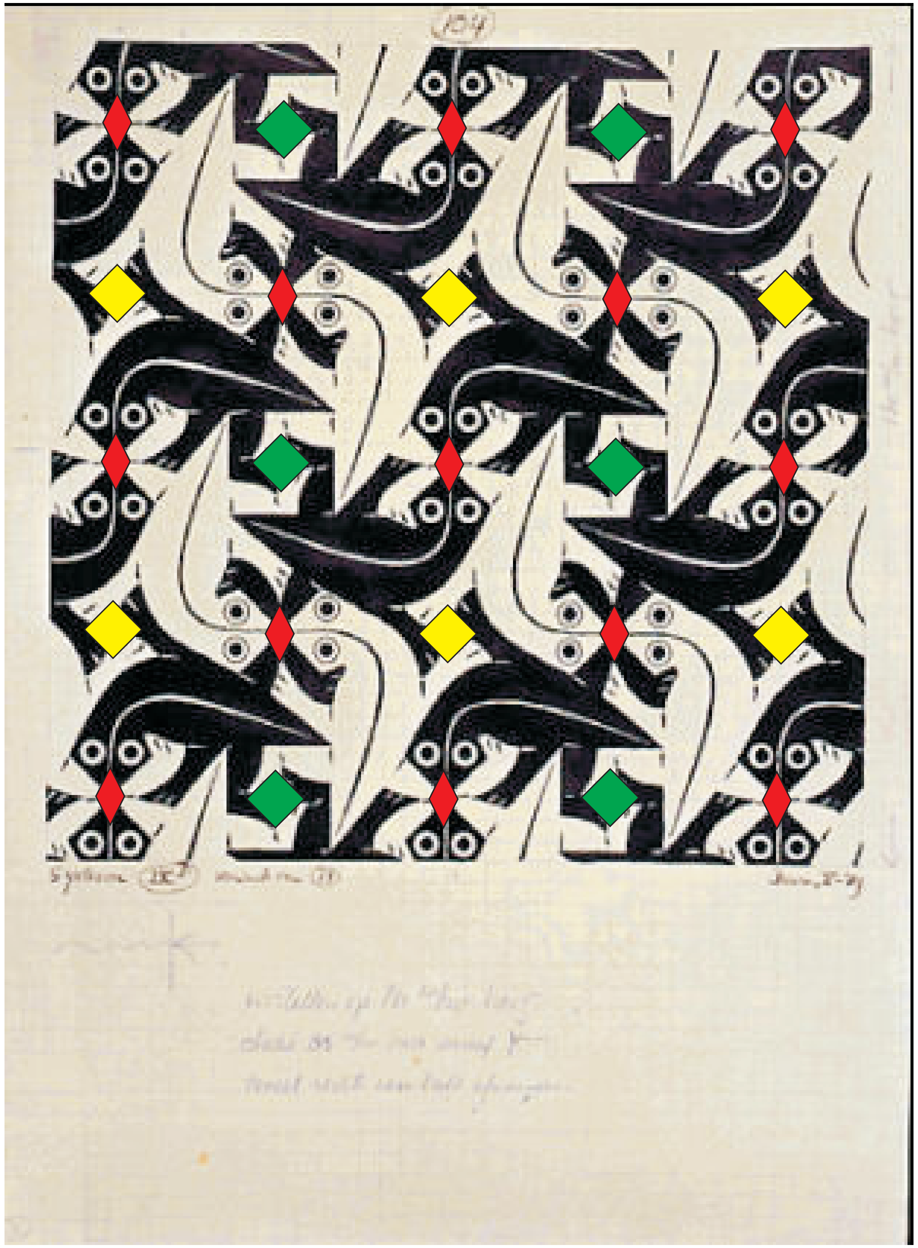


Another example of W_4, ignoring the coloring.

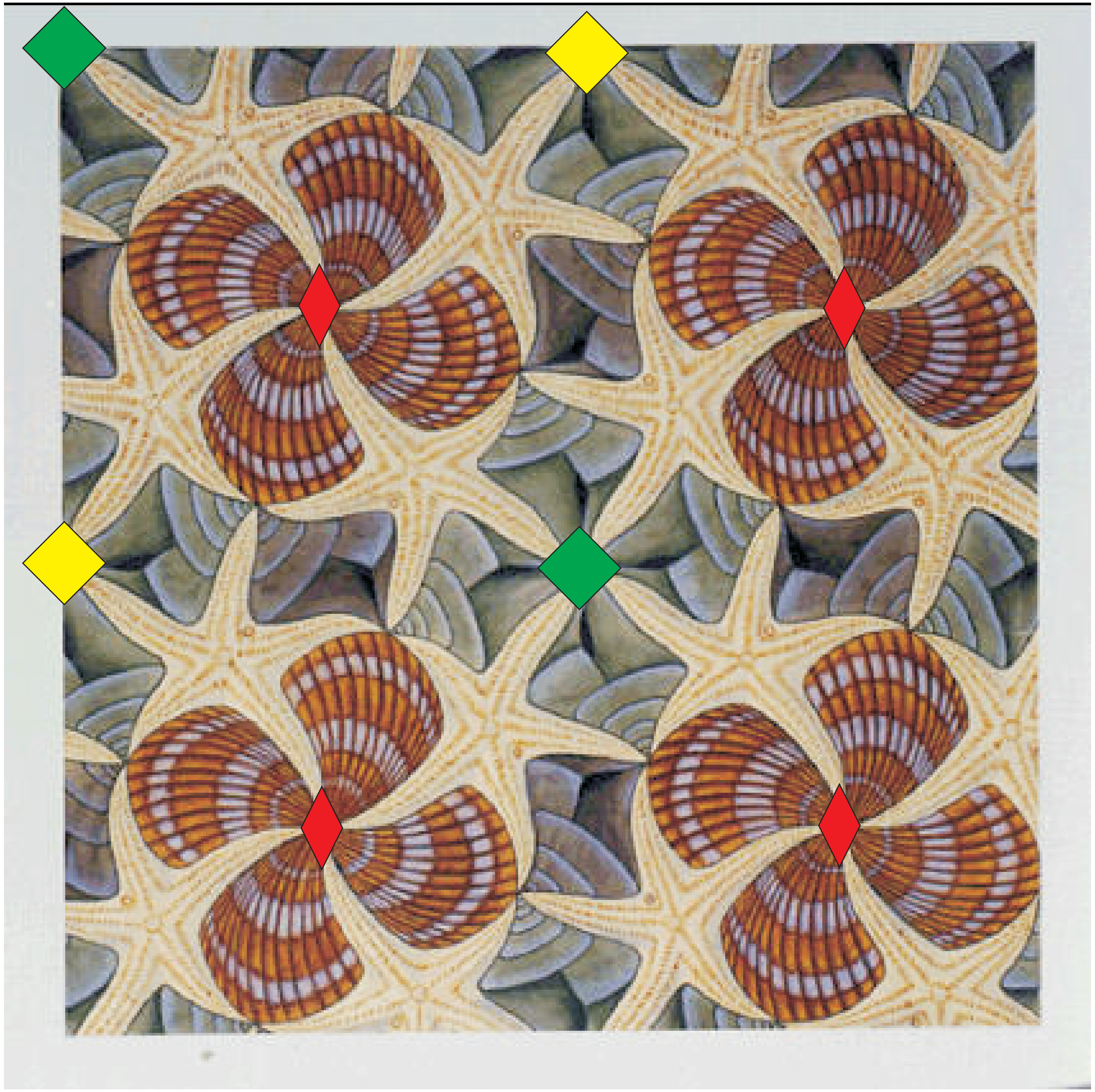


Another example of W_4, ignoring the coloring.

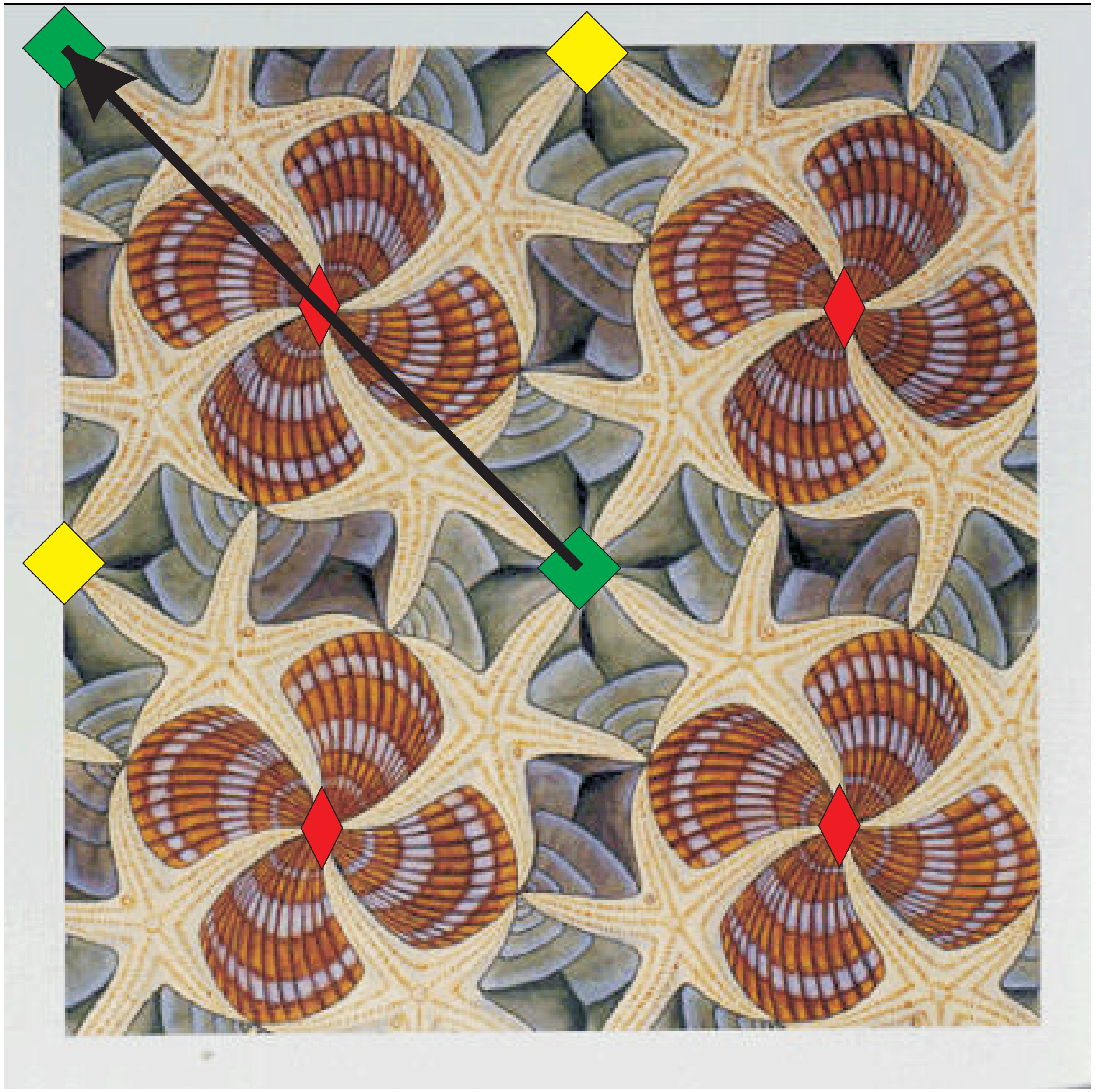


Another example of W_4, ignoring the coloring.

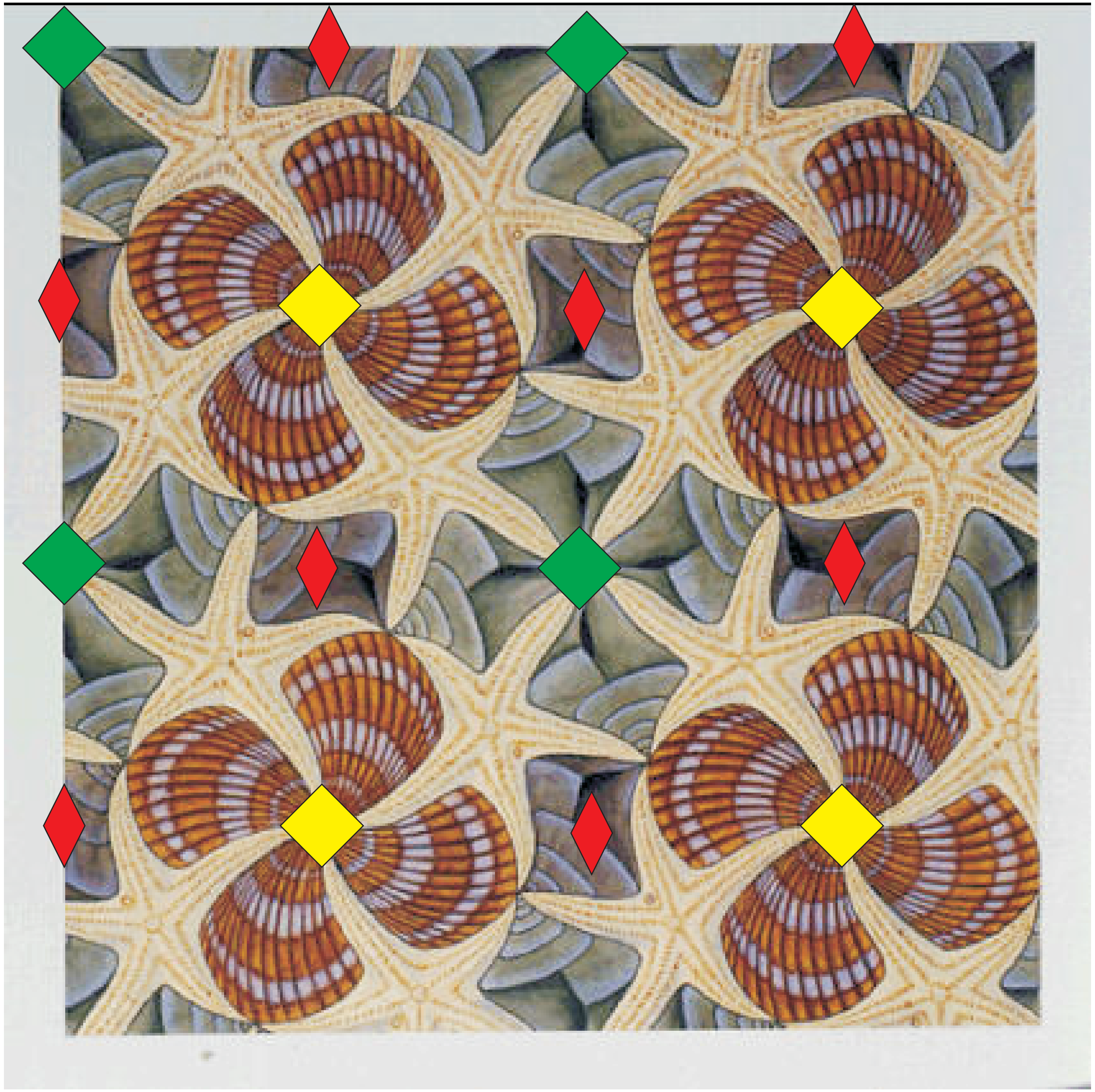
Here's an interesting example of W_4.



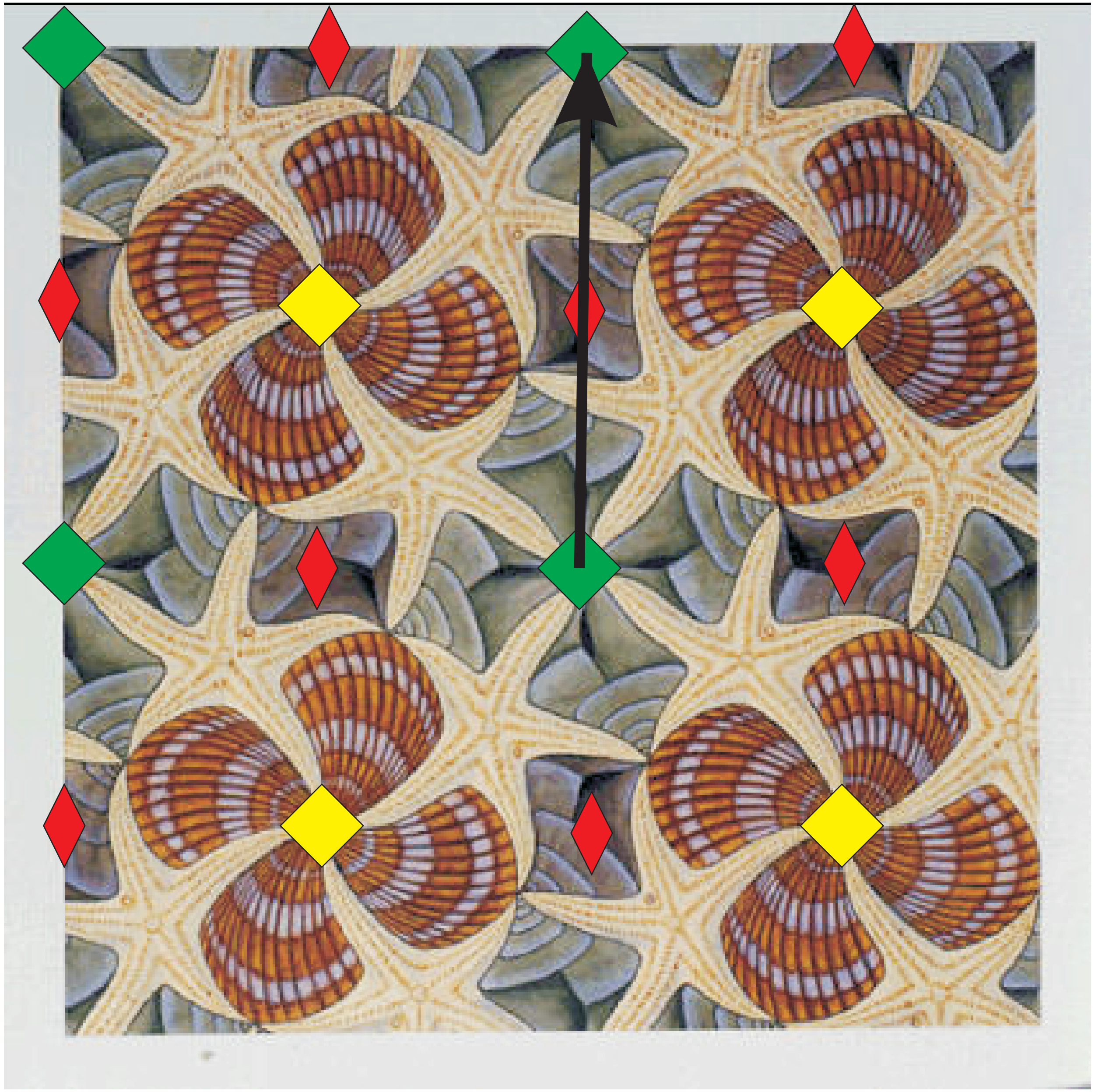
These are all the rotation centers of the picture.



And this is a shortest translation.



However, if only the shells and the stars in front are considered then we have more rotation centers. These rotations are not all valid for the grey background.

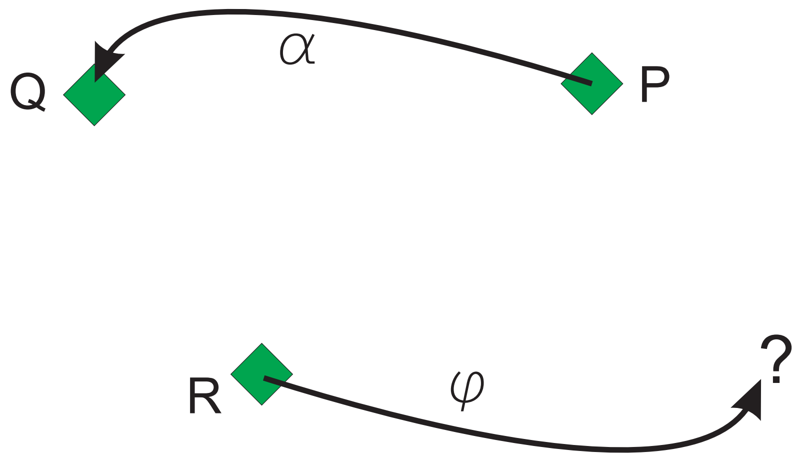


A translation for the front objects. The shells and stars also have the symmetry group W_4 but the lattice is tilted 45° and lengths are divided by $\sqrt{2}$.

Let us suppose next that group G contains some odd isometry α.

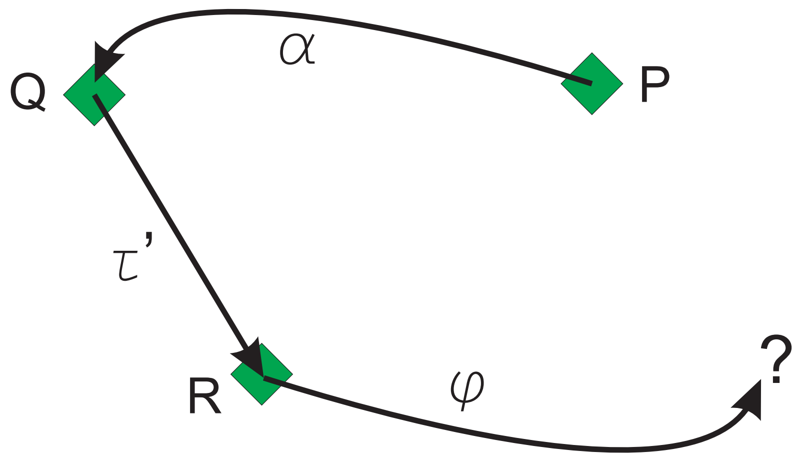
Due to conjugacy of rotations, α must map 4-fold rotation centers to 4-fold rotation centers. But there are two types of 4-fold rotation centers: lattice points and the centers of the lattice squares.

If some odd α maps some lattice point P to a lattice point Q then all odd isometries in G must map all lattice points to lattice points, and hence be symmetries of S_4.



If some odd α maps some lattice point P to a lattice point Q then all odd isometries in G must map all lattice points to lattice points, and hence be symmetries of S_4.

To see this, consider an arbitrary odd $\varphi \in G$ and an arbitrary lattice point R.



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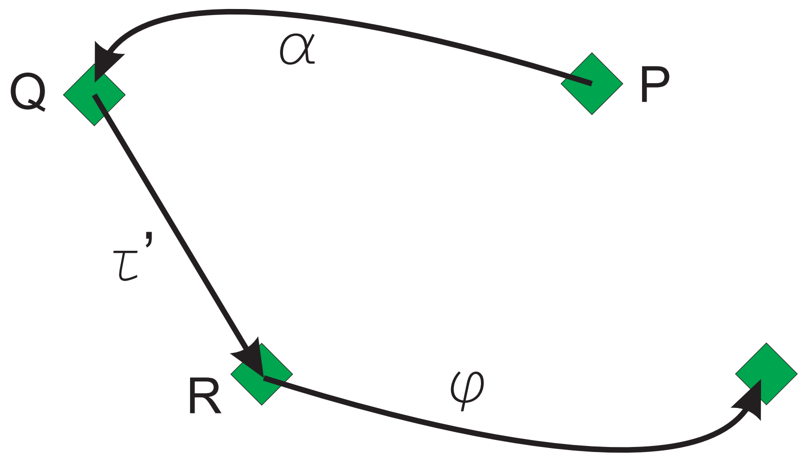
To see this, consider an arbitrary odd $\varphi \in G$ and an arbitrary lattice point R.

Let $\tau' \in G$ be the translation that takes Q to R.



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To see this, consider an arbitrary odd $\varphi \in G$ and an arbitrary lattice point R.

Let $\tau' \in G$ be the translation that takes Q to R.

Then $\varphi\tau'\alpha$ is an even isometry in G that takes point P to point $\varphi(R)$. Hence $\varphi(R)$ is a lattice point.

If one odd symmetry α of S_4 is in G then all odd symmetries of S_4 are in G because they are of the form $\alpha\beta$ where β is an even symmetry of S_4. (Recall that all even symmetries of S_4 are in G.)

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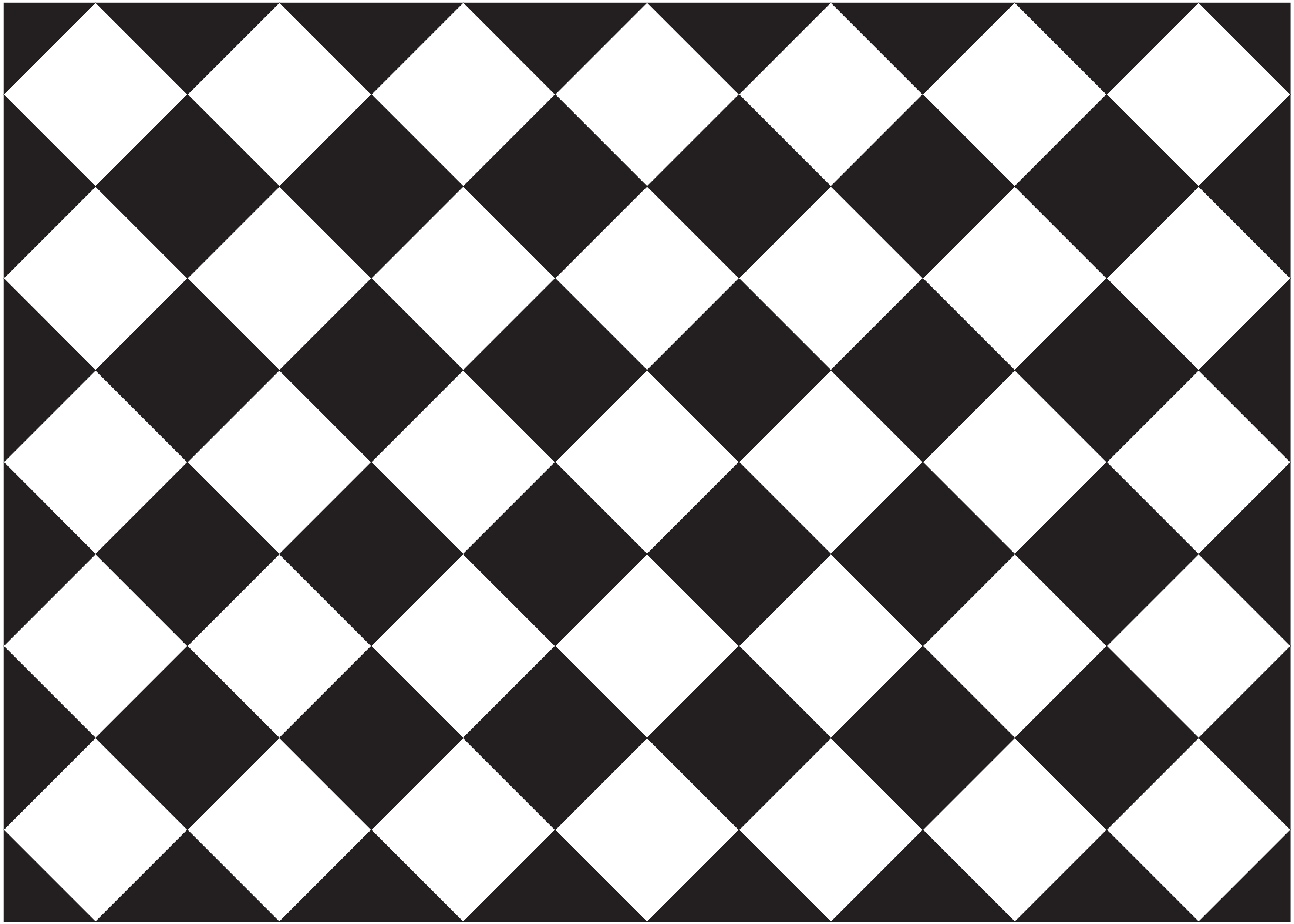
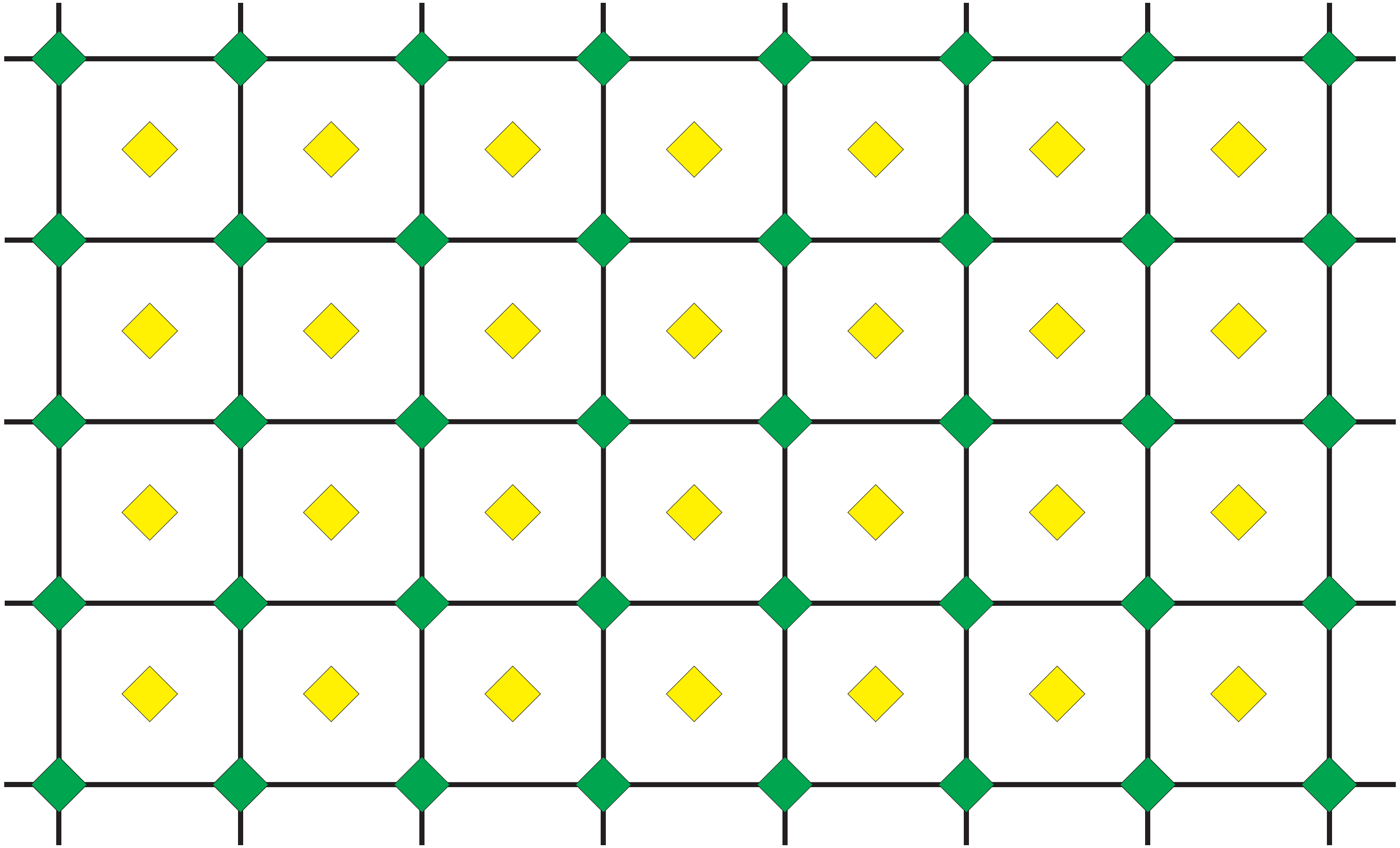
We have shown that if some odd element of G maps some lattice point to a lattice point then G is the symmetry group of the lattice S_4. We denote

$$W_4^1 = \langle \tau, \rho, \alpha \rangle.$$

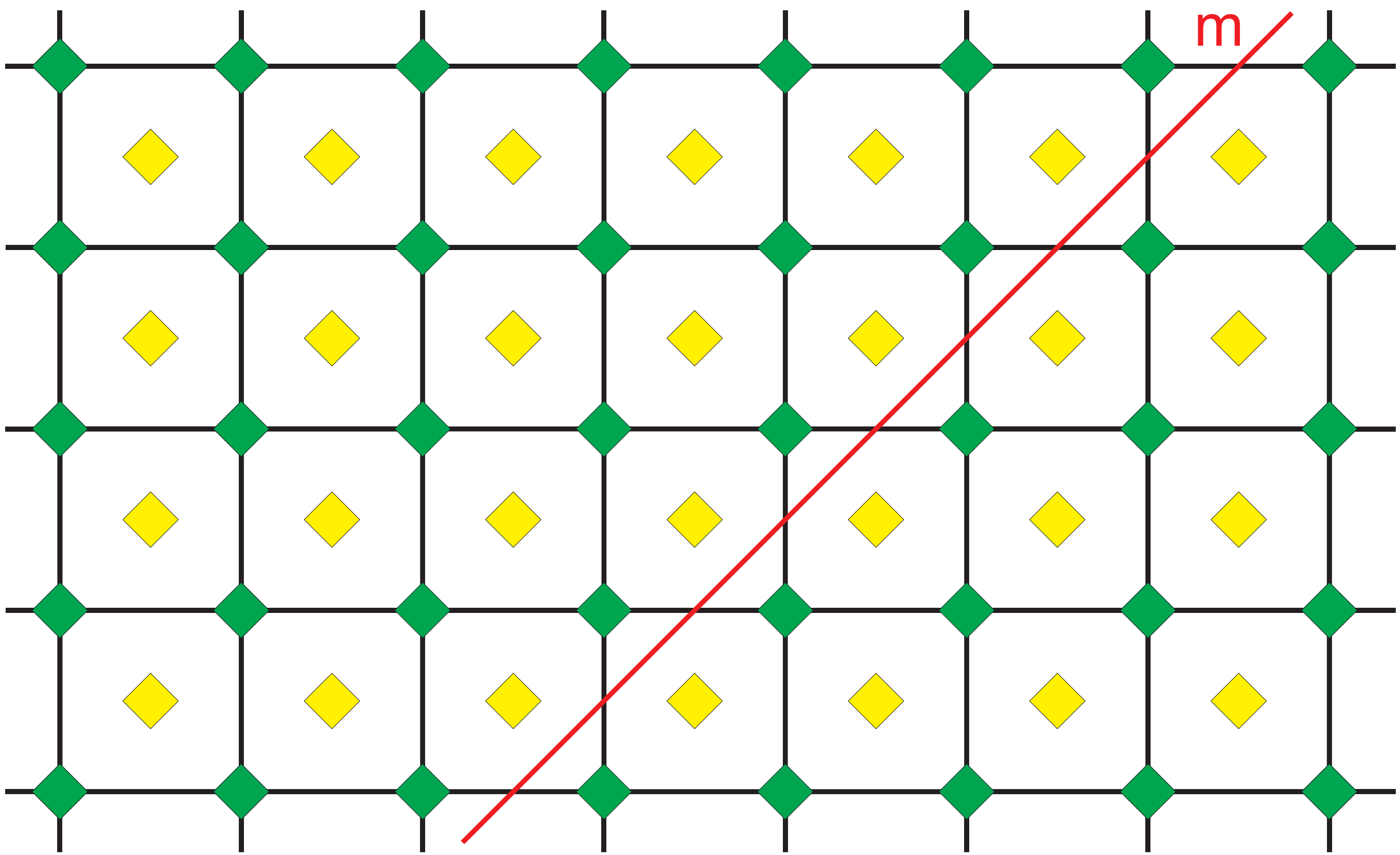
Notice that the group contains symmetry lines through all rotation centers.



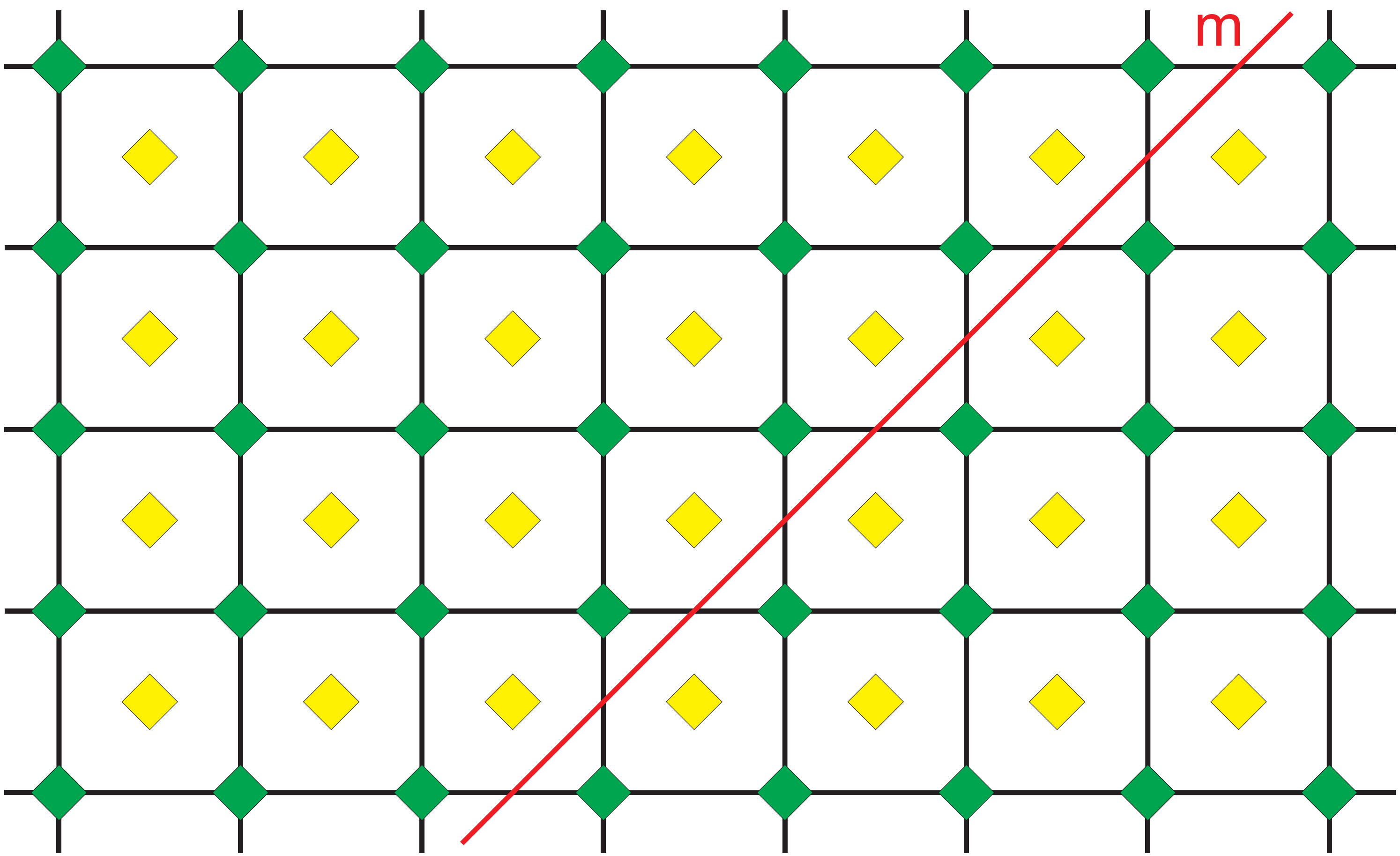
For example, the symmetries of the infinite checker board form W_4^1.



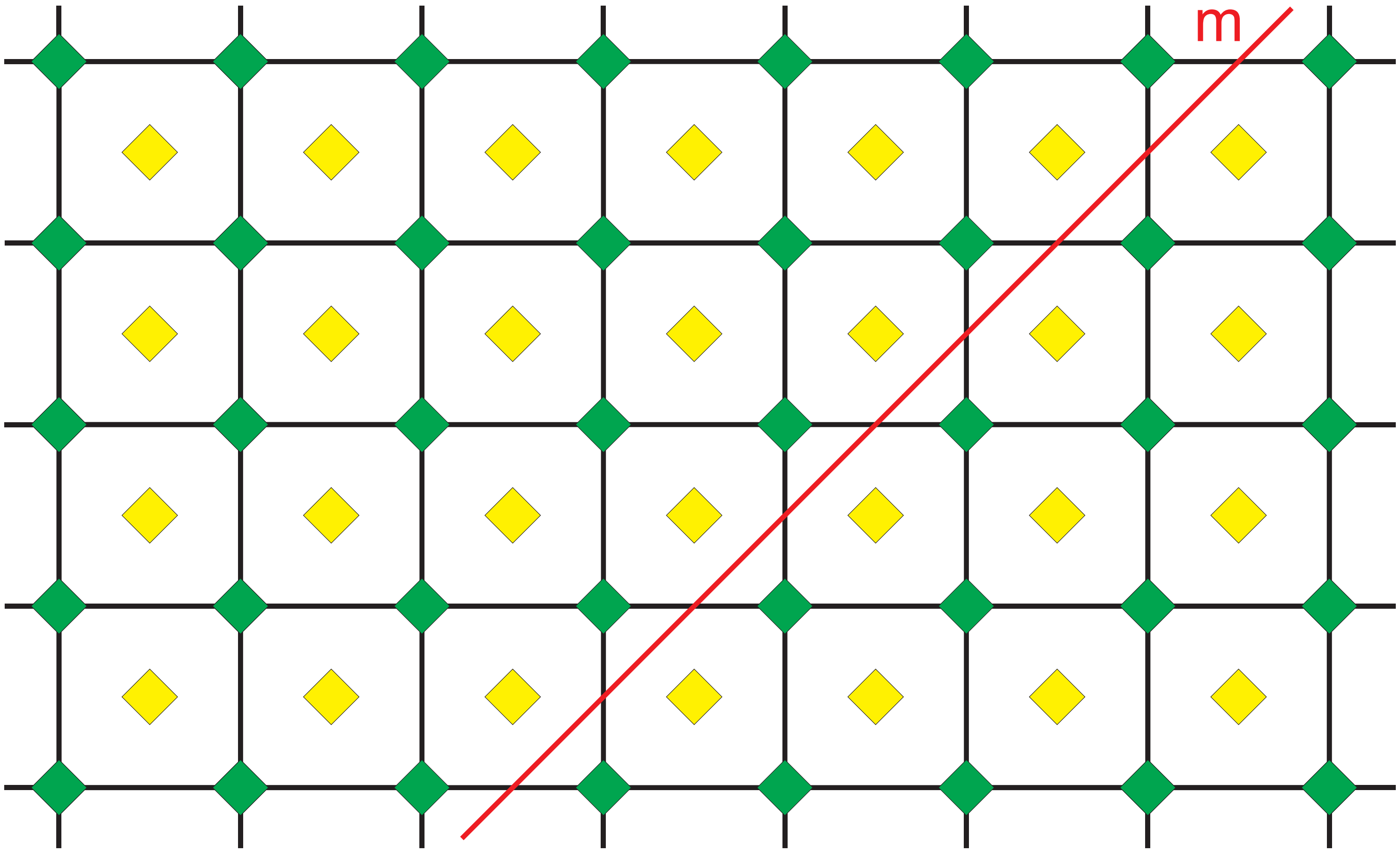
The second option for the odd elements in G is that none of them maps any lattice point into a lattice point. Hence they take all green lattice points into the yellow centers of the lattice squares. Let $\alpha \in G$ be one such isometry.



Let m be some diagonal line between the two types of rotation centers. Reflection σ_m has the property that it exchanges the green and yellow rotation centers.

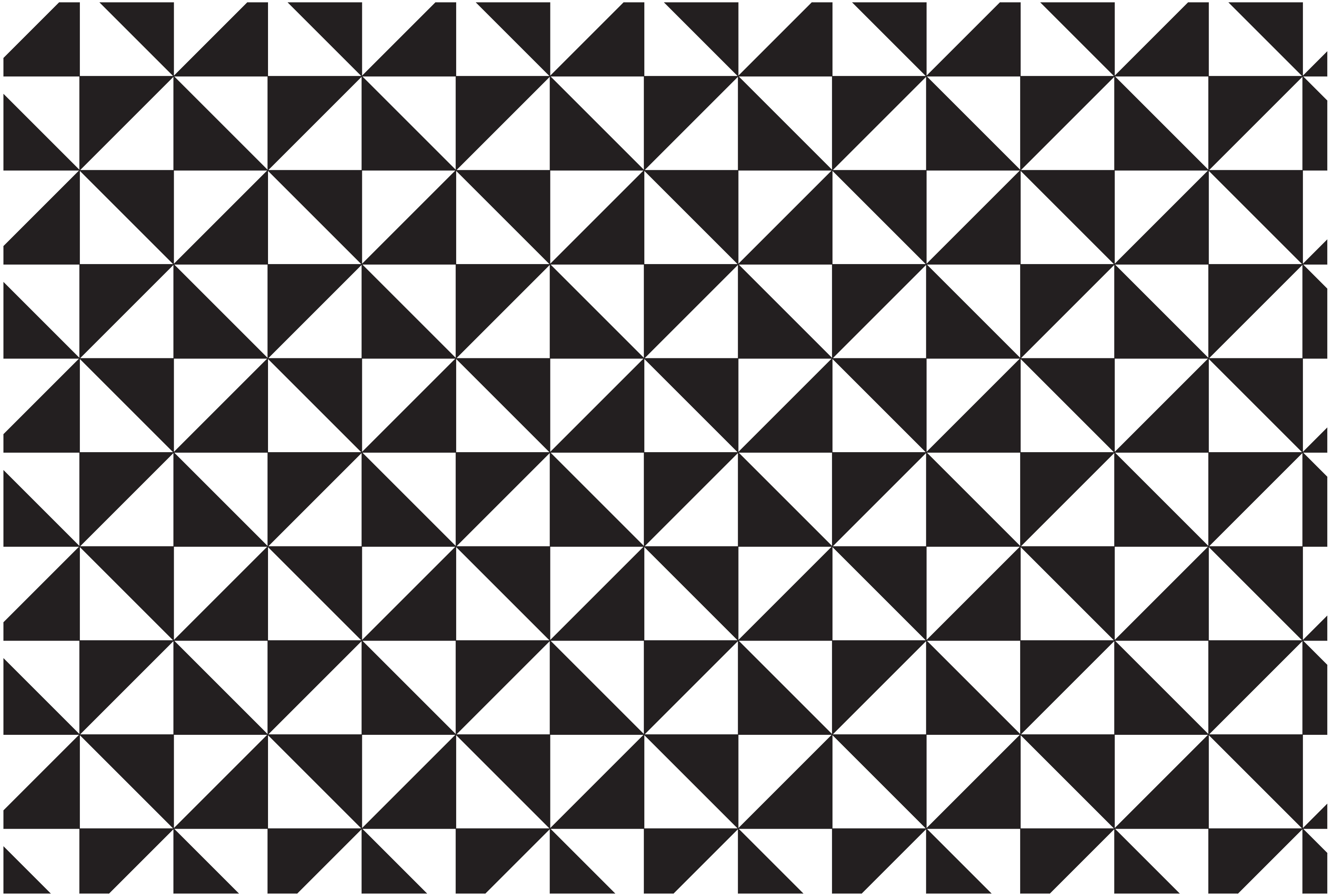


Then $\sigma_m \alpha$ is an even isometry that takes all lattice points into lattice points, so it is in group G. Therefore reflection σ_m is in G as well, and all odd isometries are now uniquely determined as the compositions of σ_m and the even elements of G.

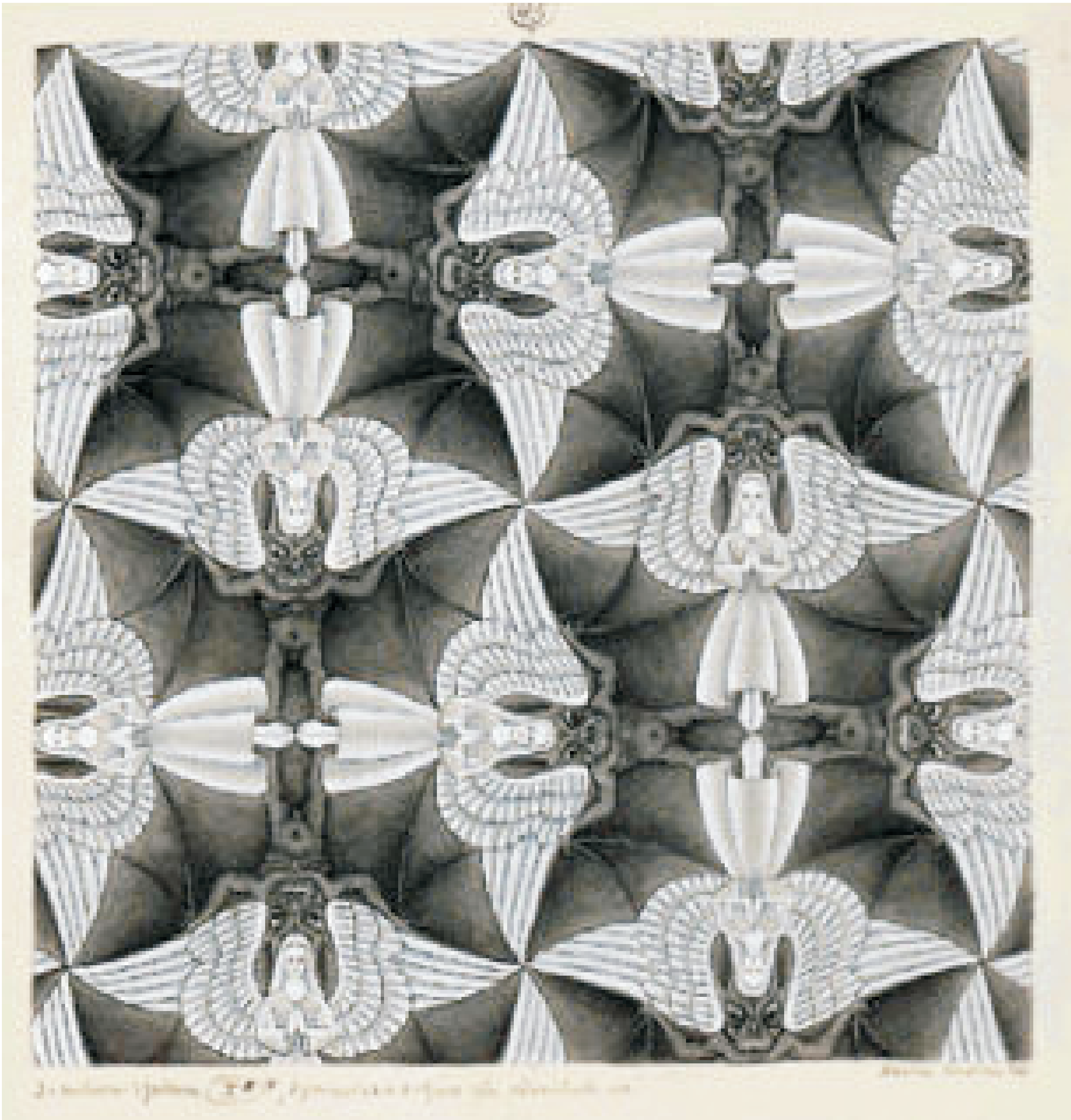


We obtain our next wallpaper group

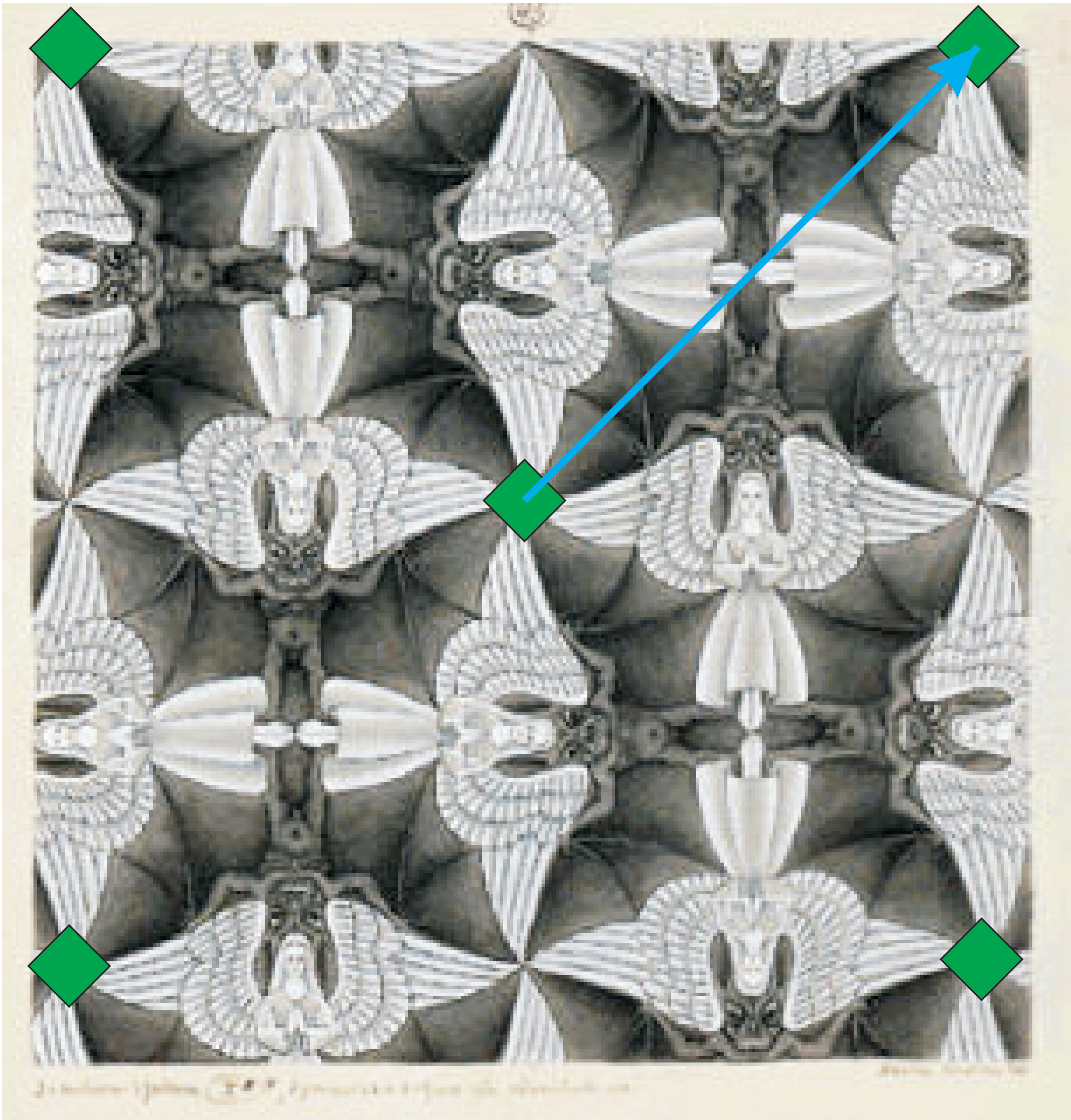
$$W_4^2 = \langle \tau, \rho, \sigma_m \rangle$$



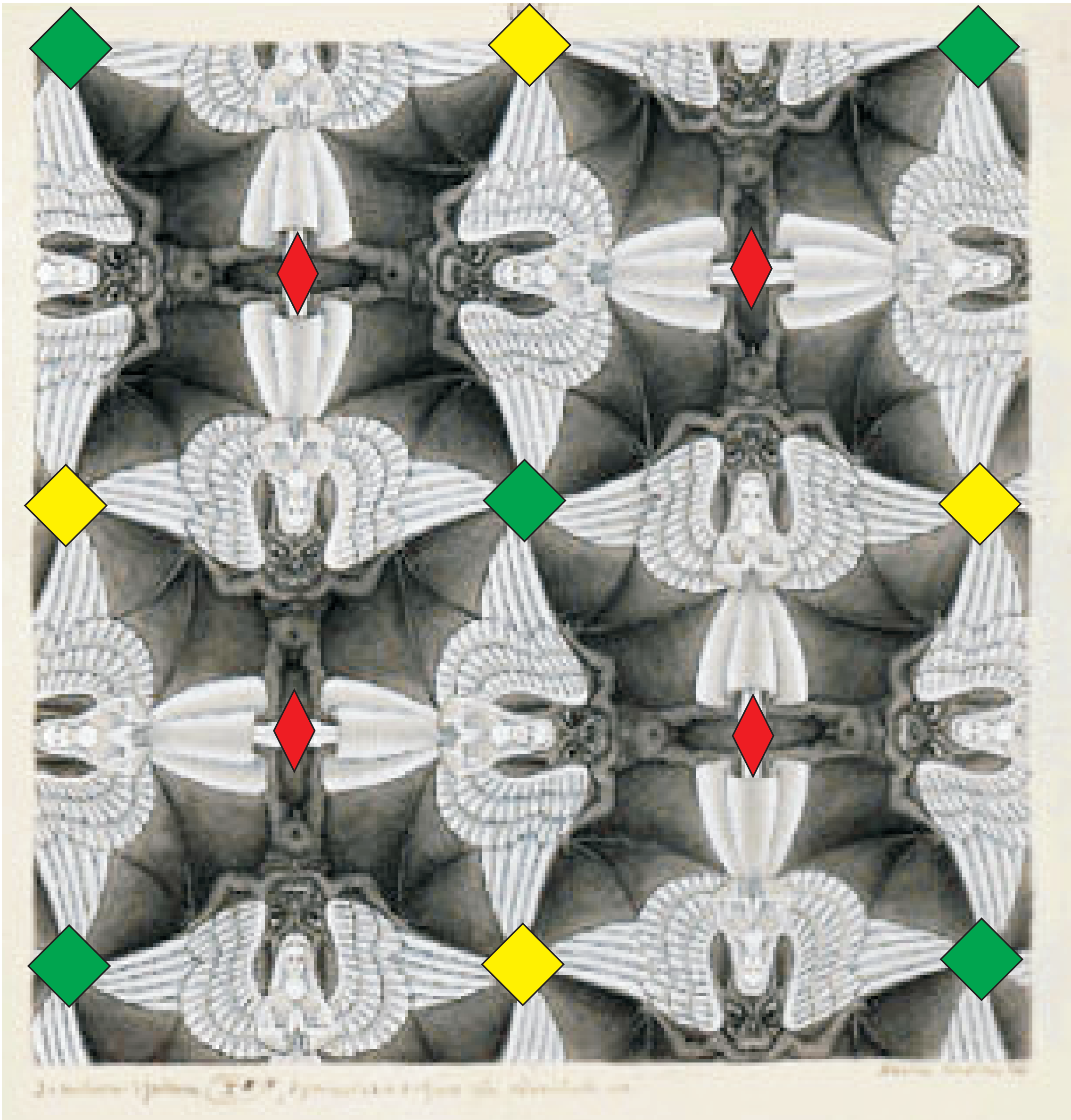
An example of a pattern with symmetry W_4^2. There are no lines of symmetry through the centers of 4-fold rotations.



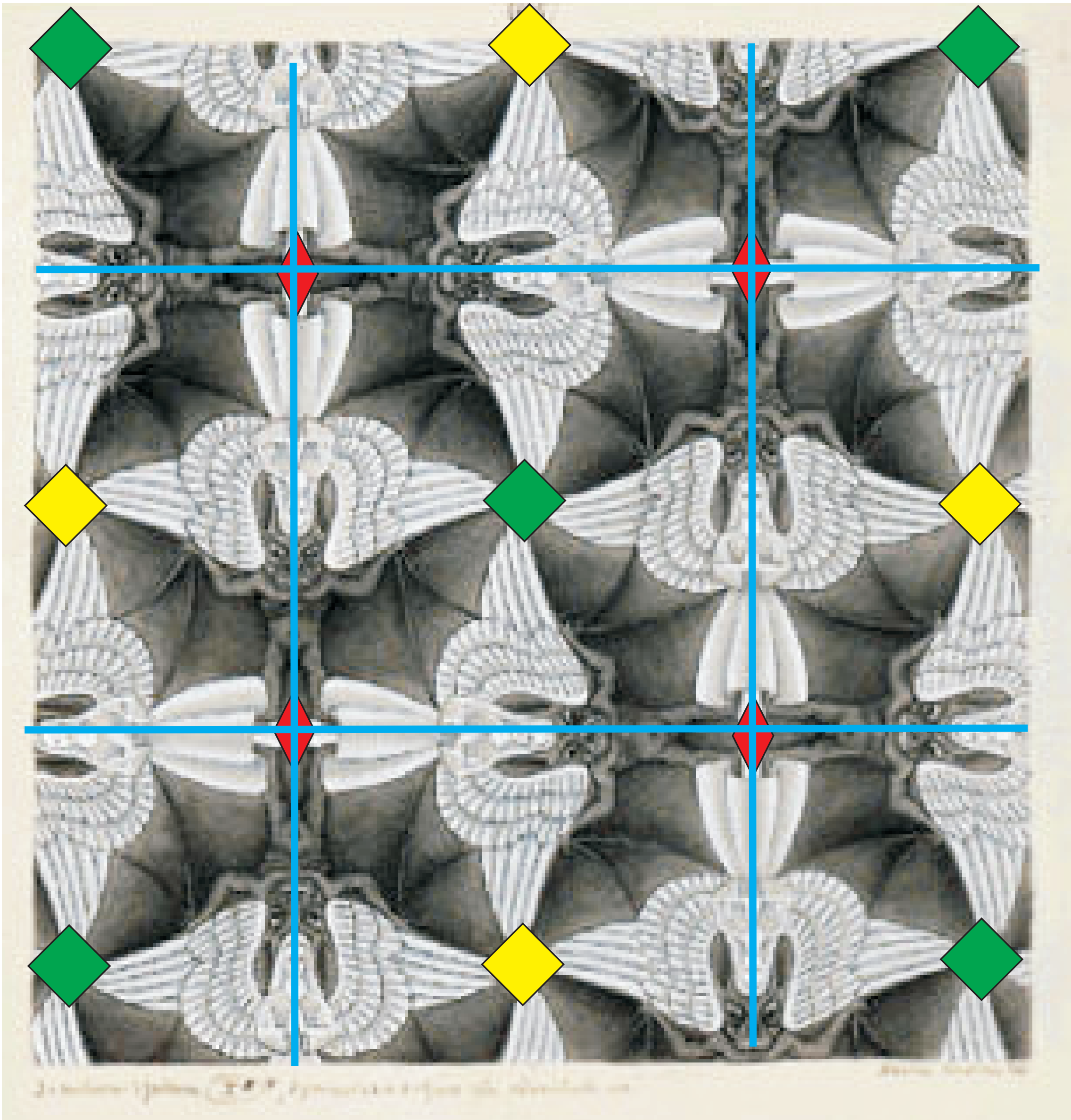
A drawing by Escher, with symmetry group is W_4^2.



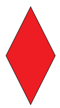
The shortest translation and the lattice points.



All rotations.

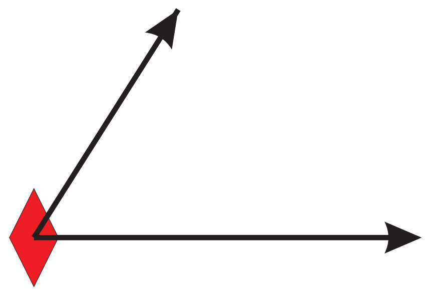


And the reflection lines. Note that the reflections swap the green and the yellow rotations. No reflection line passes through a center of a 4-fold rotation.

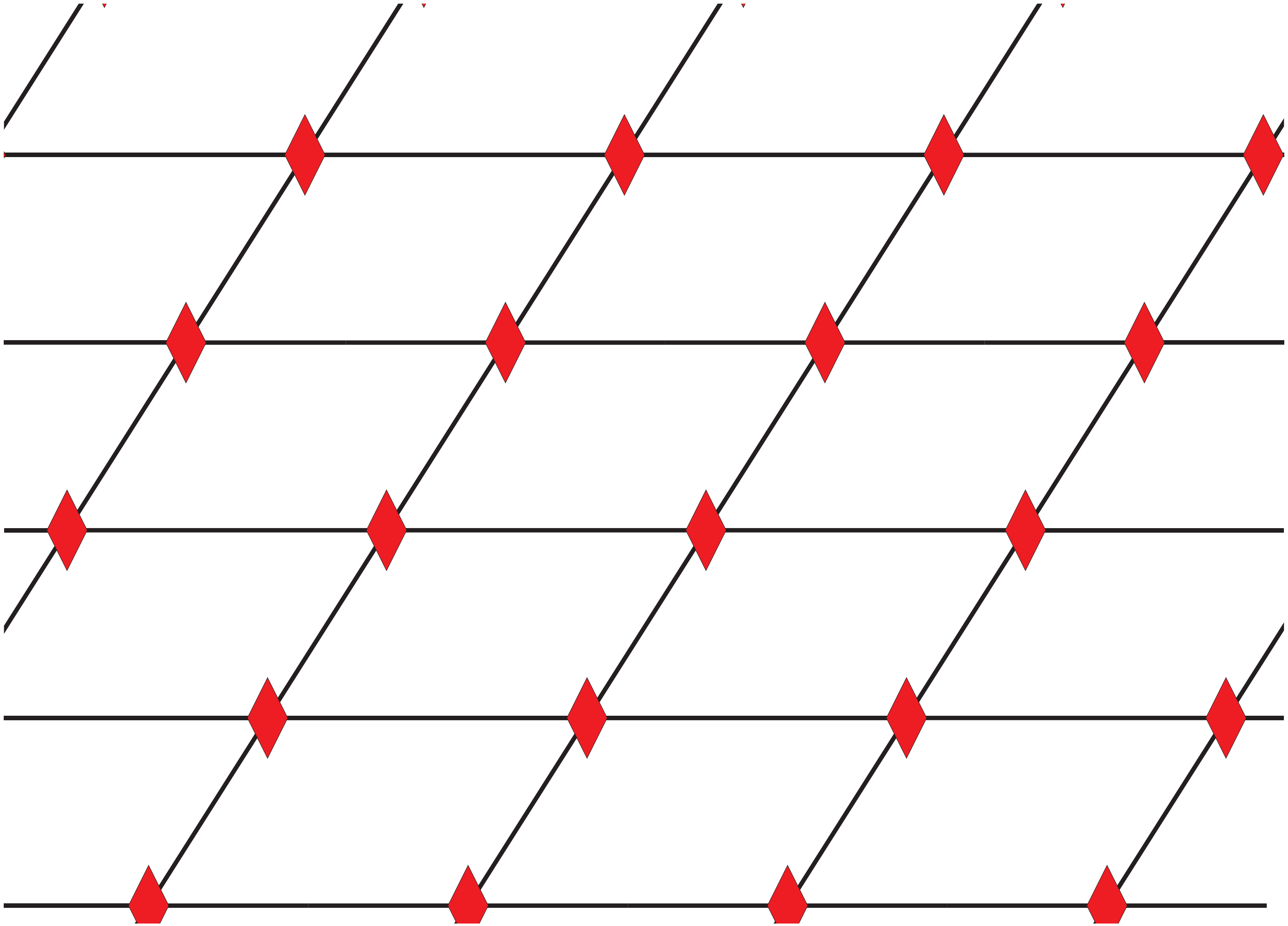


Case (4): Suppose G contains half turns but no other rotations.

Let P be the center of some half turn $\rho \in G$.

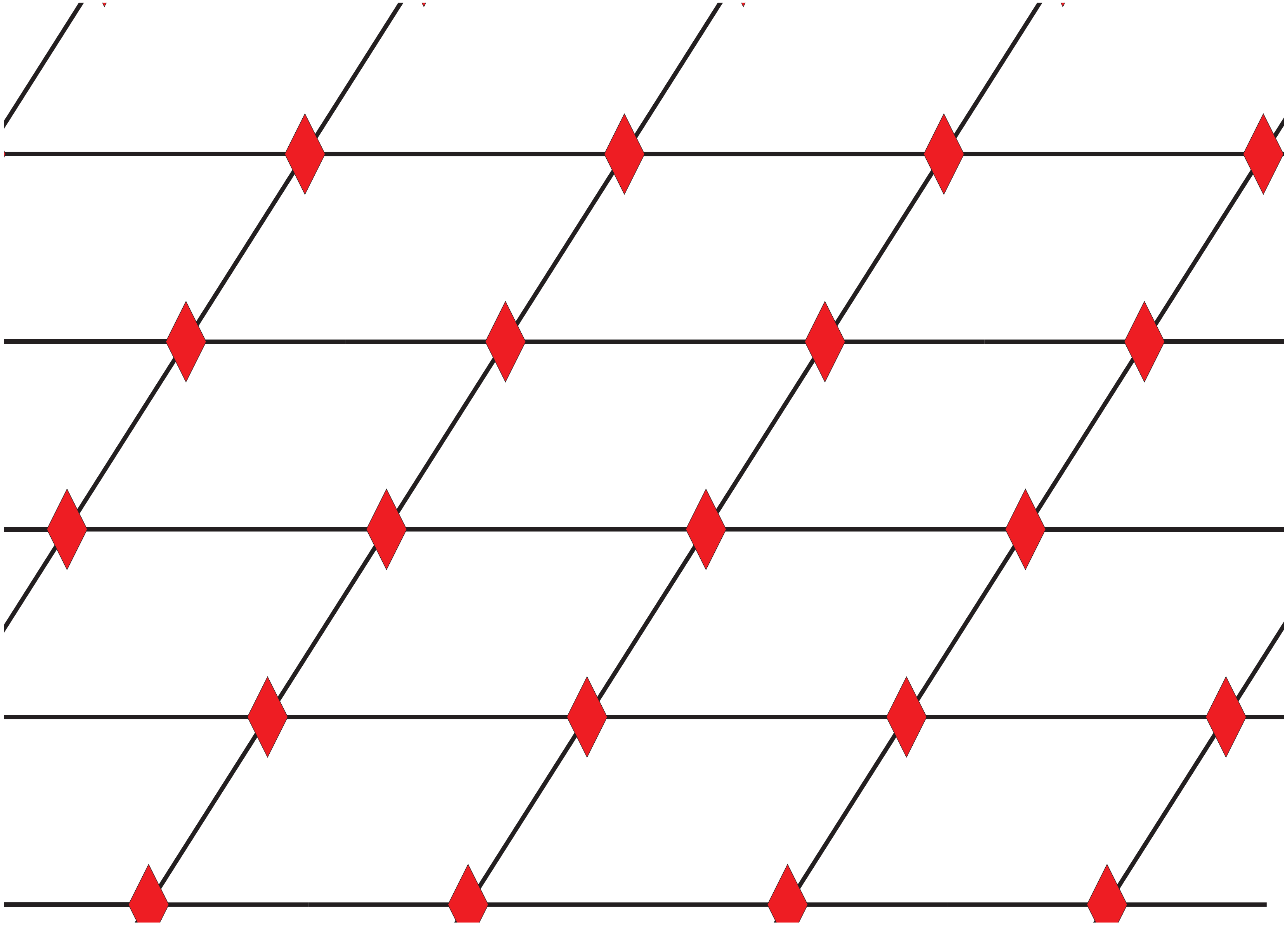


Let τ_1 and τ_2 be generators of the translations in G.



For all translation $\tau \in G$, the conjugate $\tau\rho\tau^{-1}$ of ρ is the half turn around point $\tau(P)$.

The set $\{\tau(P) \mid \tau \in G\}$ is a parallelogram lattice, which we denote by S_2.



As τ_1 and τ_2 generate all translations of G, these translations are exactly the translational symmetries of lattice S_2.

Let us show that rotations in G are exactly the half turns that are also symmetries of S_2.

All rotations $\rho' \in G$ are half turns, so $\rho'\rho = \tau'$ is a translation where ρ is our initial half turn around point P.

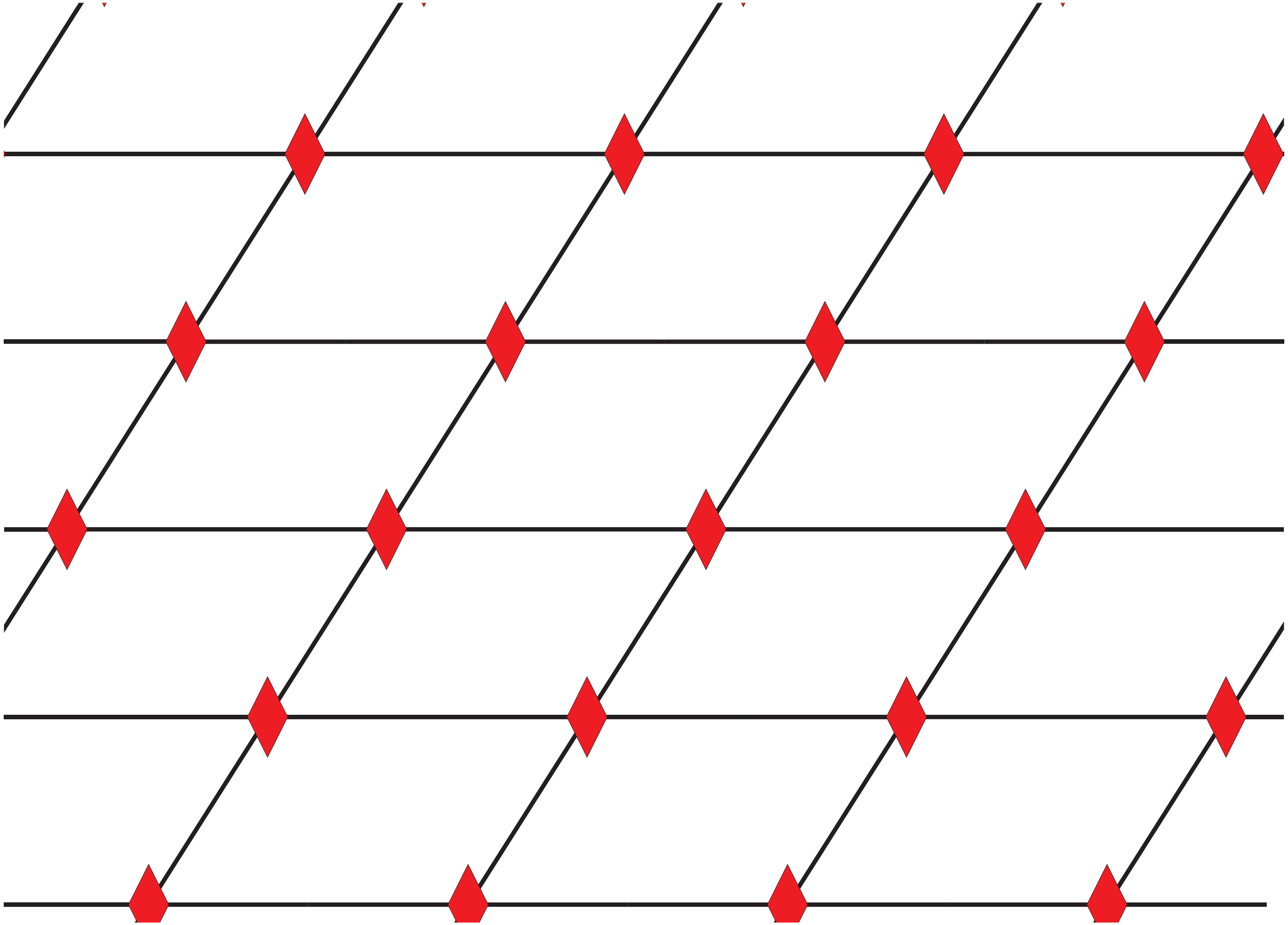
Then $\rho' = \tau'\rho$ takes lattice points to lattice points since both ρ and τ' are symmetries of the lattice.

All rotations $\rho' \in G$ are half turns, so $\rho'\rho = \tau'$ is a translation where ρ is our initial half turn around point P.

Then $\rho' = \tau'\rho$ takes lattice points to lattice points since both ρ and τ' are symmetries of the lattice.

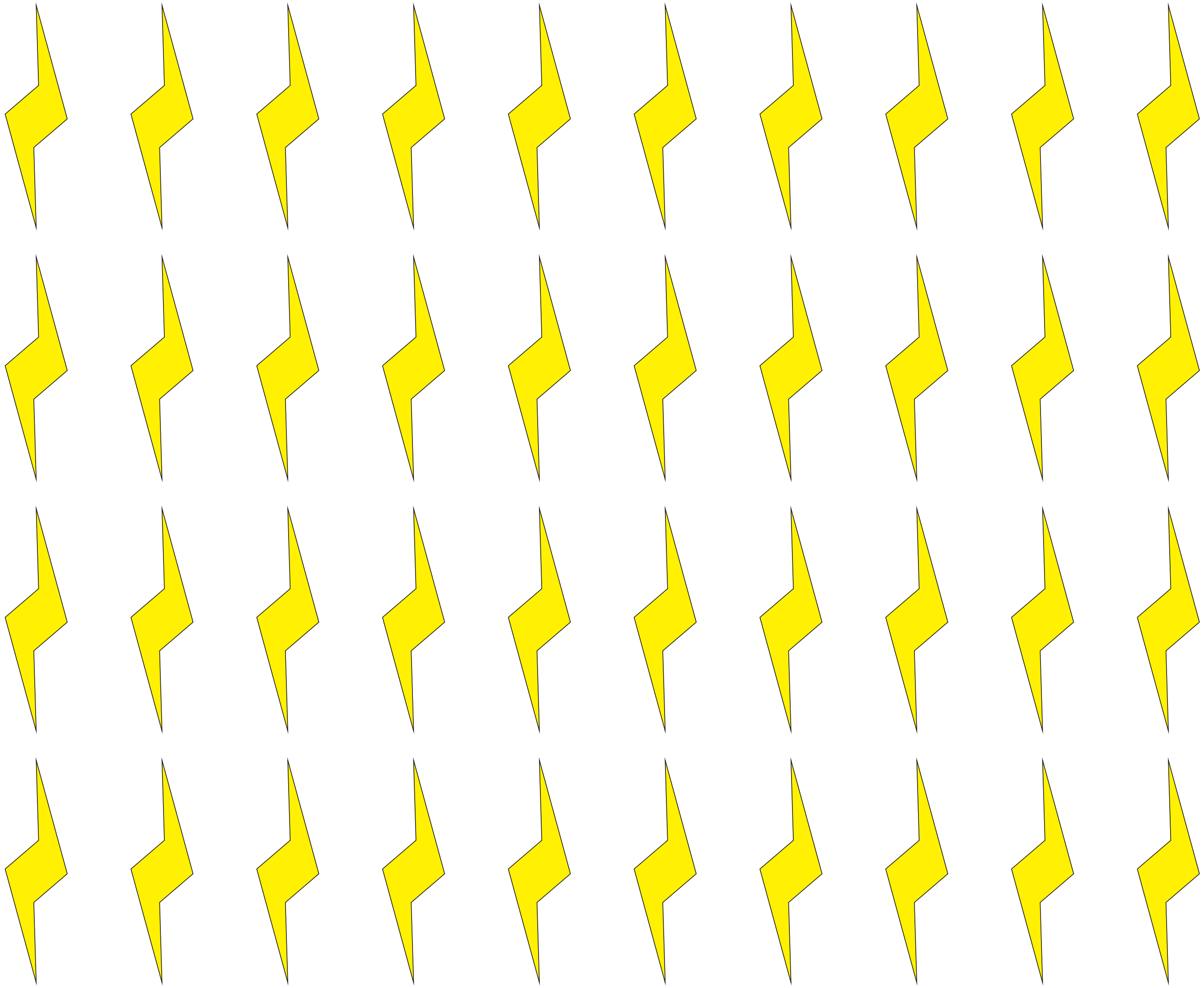
Conversely, if ρ' is a half turn and a symmetry of S_2 then $\rho'\rho$ is a translational symmetry of S_2. Hence it is in G, so ρ' is in G as well.

We have proved that the even isometries in G are exactly the symmetries of S_2 that are translations or half turns.



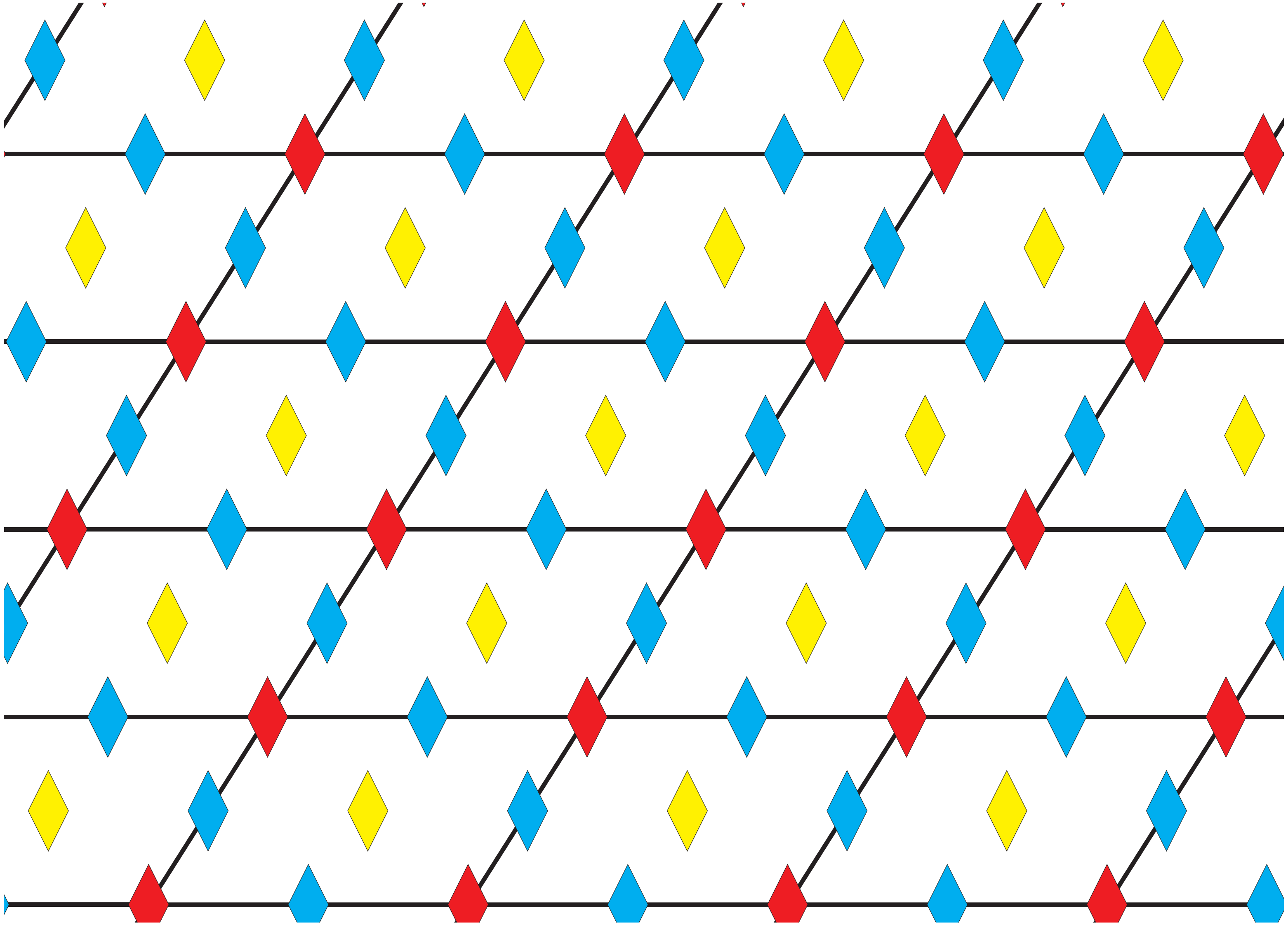
If G contains no odd isometries then G is uniquely determined: It is the wallpaper group

$$W_2 = \langle \rho, \tau_1, \tau_2 \rangle.$$

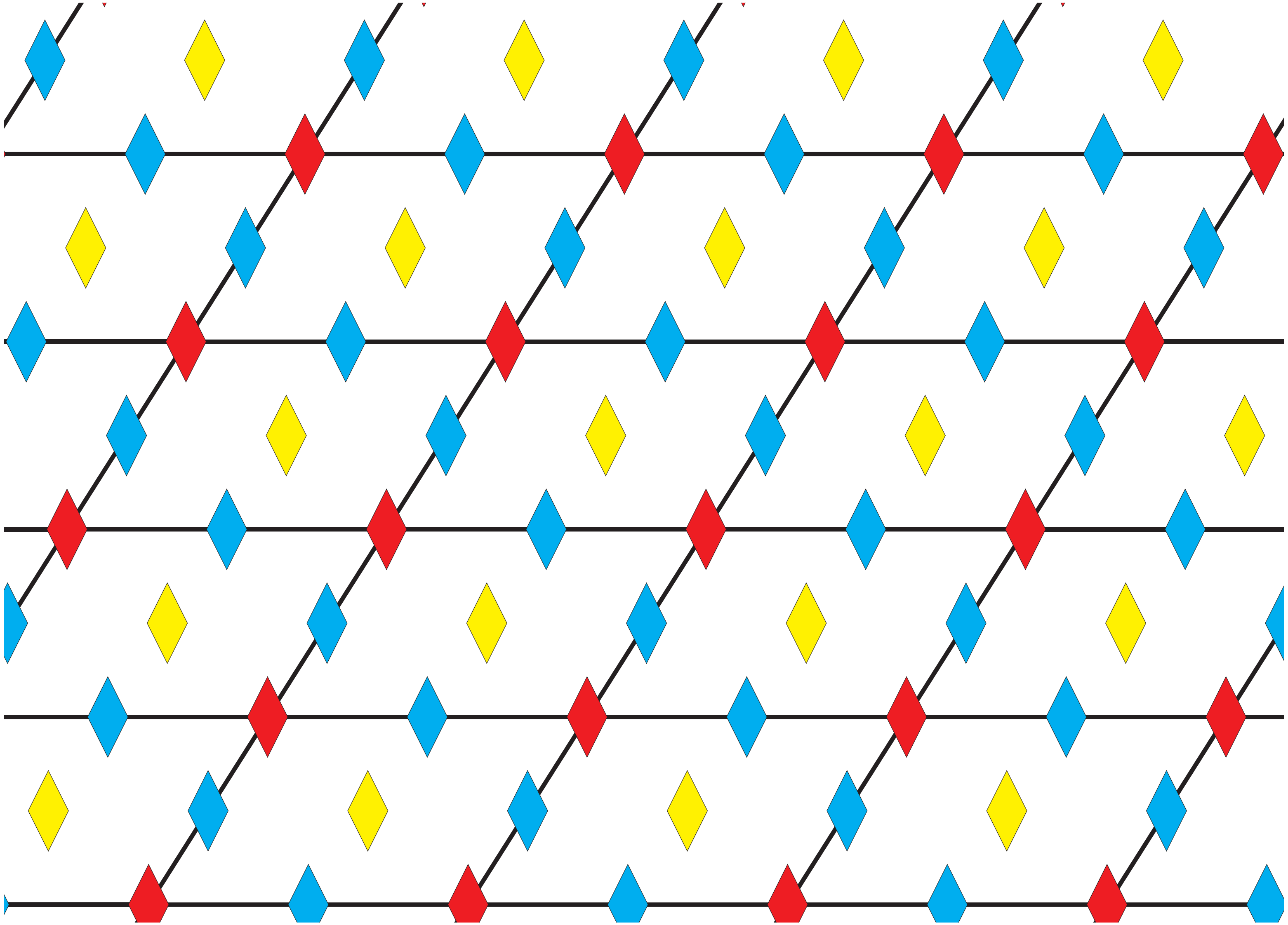


An example of a pattern whose symmetry group is

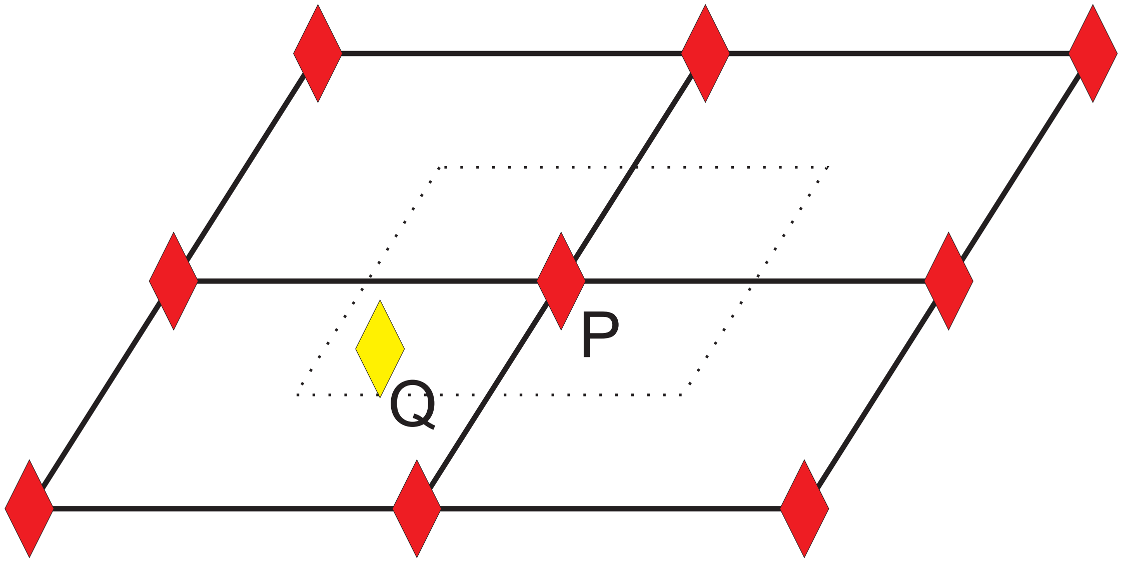
$$W_2 = \langle \rho, \tau_1, \tau_2 \rangle.$$



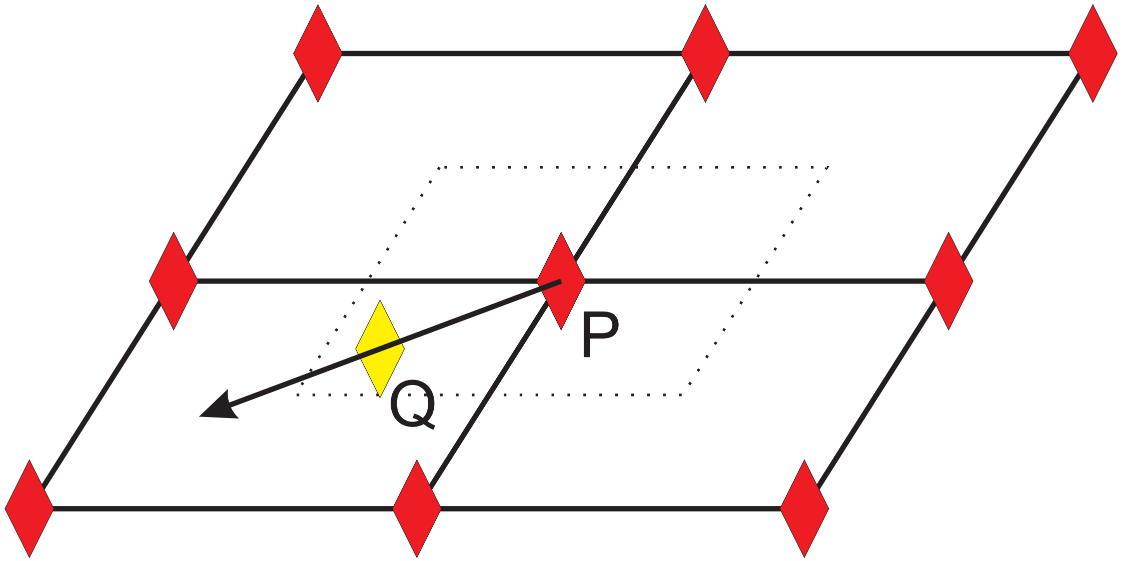
In addition to the half turns around the lattice points, the group also contains half turns around midpoints between adjacent lattice points, as well as around the centers of the parallelograms.



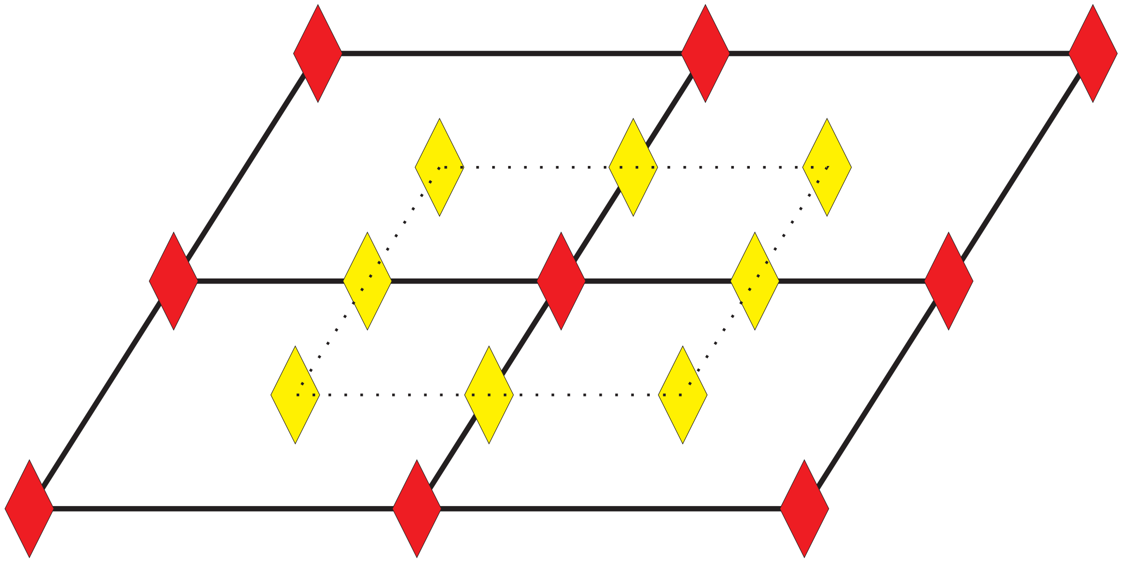
Let us prove that these are the only centers of half turns.



Let Q be an arbitrary center of a half turn, and let P be the lattice point at the center of a translated parallelogram containing Q.



The composition $\sigma_Q \sigma_P$ of the two half turns is the translation by the vector $2\overrightarrow{PQ}$. All translations of G map lattice points to lattice points, so the image of P must be one of the eight lattice points surrounding P.



The composition $\sigma_Q \sigma_P$ of the two half turns is the translation by the vector $2\overrightarrow{PQ}$. All translations of G map lattice points to lattice points, so the image of P must be one of the eight lattice points surrounding P.

This is only possible if Q is the center of a lattice parallelogram or a midpoint between two adjacent lattice points. So these are all the centers of half turns in G.

Here is an example of a painting by Escher with symmetry group W_2 (ignoring colors).

Two generating translations and a center of a half turn.

The generated lattice S_2.

All centers of half turns.

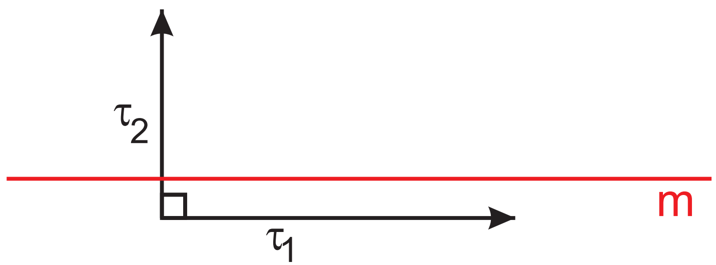
Suppose then that G also contains some odd isometry. The following lemma states that in this case the lattice S_2 is rectangular or rhombic:

Lemma: Let G be any wallpaper group that contains an odd isometry with axis m. Then there are translations $\tau_1, \tau_2 \in G$ that generate all translations of G such that one of the following holds:

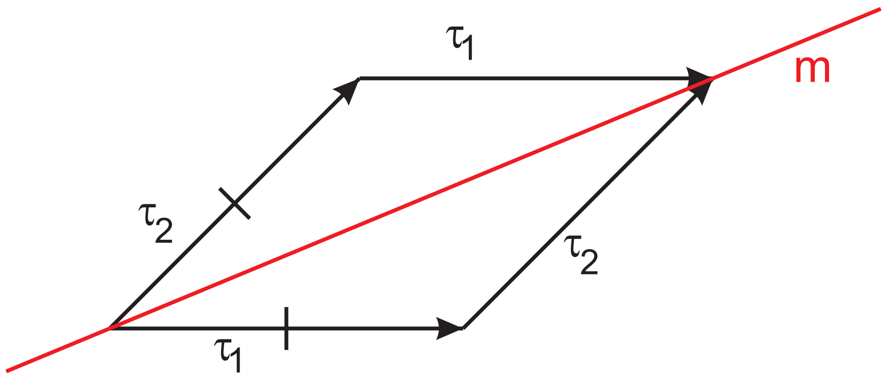
(1) $\tau_1 \parallel m$ and $\tau_2 \perp m$, or

(2) $|\tau_1| = |\tau_2|$ and $\tau_1\tau_2 \parallel m$.

Moreover, in case (2), group G contains a reflection.



(1)



(2)

Even isometries of G are already known to us (symmetries of S_2 that are translations or half turns), so fixing one odd isometry α uniquely determines all odd isometries: They are the compositions of α and the even elements of G.

Now we have the following cases:

(a) G contains some reflection σ_m. Based on the lemma we have two cases:

(a1) $\tau_1 \parallel m$ and $\tau_2 \perp m$,

(a2) $|\tau_1| = |\tau_2|$ and $\tau_1\tau_2 \parallel m$.

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(b) G does not contain a reflection but it contains a glide reflection γ with axis m. Case (1) of the lemma must hold, so

(b1) $\tau_1 \parallel m$ and $\tau_2 \perp m$.

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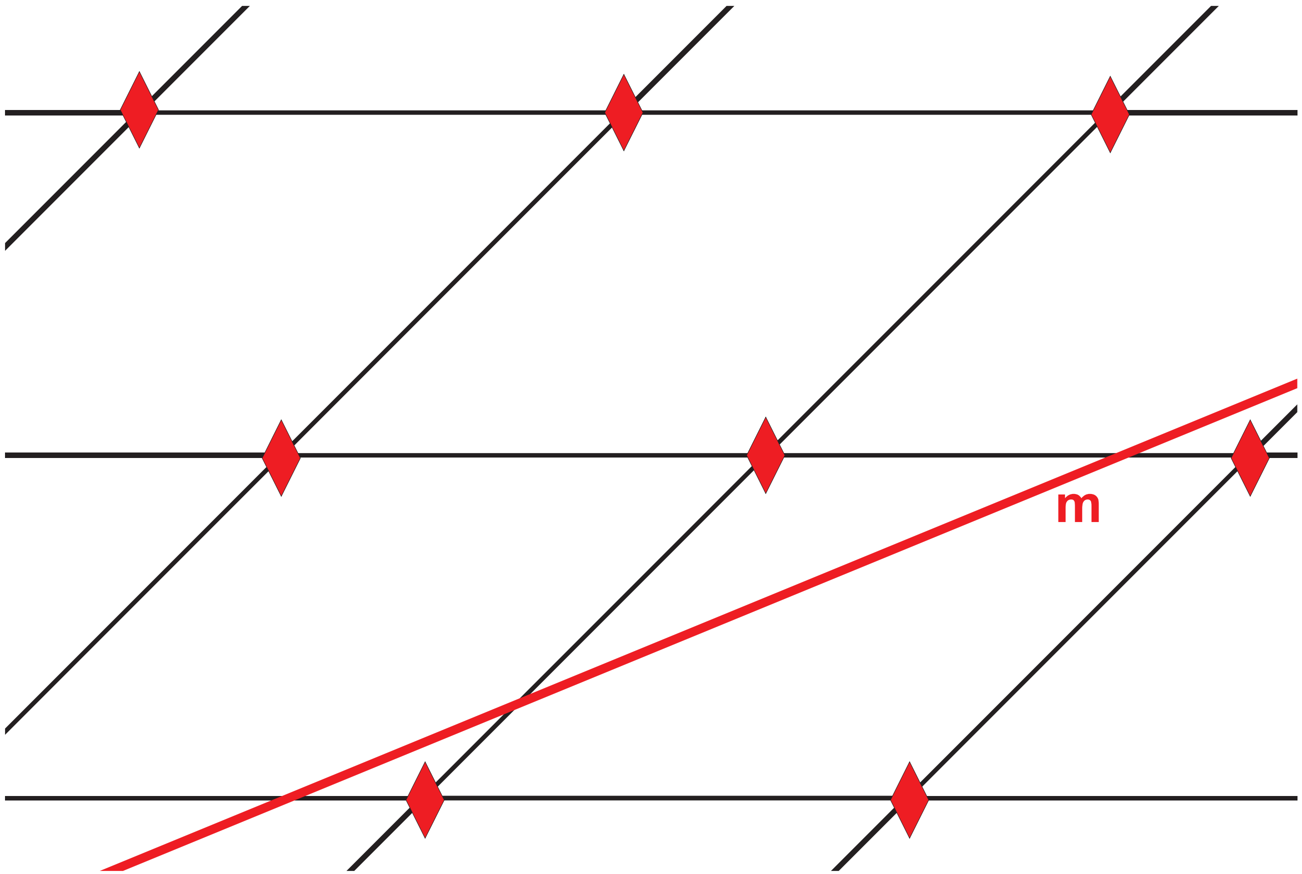
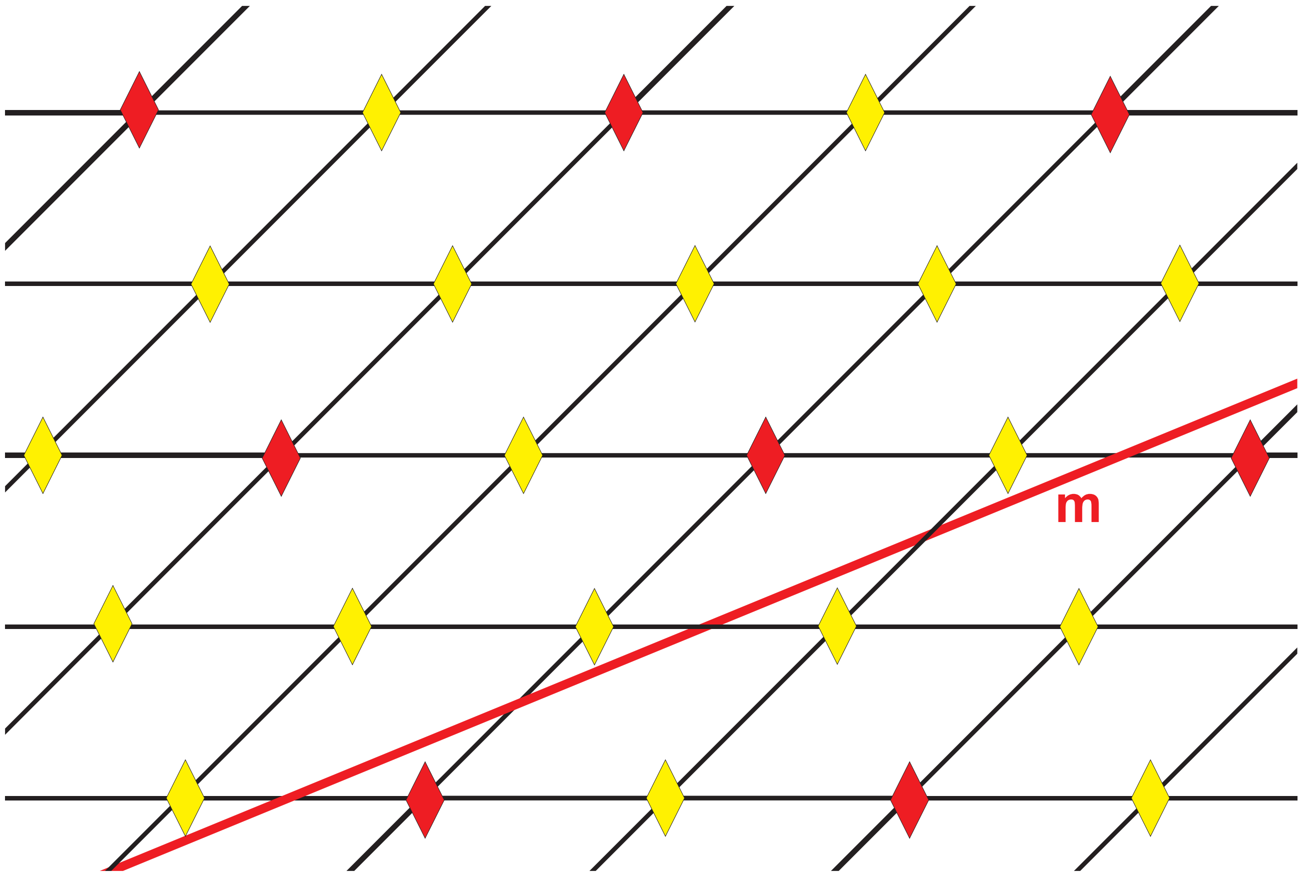
(b) G does not contain a reflection but it contains a glide reflection γ with axis m. Case (1) of the lemma must hold, so

(b1) $\tau_1 \parallel m$ and $\tau_2 \perp m$.

It turns out that cases (a2) and (b1) lead to one new wallpaper group, while case (a1) leads to two different wallpaper groups.

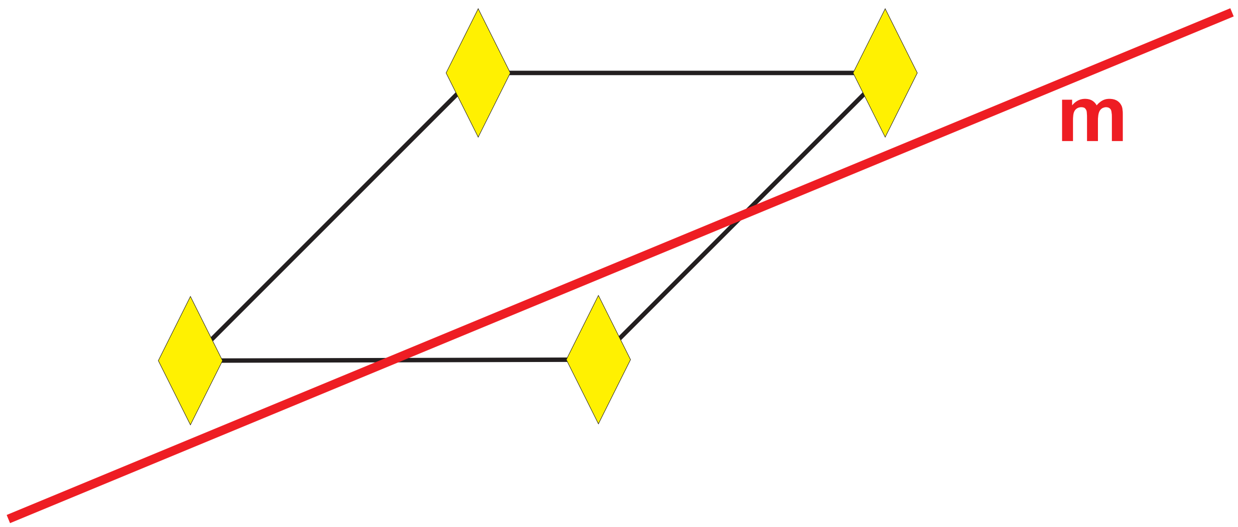


(a2) $\sigma_m \in G$ and $|\tau_1| = |\tau_2|$ and $\tau_1\tau_2 \parallel m$. In this case the lattice is rhombic, and m is parallel to a diagonal of each rhombus.

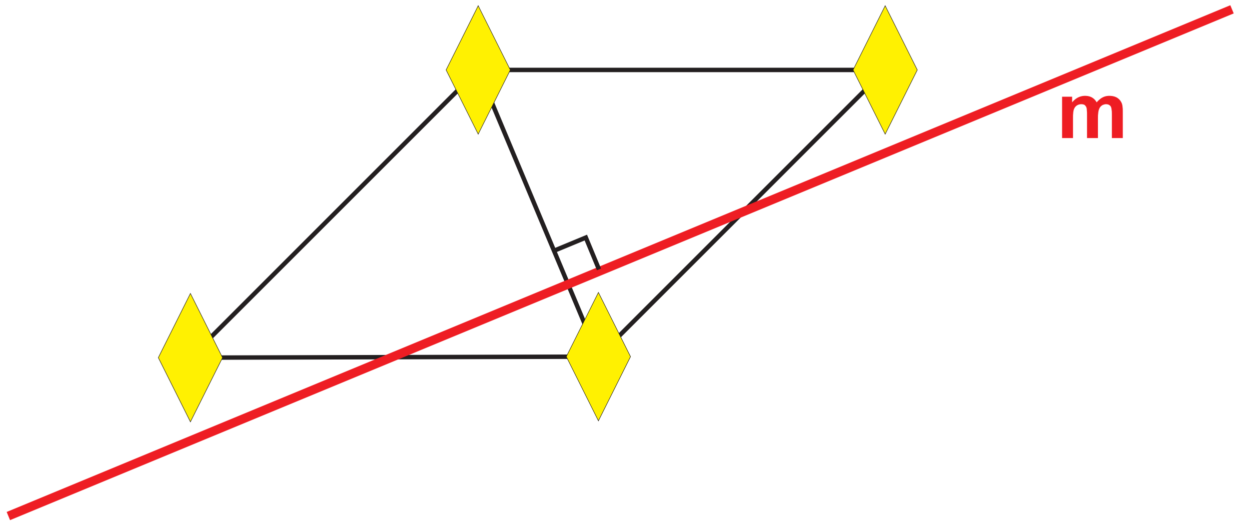


(a2) $\sigma_m \in G$ and $|\tau_1| = |\tau_2|$ and $\tau_1\tau_2 \parallel m$. In this case the lattice is rhombic, and m is parallel to a diagonal of each rhombus.

By connecting all centers of half turns (including the ones that are not lattice points) we obtain a lattice of "mini-rhombi" of quarter size.

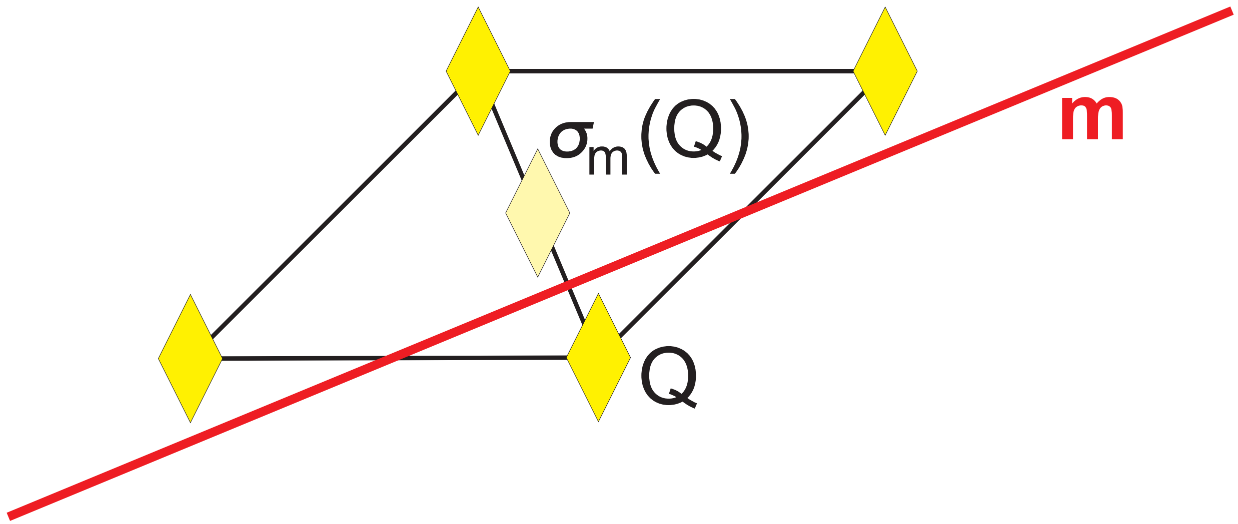


Consider a mini-rhombus that is intersected by the line m of reflection. We want to prove that m contains one of the corners of the rhombus, so suppose line m properly intersects the rhombus, not just a corner of it.



Consider a mini-rhombus that is intersected by the line m of reflection. We want to prove that m contains one of the corners of the rhombus, so suppose line m properly intersects the rhombus, not just a corner of it.

The diagonals of a rhombus are perpendicular bisectors of each other, so one of them is perpendicular to m.



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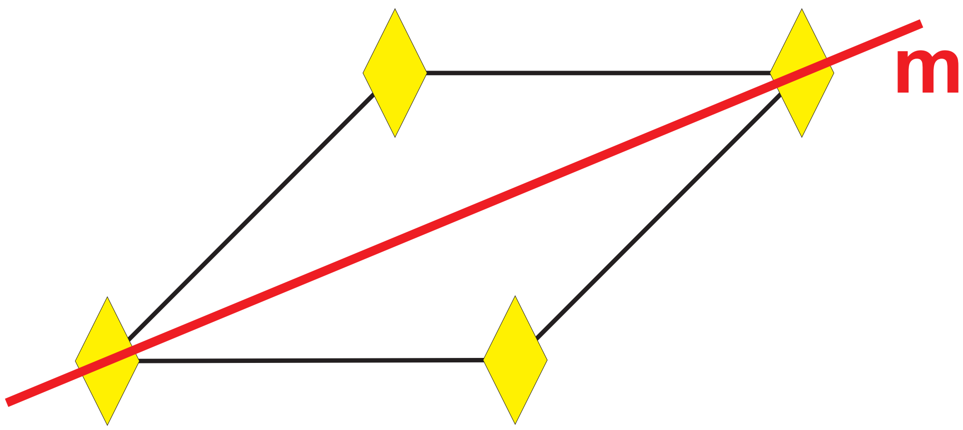
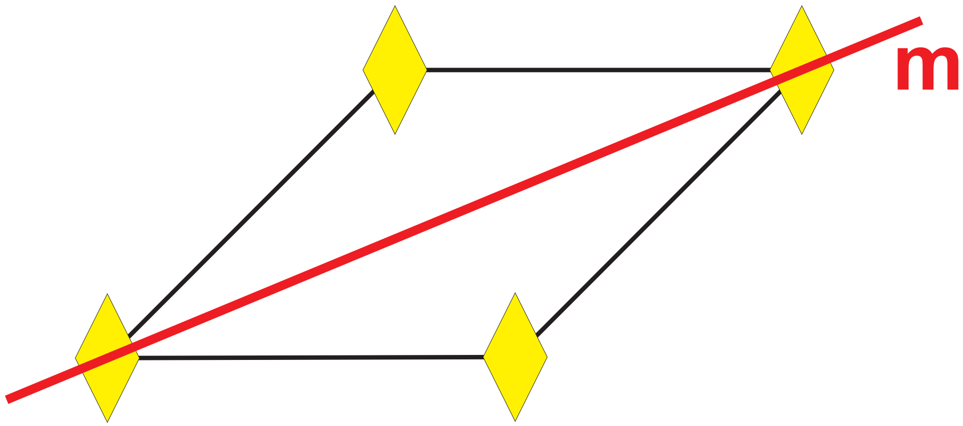
The diagonals of a rhombus are perpendicular bisectors of each other, so one of them is perpendicular to m.

If m is not a bisector of the diagonal then a corner Q of the rhombus is reflected by σ_m inside the rhombus. This contradicts the fact that there are no centers of half turns inside the rhombus.



So line m is the perpendicular bisector of a diagonal, and hence contains the other diagonal. It passes through two corners of the mini-rhombus.

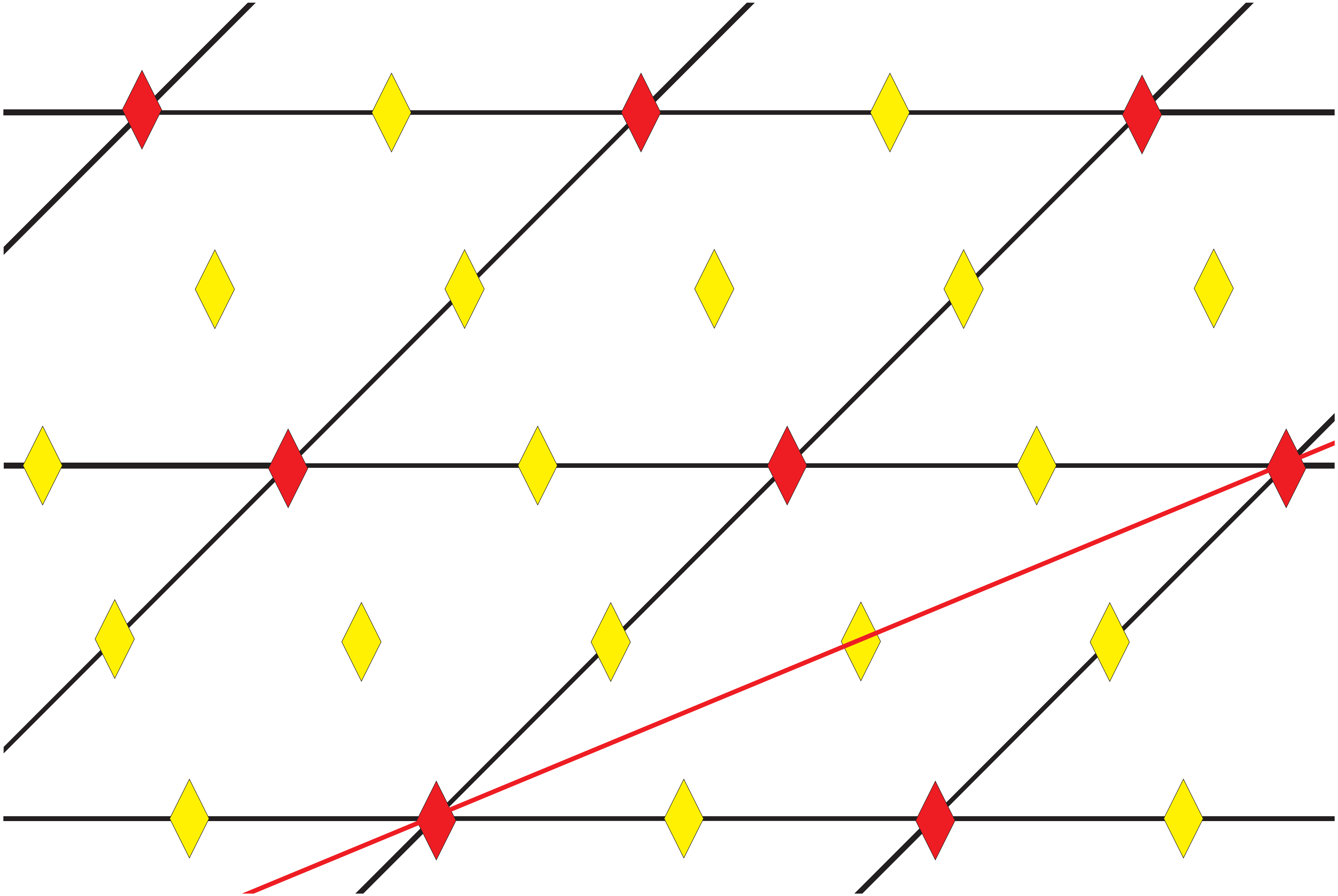


So line m is the perpendicular bisector of a diagonal, and hence contains the other diagonal. It passes through two corners of the mini-rhombus.

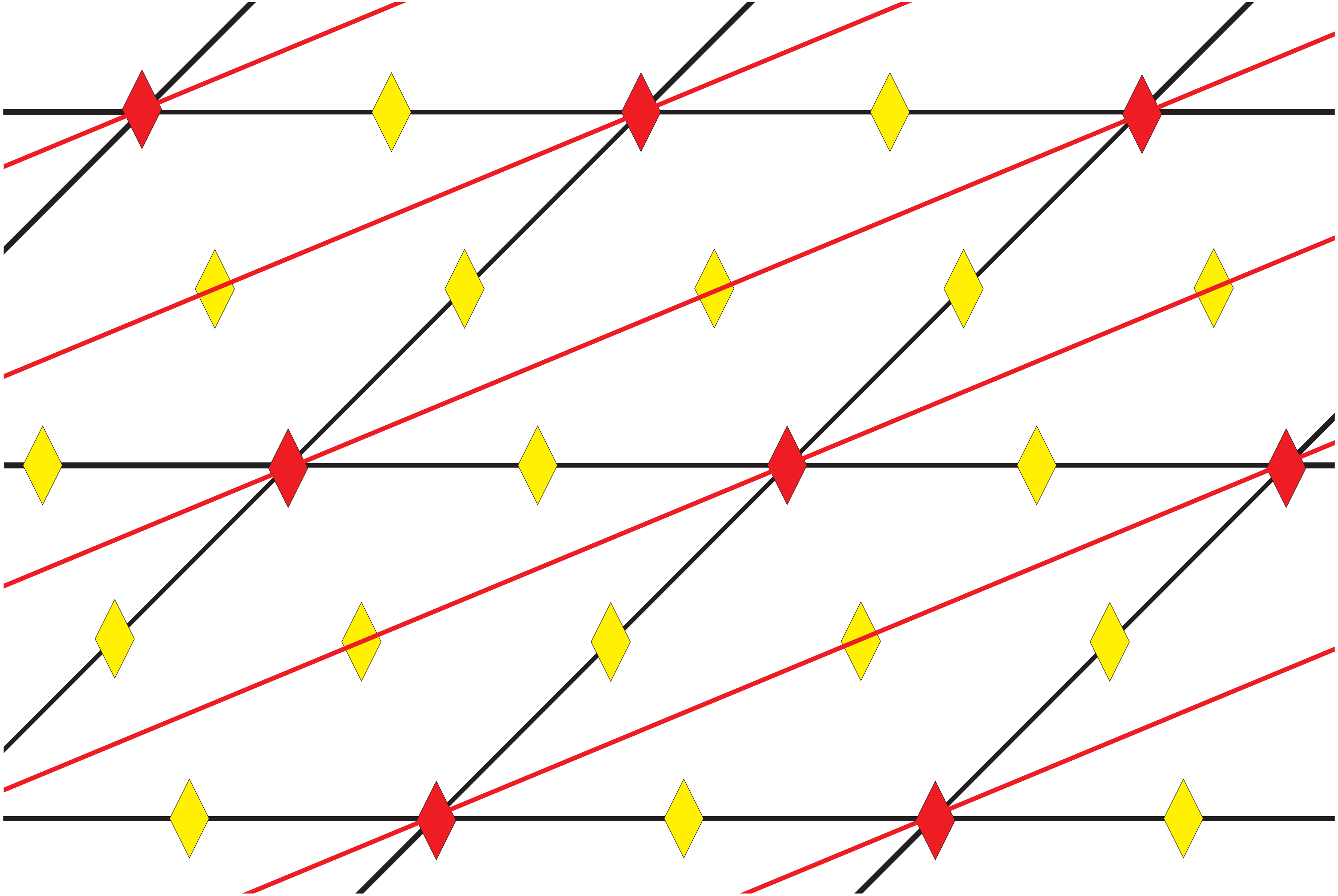
We conclude that the reflection line m necessarily contains centers of half turn. As the original center P of a half turn was arbitrary, we can select it so that $P \in m$. The groups is uniquely determined:

$$W_2^1 = \langle \tau_1, \tau_2, \sigma_P, \sigma_m \rangle,$$

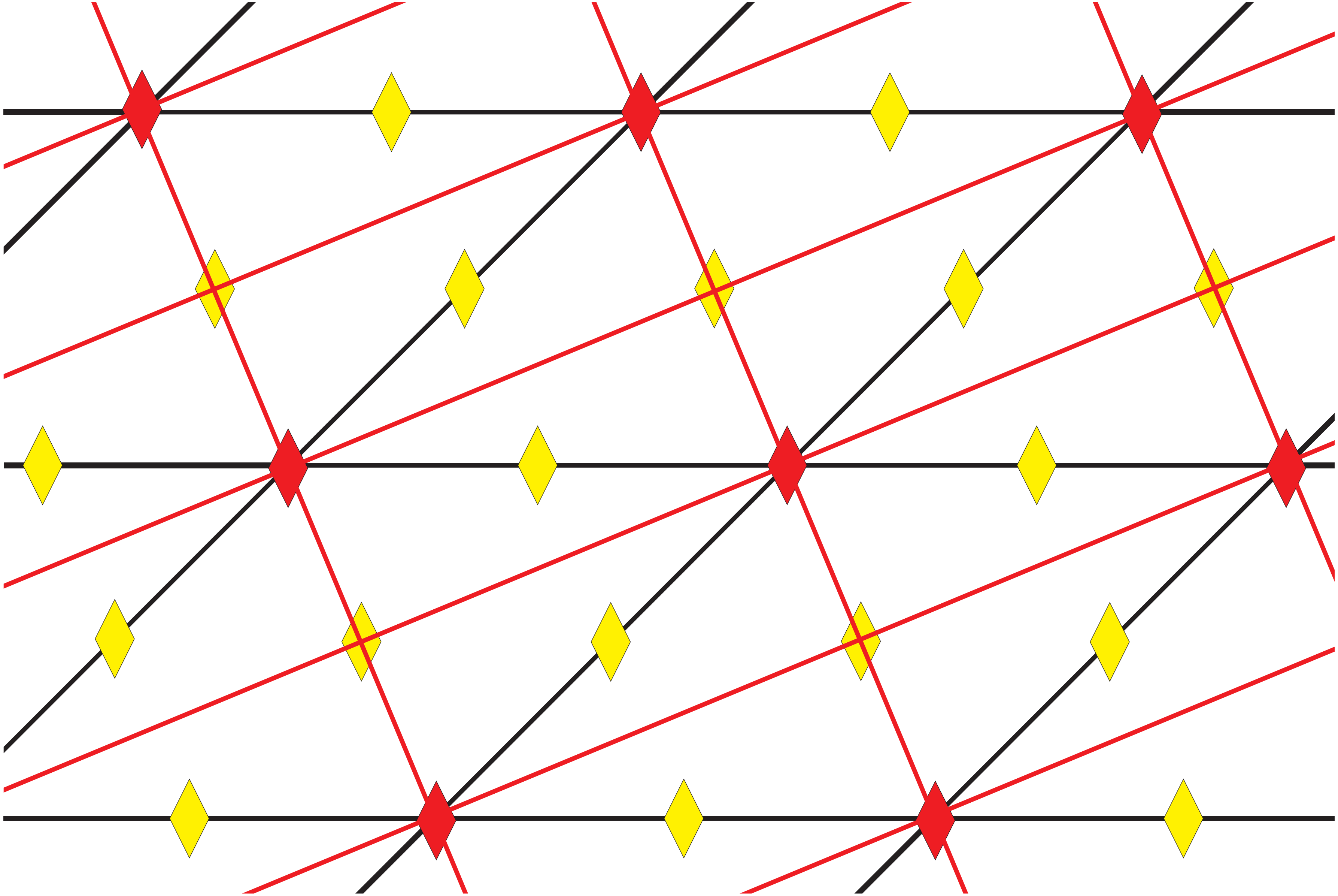
where $|\tau_1| = |\tau_2|$ and $P \in m$ and $\tau_1 \tau_2(P) \in m$.



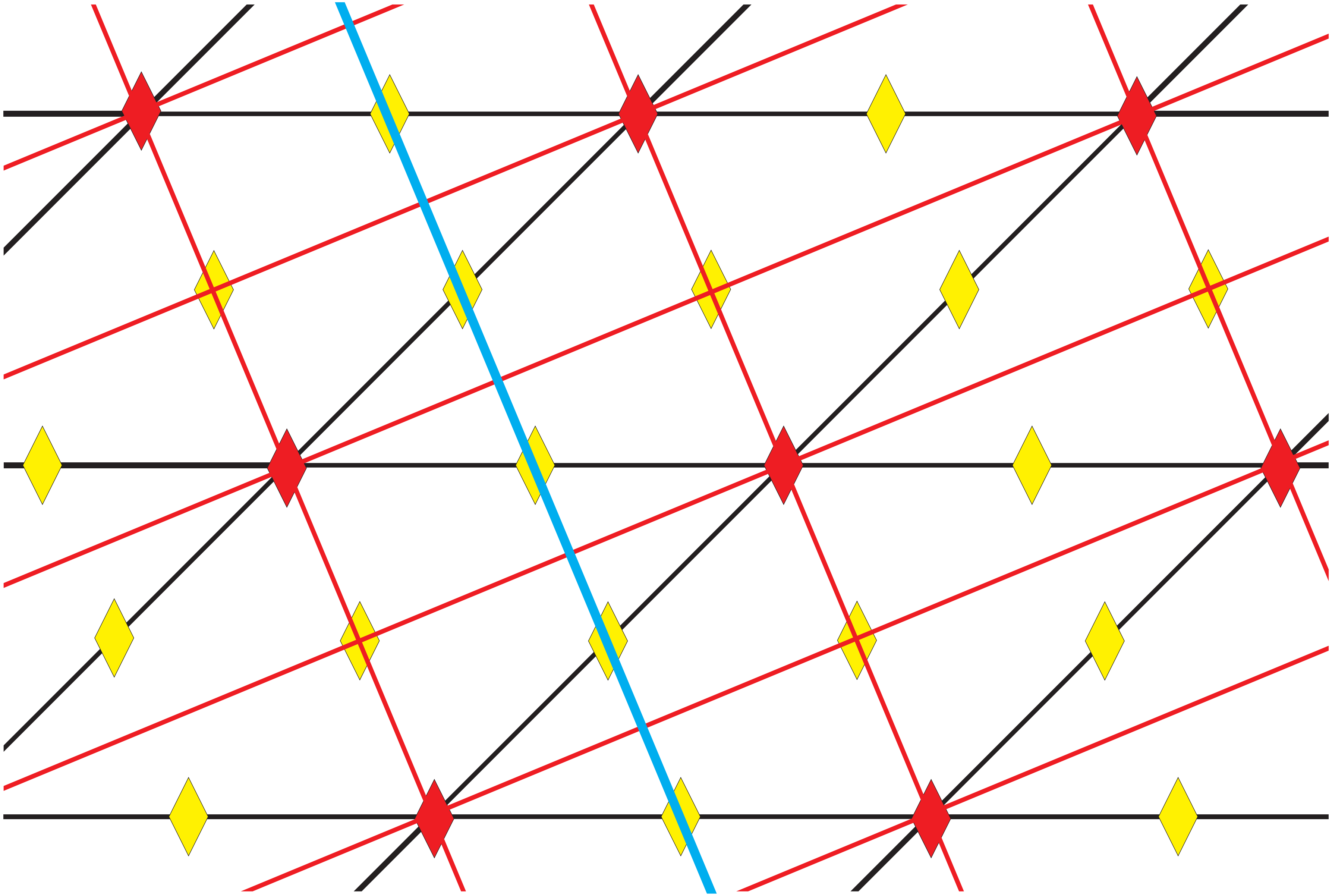
In group G a reflection line m follows diagonals of rhombi.



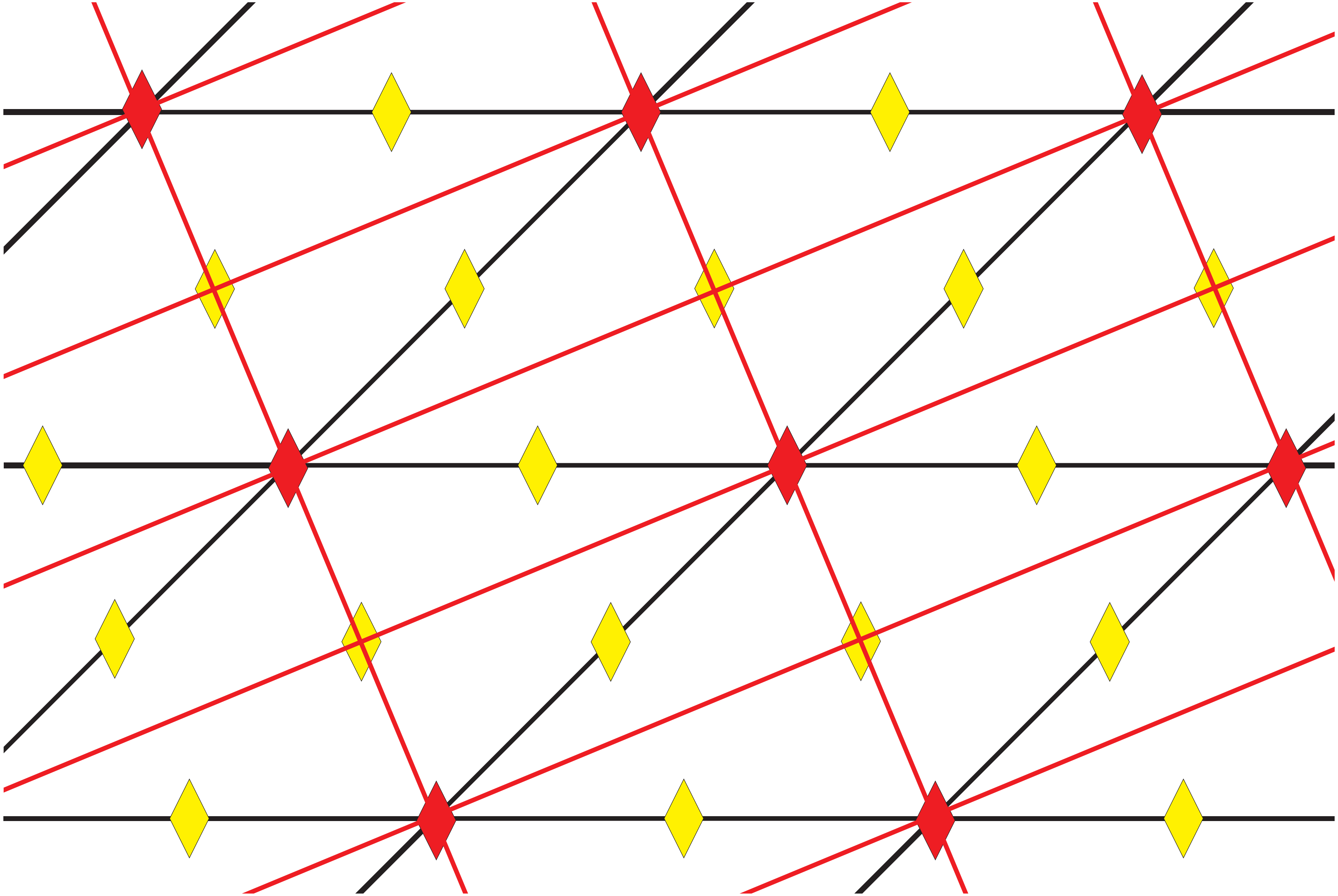
The conjugates of the reflection with respect to translations give reflections on all parallel lines through lattice points.



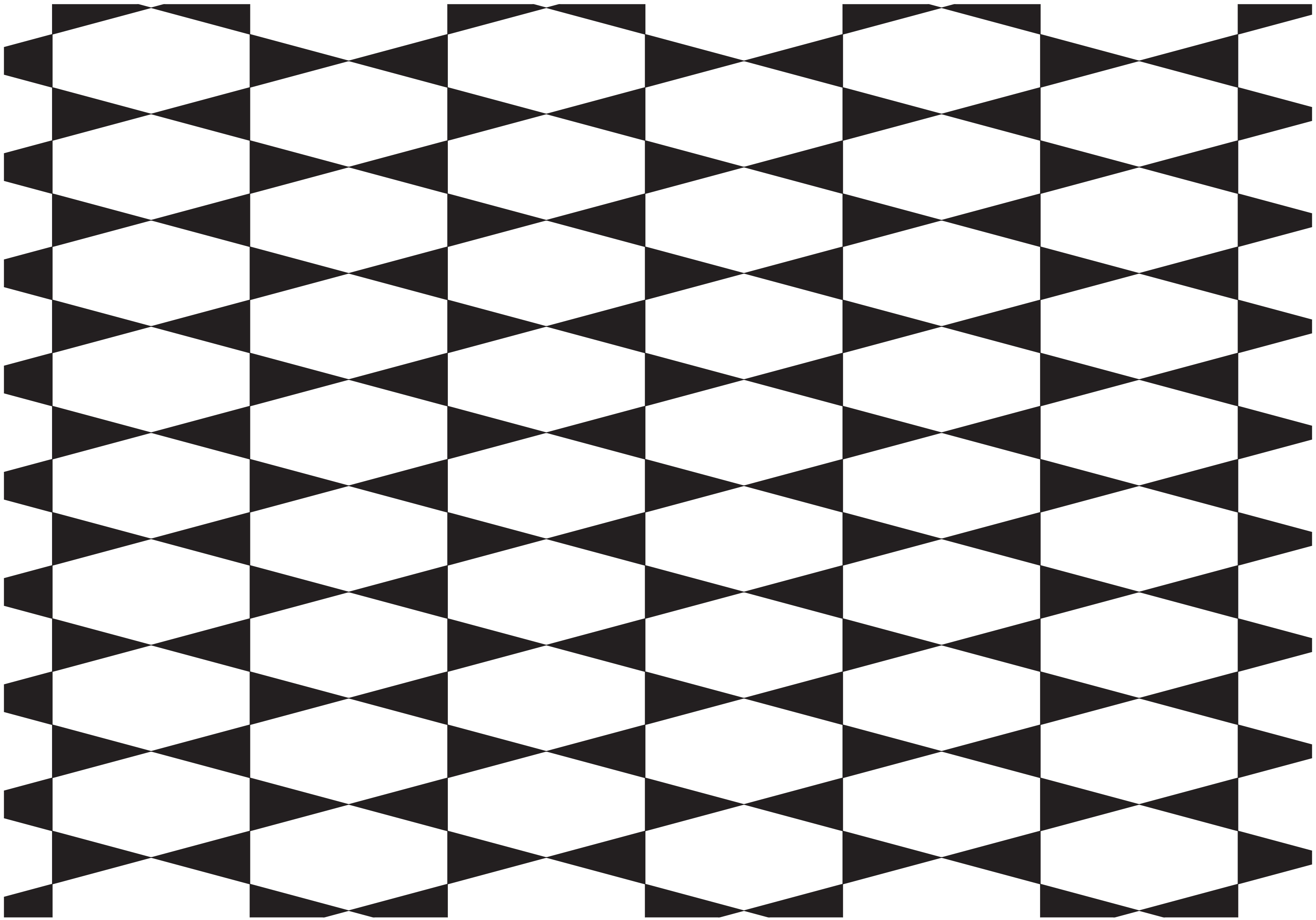
And the compositions of the halfturns and the reflections give reflections on perpendicular lines through the lattice points.



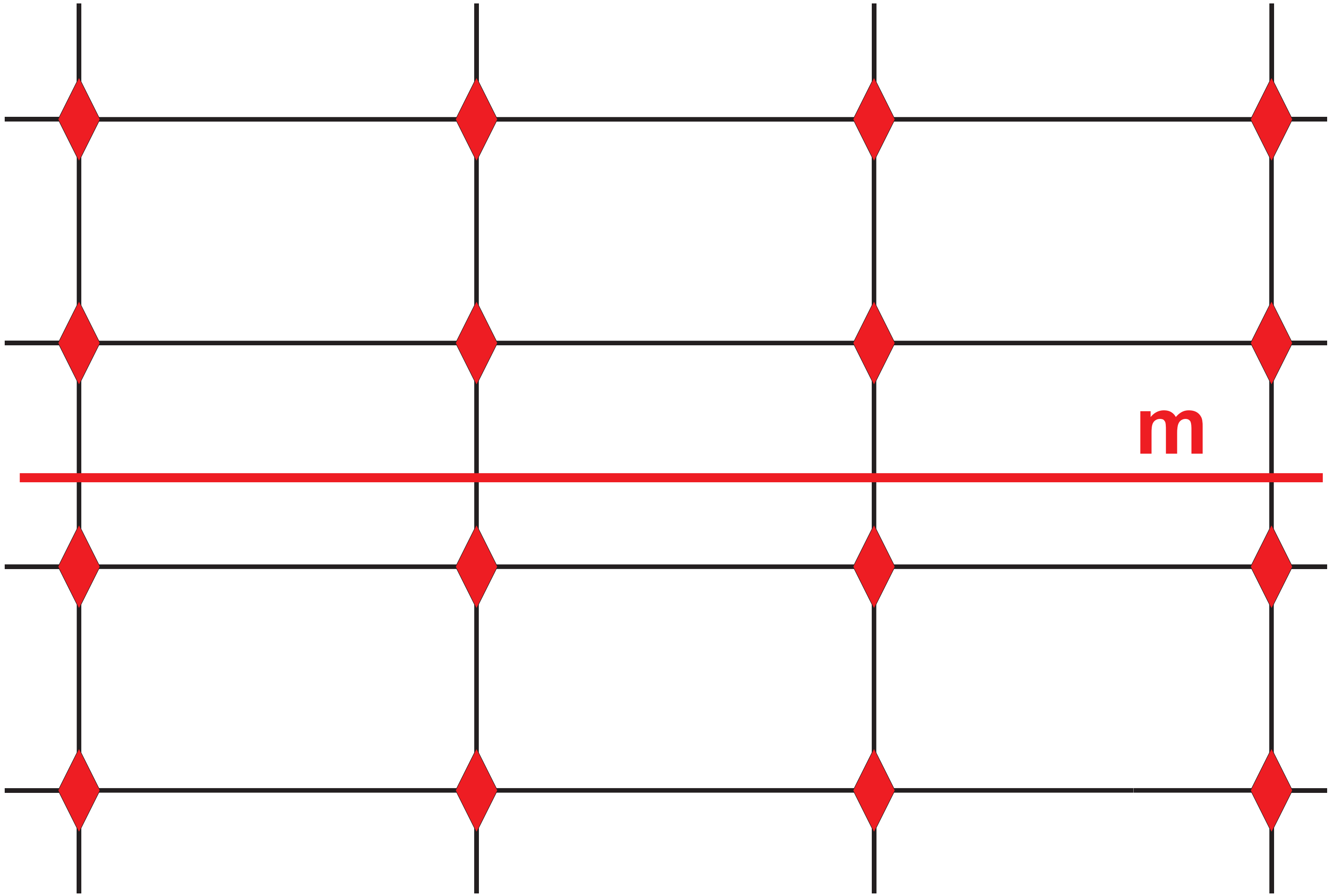
No other reflections exist: any other reflection line creates a wrong rotation at the intersection point with an existing reflection line.



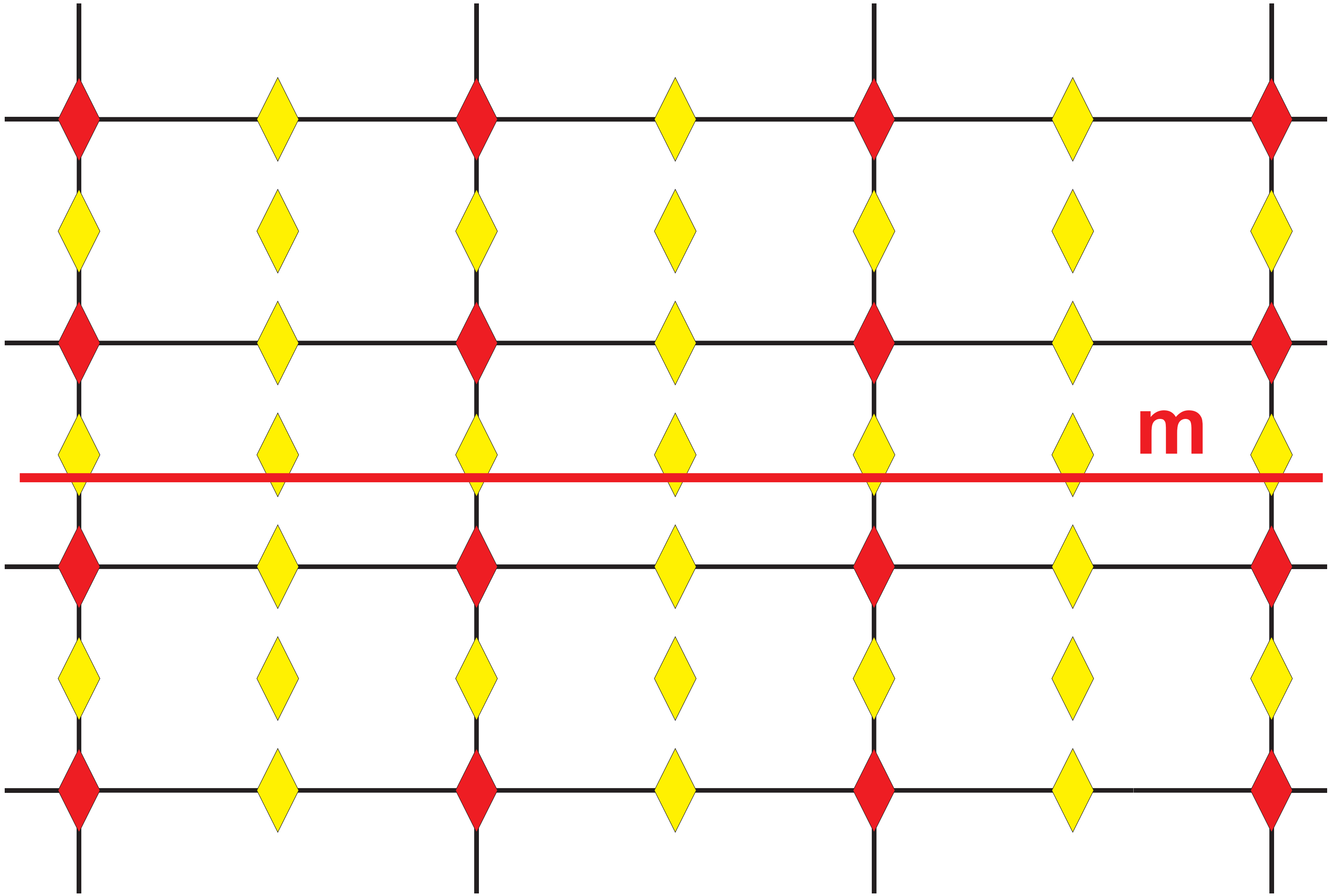
We conclude that there are centers of half turn that are not on any line of reflection.



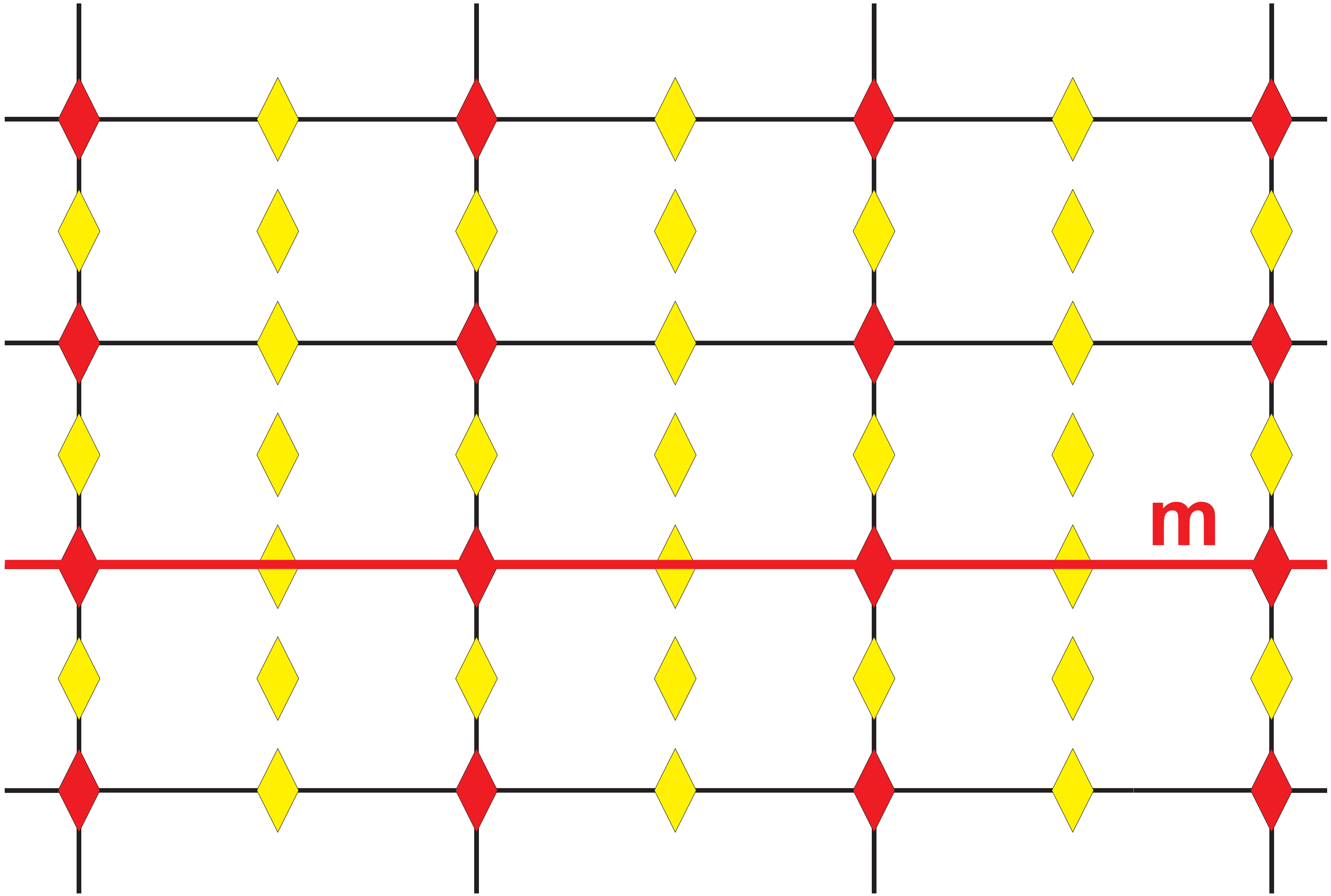
An example of a pattern with symmetry group W_2^1.



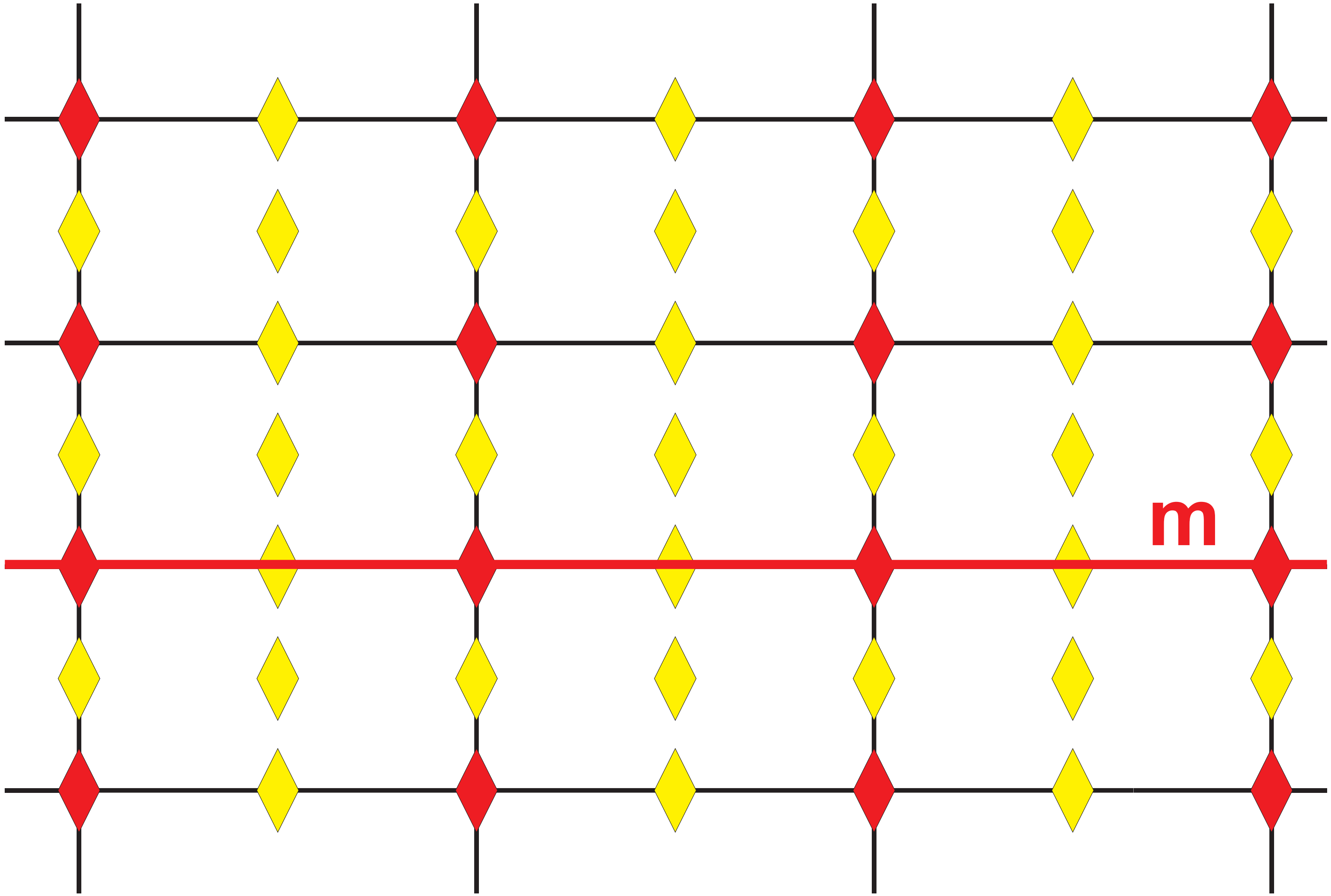
Case (a1): $\sigma_m \in G$ and $\tau_1 \parallel m$ and $\tau_2 \perp m$.



There are two possibilities: Line m contains a center of a half turn, or it does not.



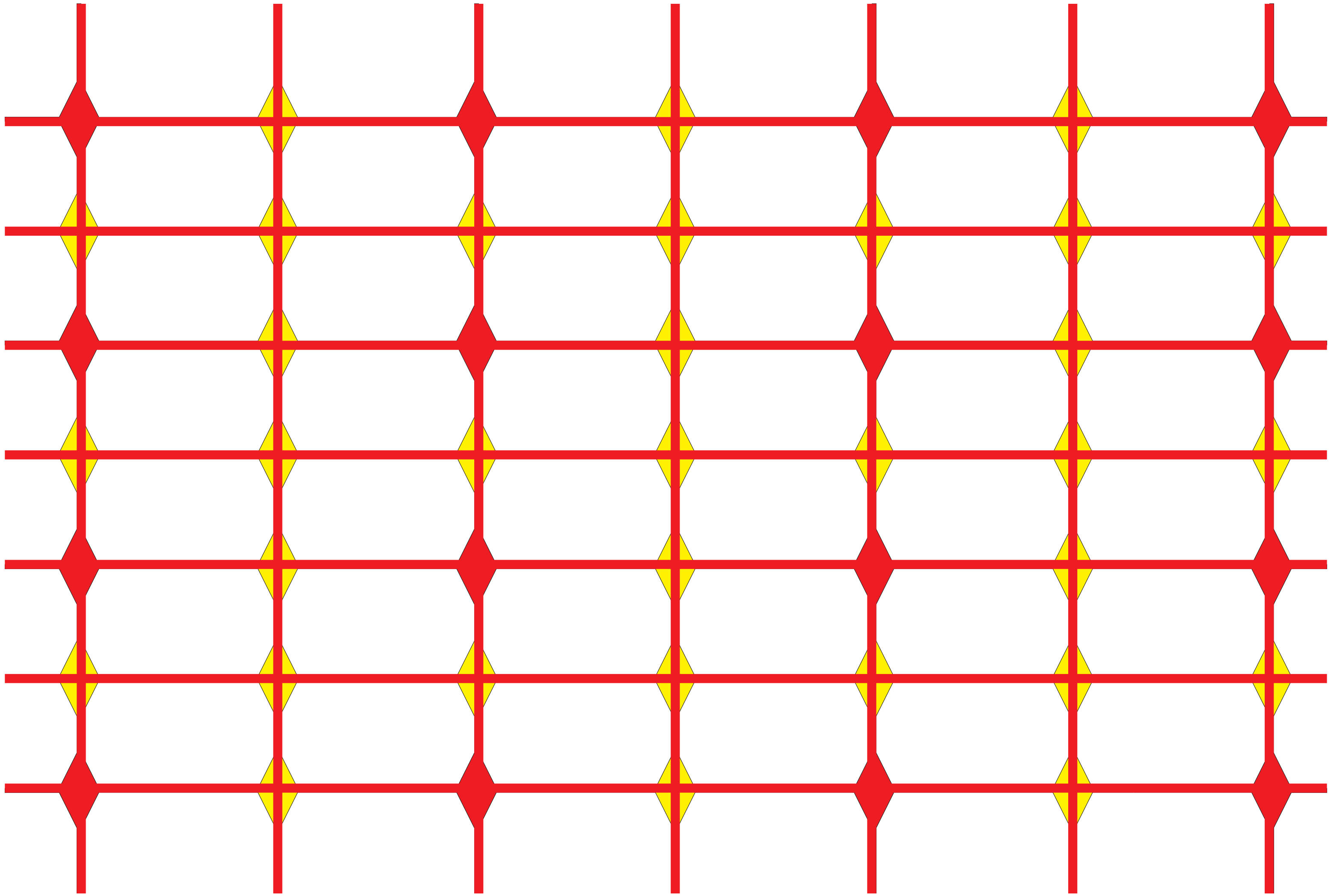
Suppose first that line m contains a center of a half turn. As the original center P of a half turn was arbitrary, we can choose it so that $P \in m$.



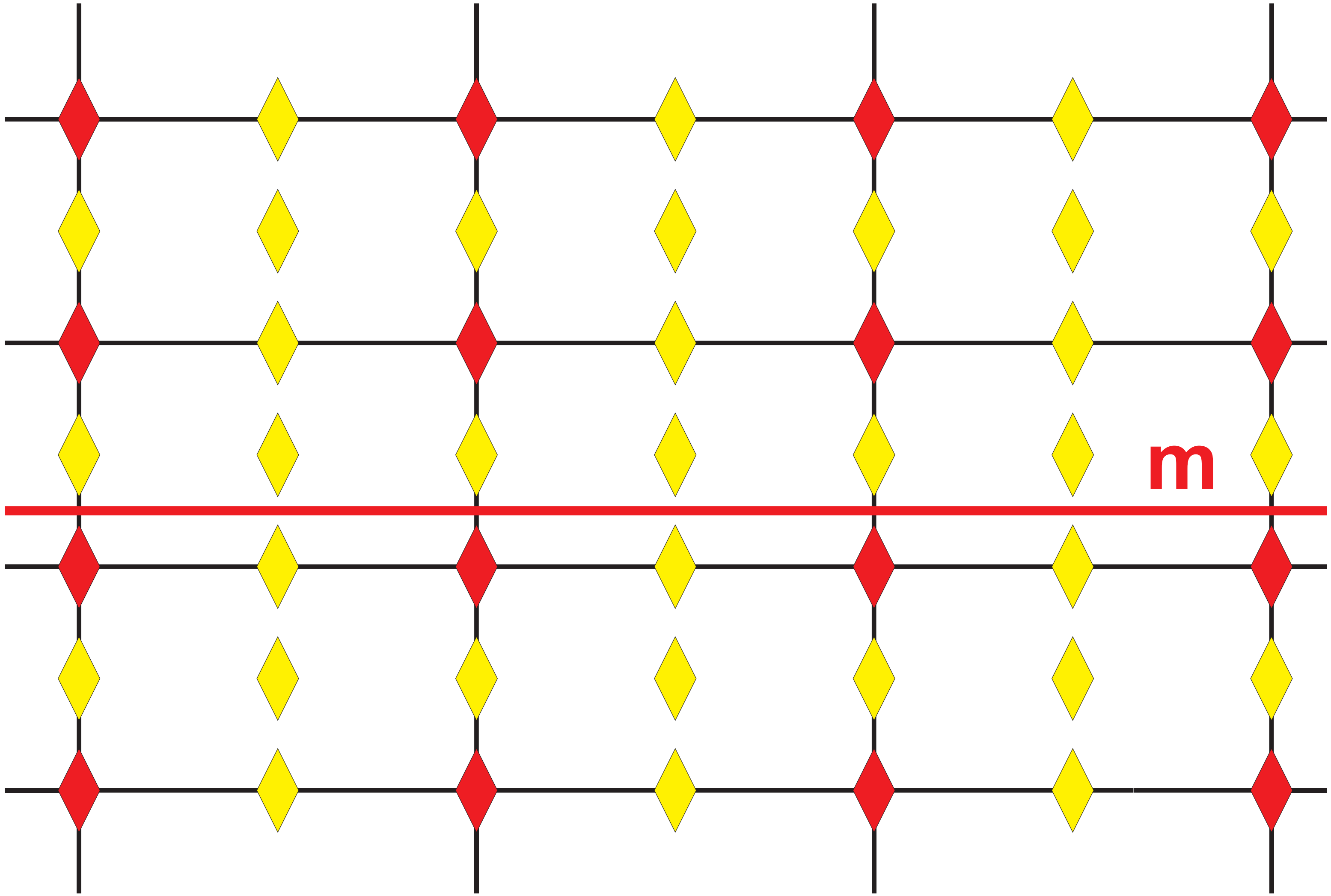
This gives the wallpaper group

$$W_2^2 = \langle \tau_1, \tau_2, \sigma_P, \sigma_m \rangle,$$

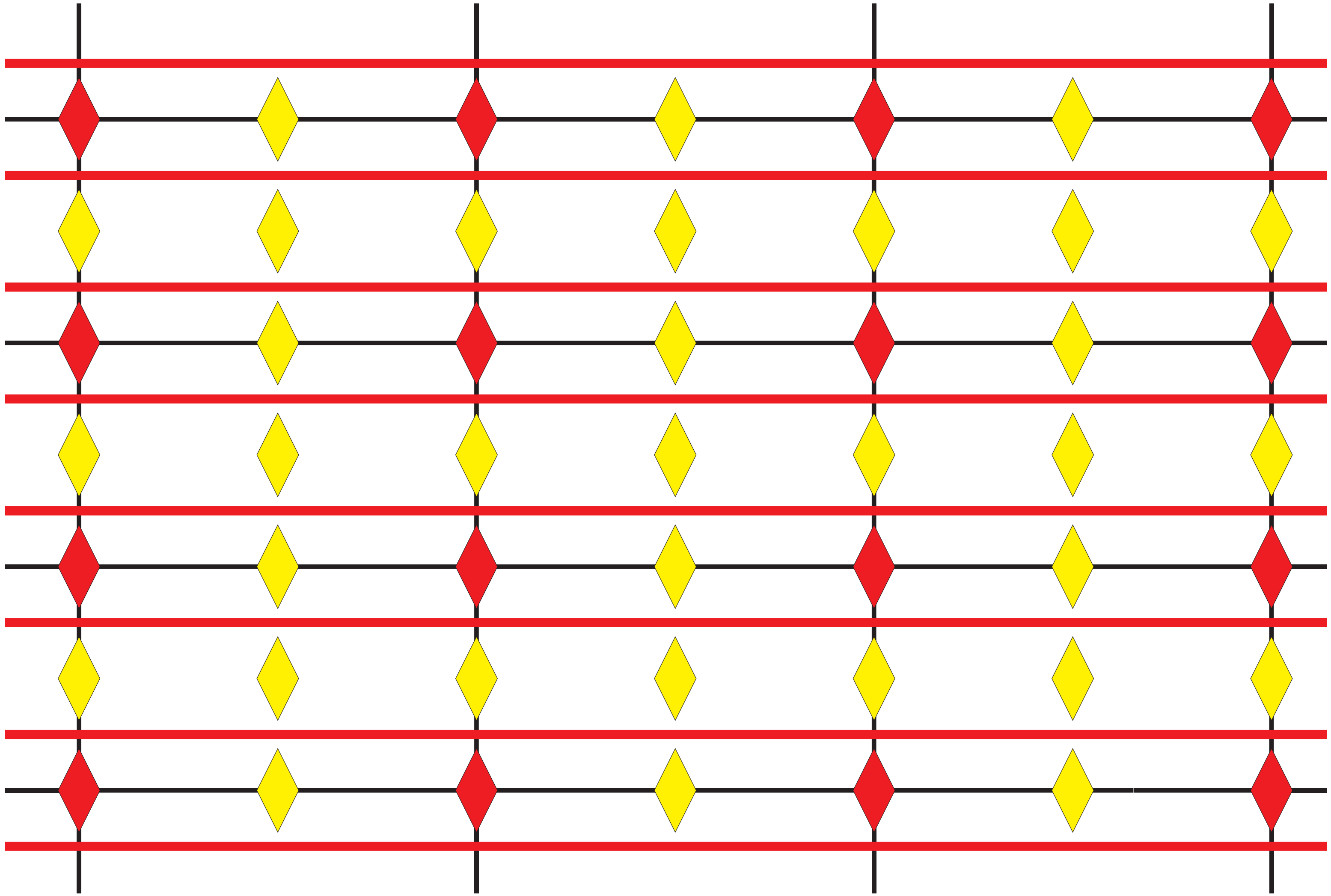
where $\tau_1 \perp \tau_2$ and $P \in m$ and $\tau_1(P) \in m$.



There are perpendicular lines of reflection through all centers of half turn, obtained as the compositions of σ_m and translations and half turns of G.

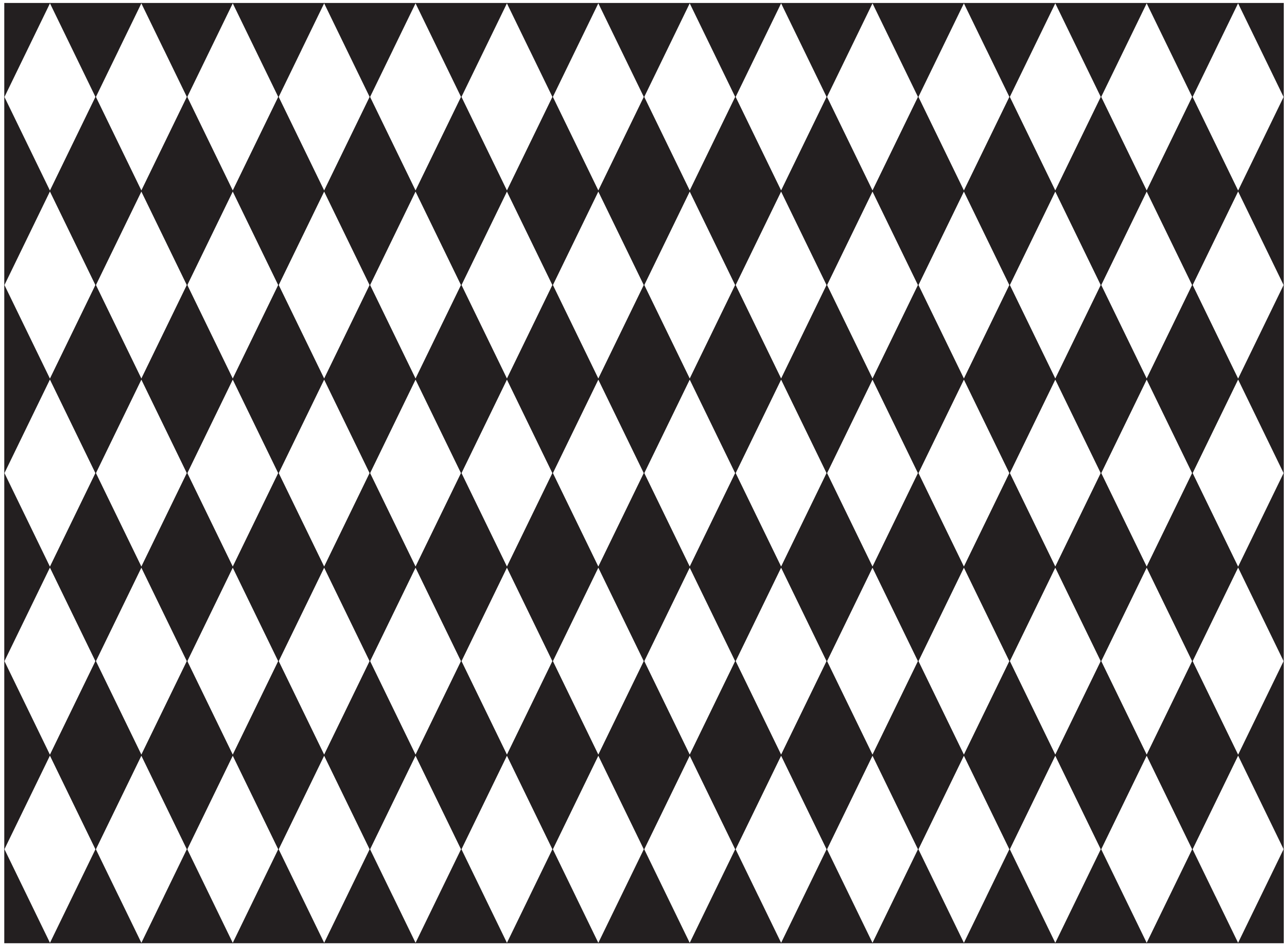


The other option is that line m does not contain any centers of half turn. Then m must be in the middle between two horizontal rows of such points.

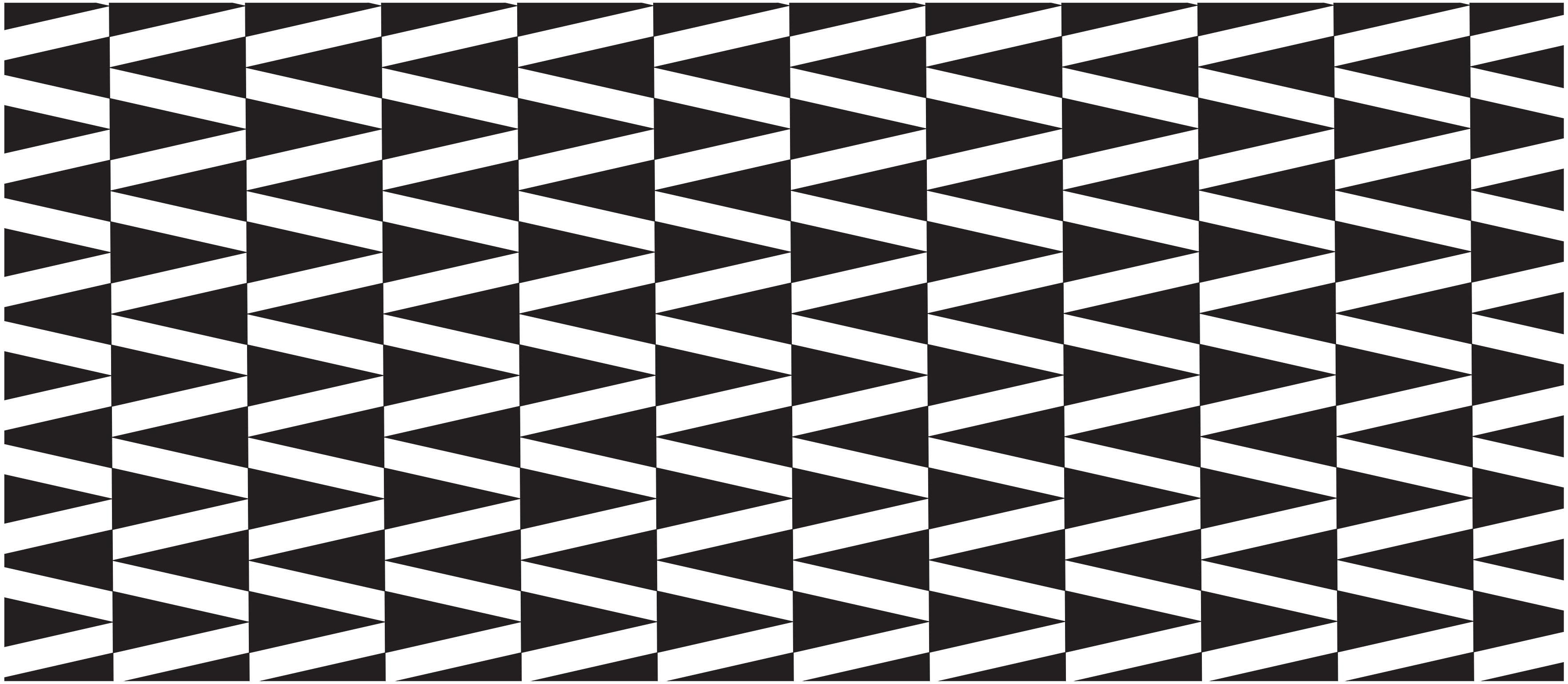


We obtain wallpaper group $W_2^3 = \langle \tau_1, \tau_2, \sigma_P, \sigma_m \rangle$.

In this case the lines of reflection are all parallel and run in between the centers of half turns.



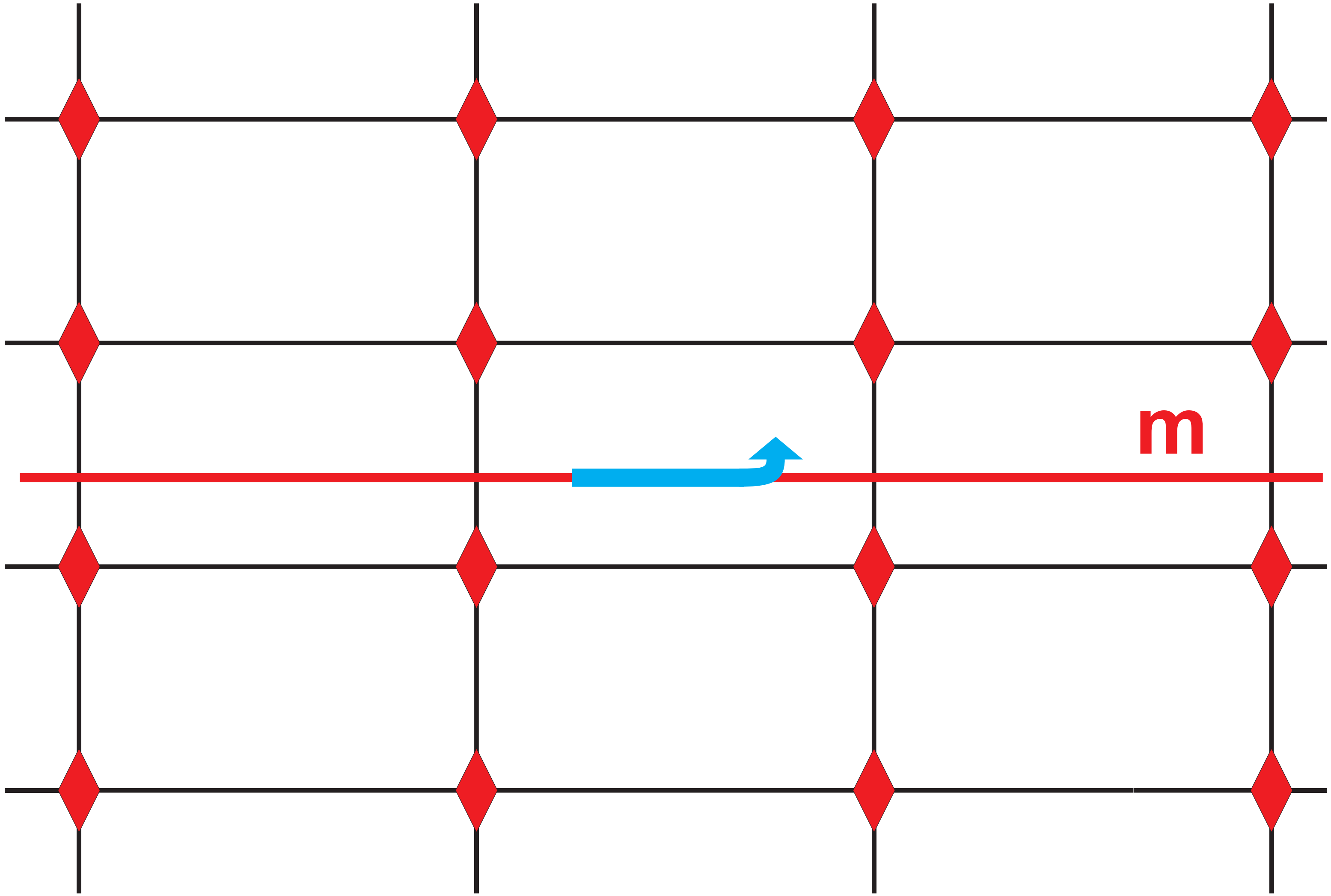
An example of a pattern with symmetry group W_2^2.



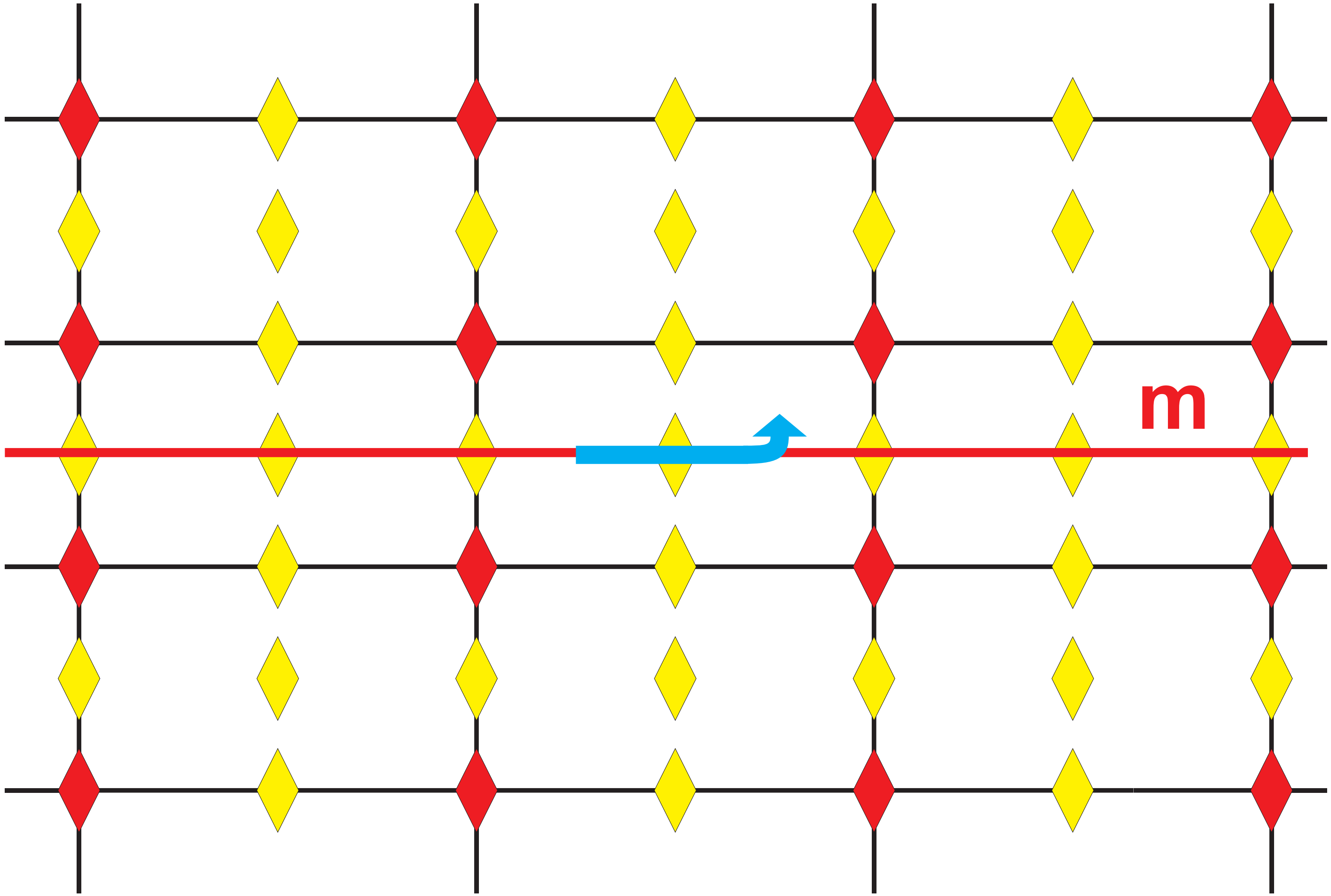
And an example of a pattern with symmetry group W_2^3.

A way to differentiate groups W_2^1, W_2^2 and W_2^3:

- In W_2^2 there are lines of reflection through all centers of half turns,
- In W_2^1 there are lines of reflection through some but not all centers of half turns,
- In W_2^3 there are no lines of reflection through any centers of half turns. All lines of reflection are parallel.

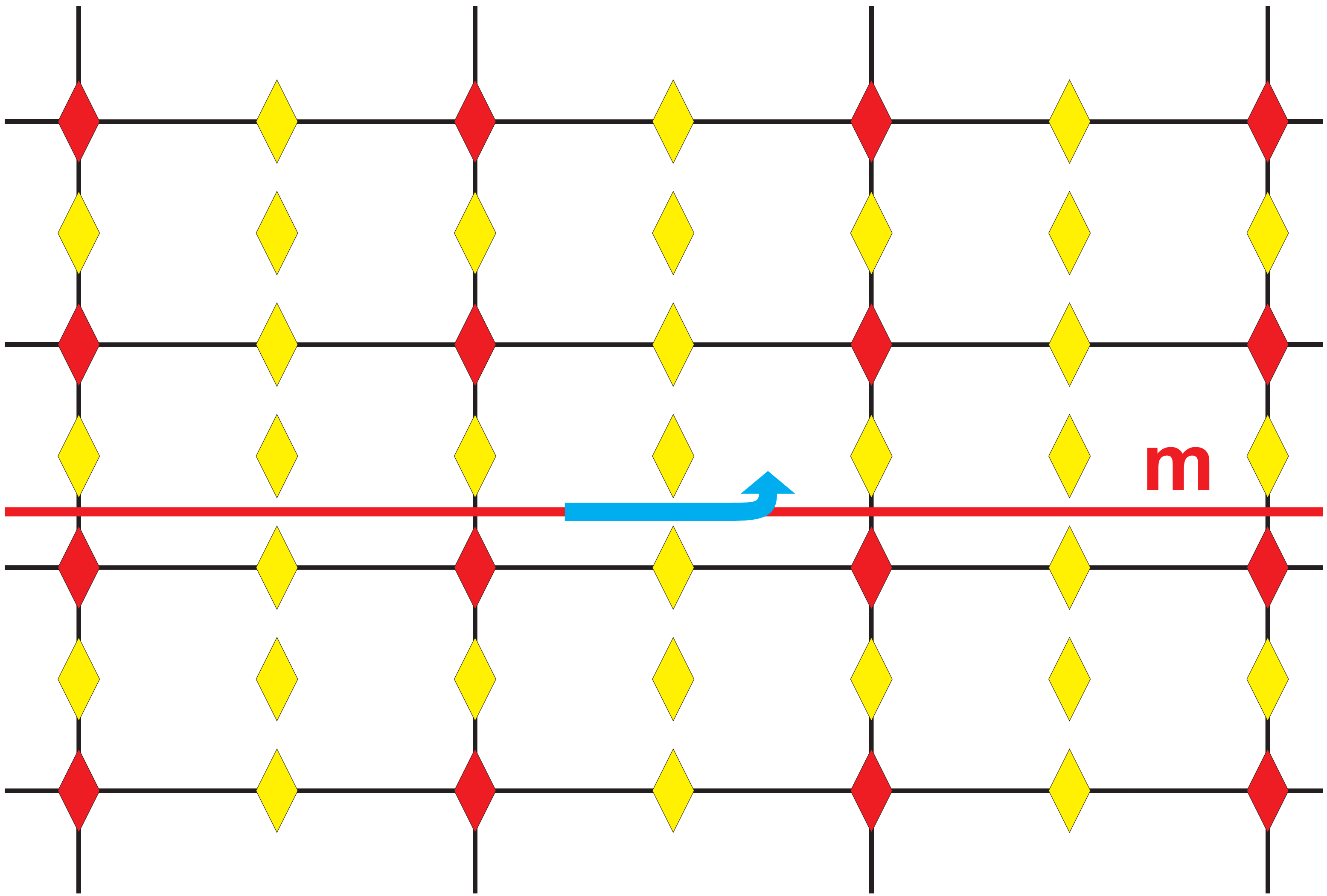


Case (b1): G does not contain a reflection but it contains a glide reflection γ with axis m. Case (1) of the lemma holds, so $\tau_1 \parallel m$ and $\tau_2 \perp m$.



In this case line m cannot contain any center of a half turn.

Reason: If $P \in m$ then $\gamma\sigma_P$ is a reflection, and we assumed there are no reflections in G.



So m must be a line in the middle between two rows of symmetry points.

As γ^2 is a translation parallel to τ_1, we have

$$\gamma^2 = \tau_1^i$$

for some $i \in \mathbb{Z}$. If $i = 2j$ is even then $\gamma = \sigma_m \tau_1^j$, which implies that $\sigma_m = \gamma \tau_1^{-j}$ is in G, a contradiction.

As γ^2 is a translation parallel to τ_1, we have

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for some $i \in \mathbb{Z}$. If $i = 2j$ is even then $\gamma = \sigma_m \tau_1^j$, which implies that $\sigma_m = \gamma \tau_1^{-j}$ is in G, a contradiction.

So i is an odd integer, say

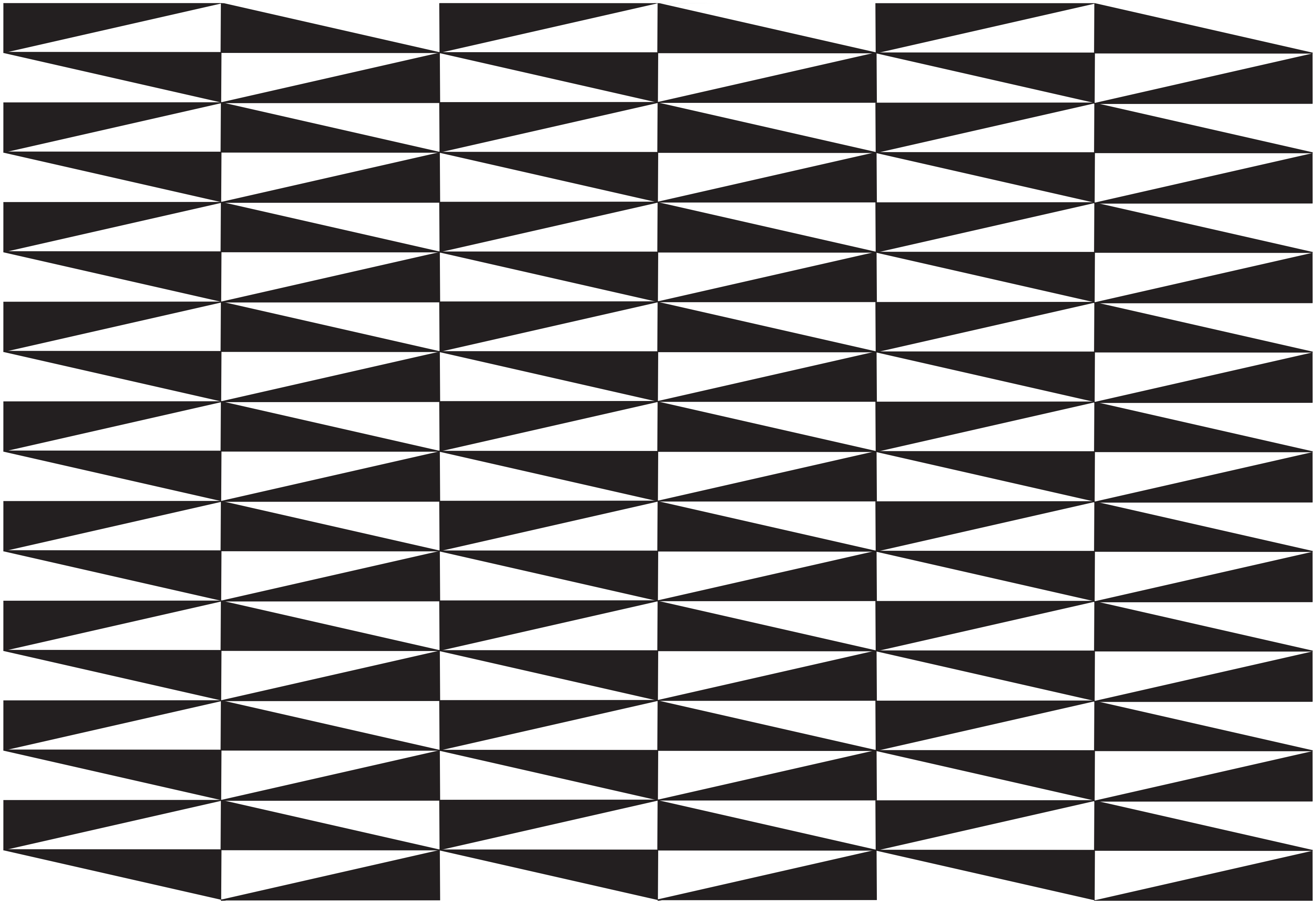
$$\gamma^2 = \tau_1^{2j+1}.$$

Then $\gamma \tau_1^{-j}$ is another glide reflection in G whose translation is exactly half of τ_1.

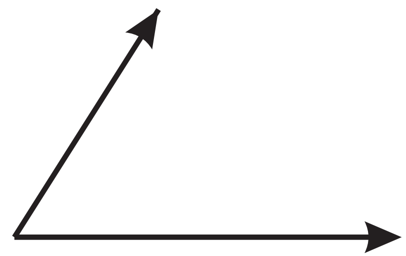
We have the wallpaper group

$$W_2^4 = \langle \tau_2, \sigma_P, \gamma \rangle,$$

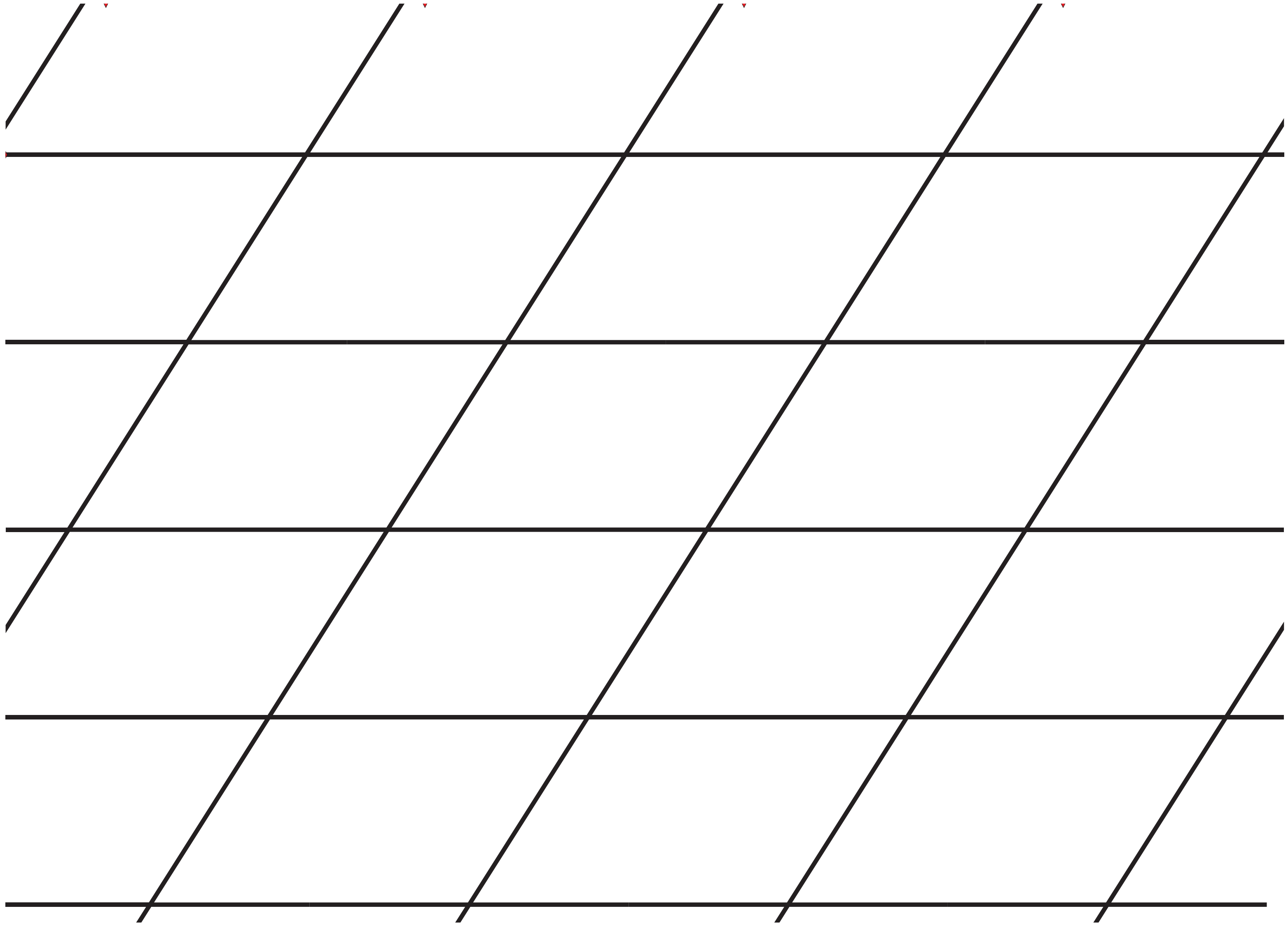
where $\gamma^2 = \tau_1$ is perpendicular to τ_2.



Here is an example of a pattern with symmetry group W_2^4.

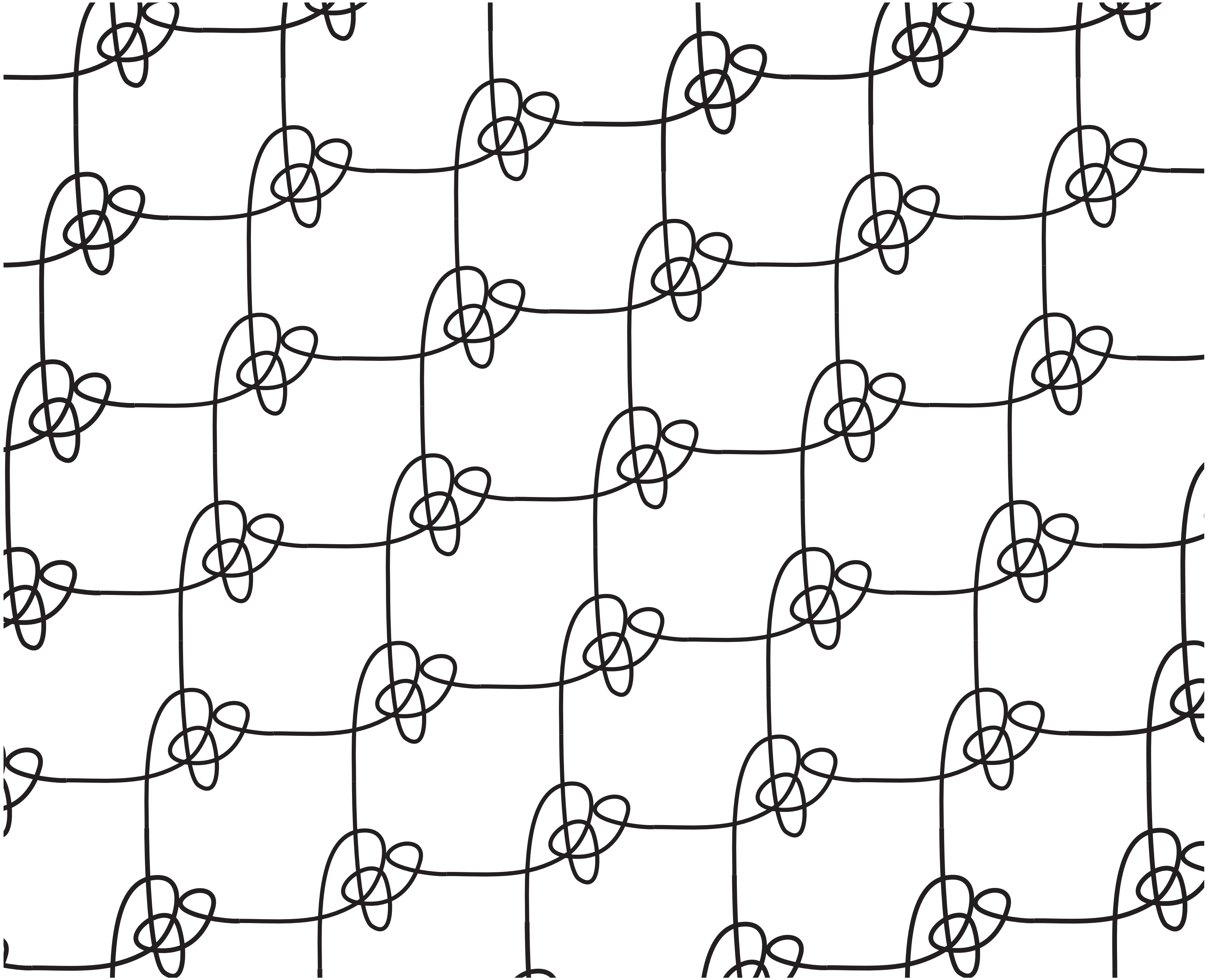


Case (5): Suppose G has no non-trivial rotations. Let τ_1 and τ_2 be translations that generate all translations of G.



Even isometries in G are precisely the translations. If there are no odd isometries in G then we have the wallpaper group

$$W_1 = \langle \tau_1, \tau_2 \rangle.$$



Group $W_1 = \langle \tau_1, \tau_2 \rangle$ is the symmetry group of this pattern, for example.

Suppose then that there are some odd isometries in G. Based on our lemma, we have the following cases:

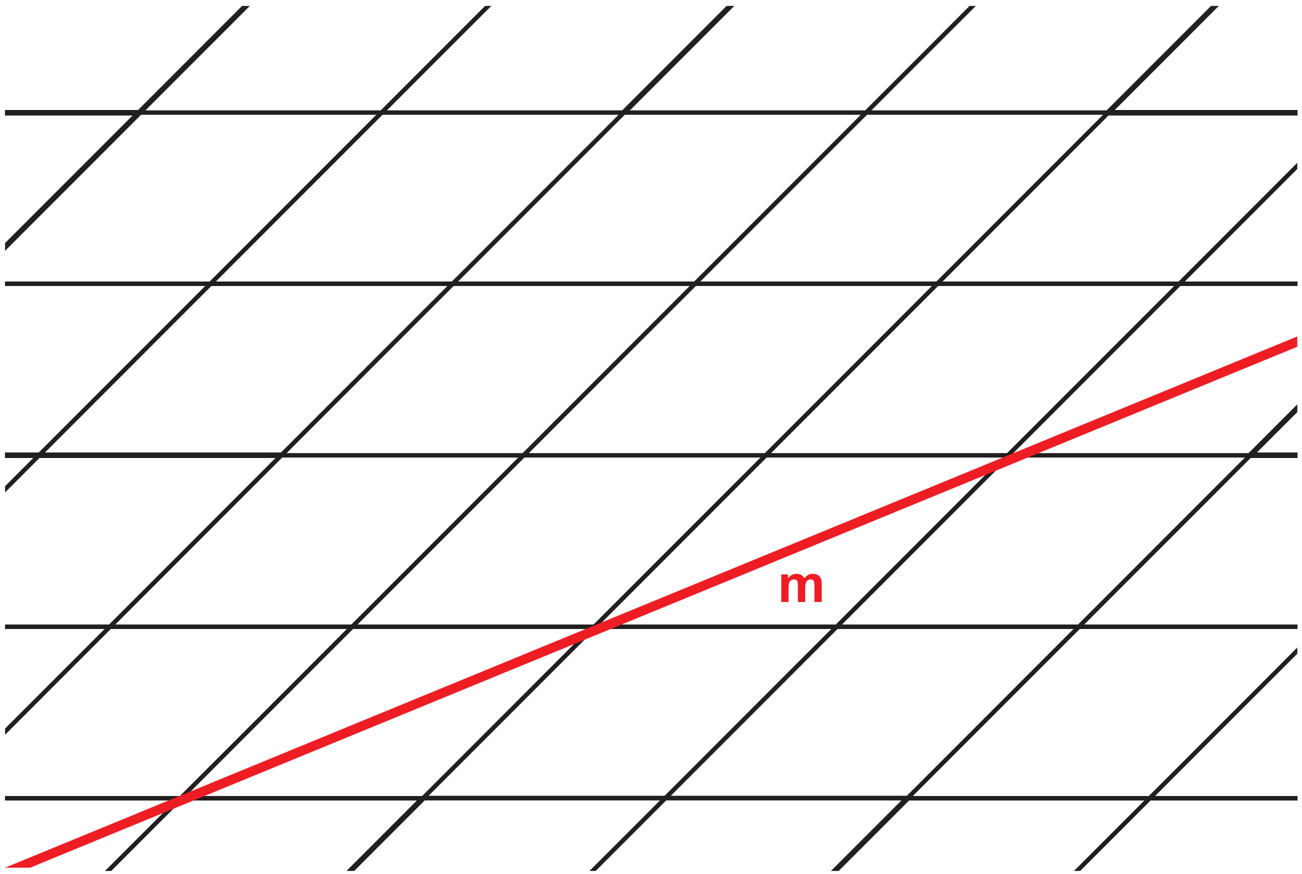
(a) G contains some reflection σ_m. Two subcases:

(a1) $\tau_1 \parallel m$ and $\tau_2 \perp m$,

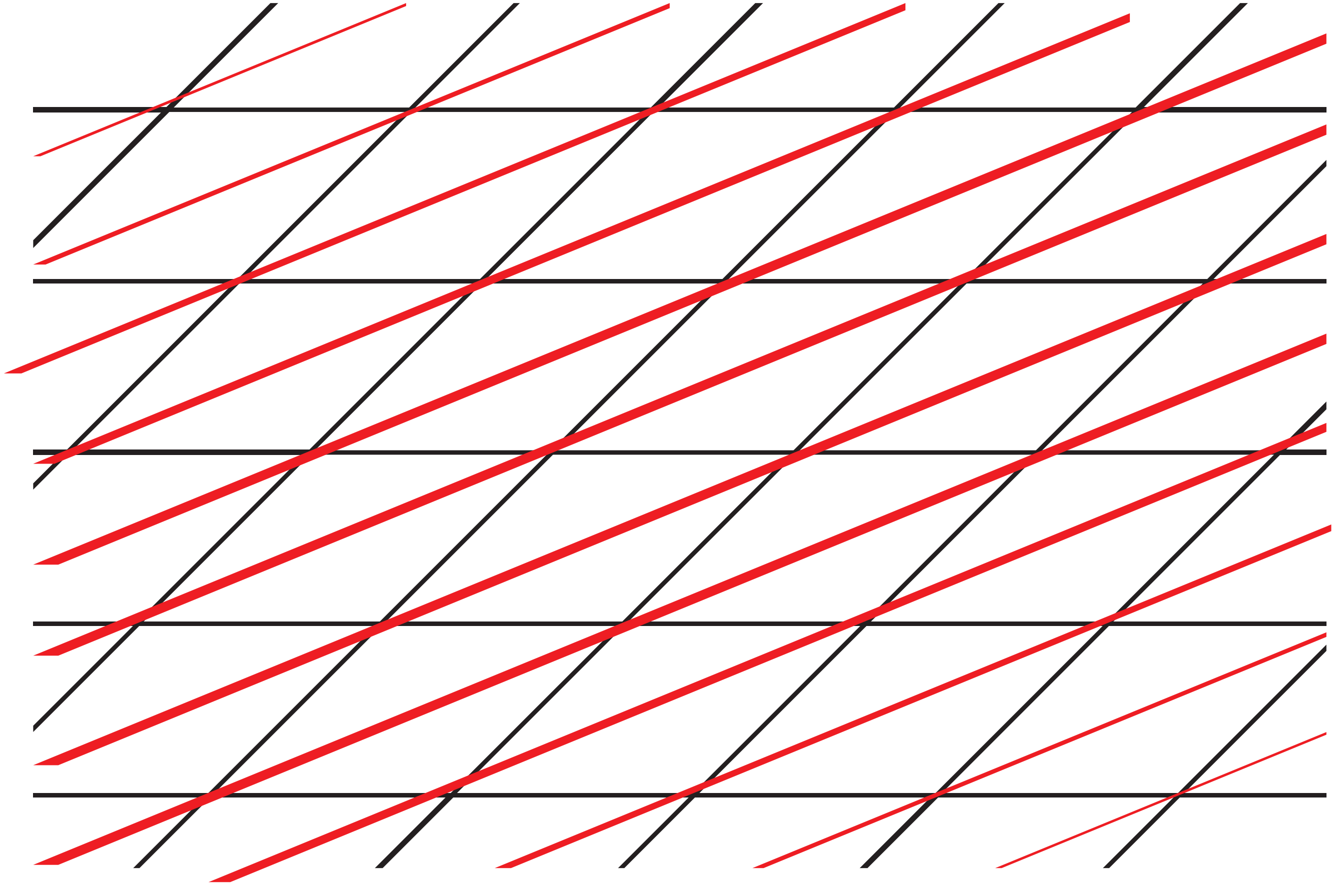
(a2) $|\tau_1| = |\tau_2|$ and $\tau_1\tau_2 \parallel m$.

(b) G does not contain a reflection but it contains a glide reflection γ with axis m. Case (1) of the lemma must hold, so

(b1) $\tau_1 \parallel m$ and $\tau_2 \perp m$.

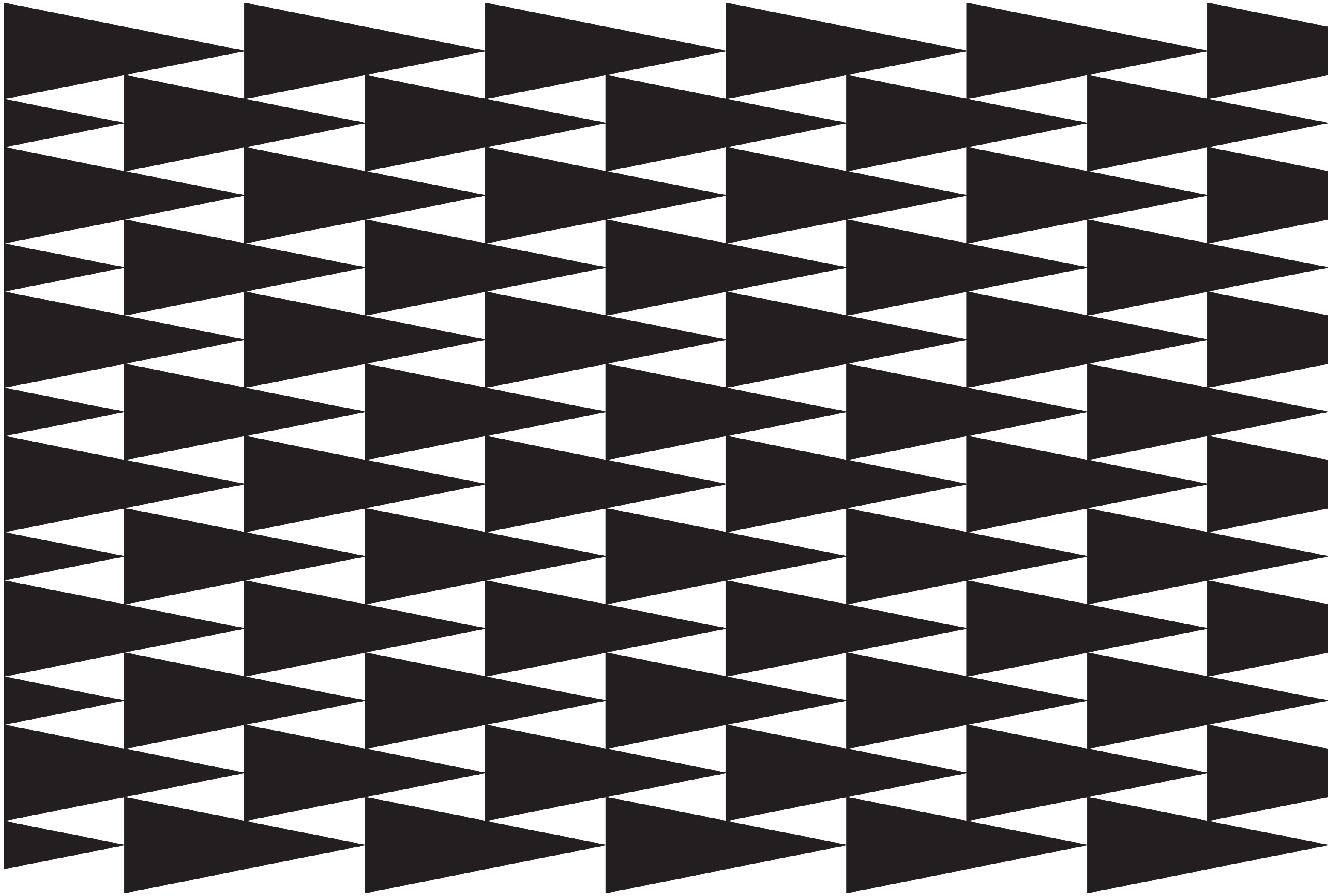


Let us begin with case (a2): $\sigma_m \in G$ and $|\tau_1| = |\tau_2|$ and $\tau_1\tau_2 \parallel m$. In this case the lattice is rhombic, and m is parallel to a diagonal of each rhombus.



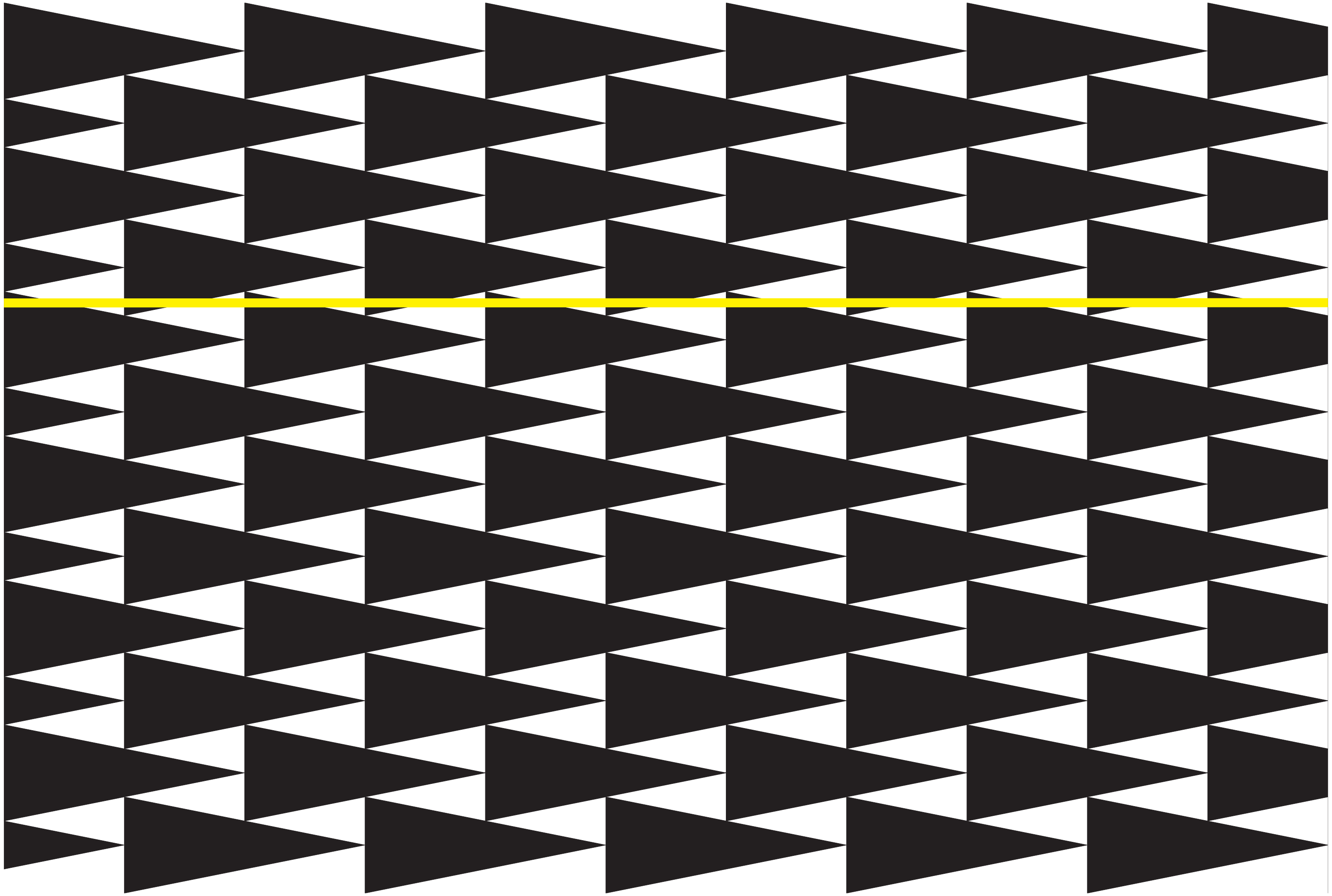
We have the symmetry group

$$W_1^1 = \langle \tau_1, \tau_2, \sigma_m \rangle.$$



An example of a pattern whose symmetry group is

$$W_1^1 = \langle \tau_1, \tau_2, \sigma_m \rangle.$$

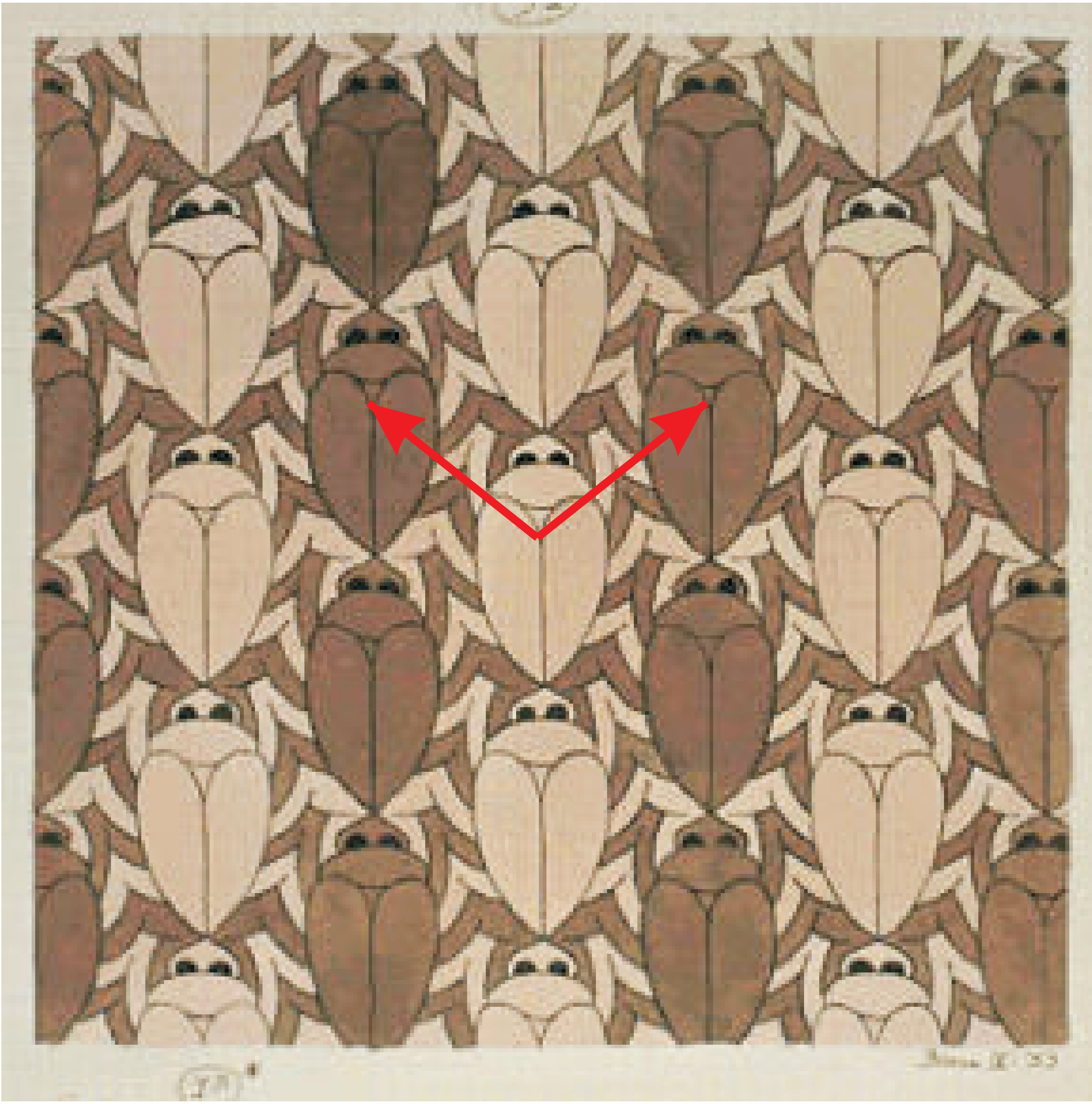


An example of a pattern whose symmetry group is

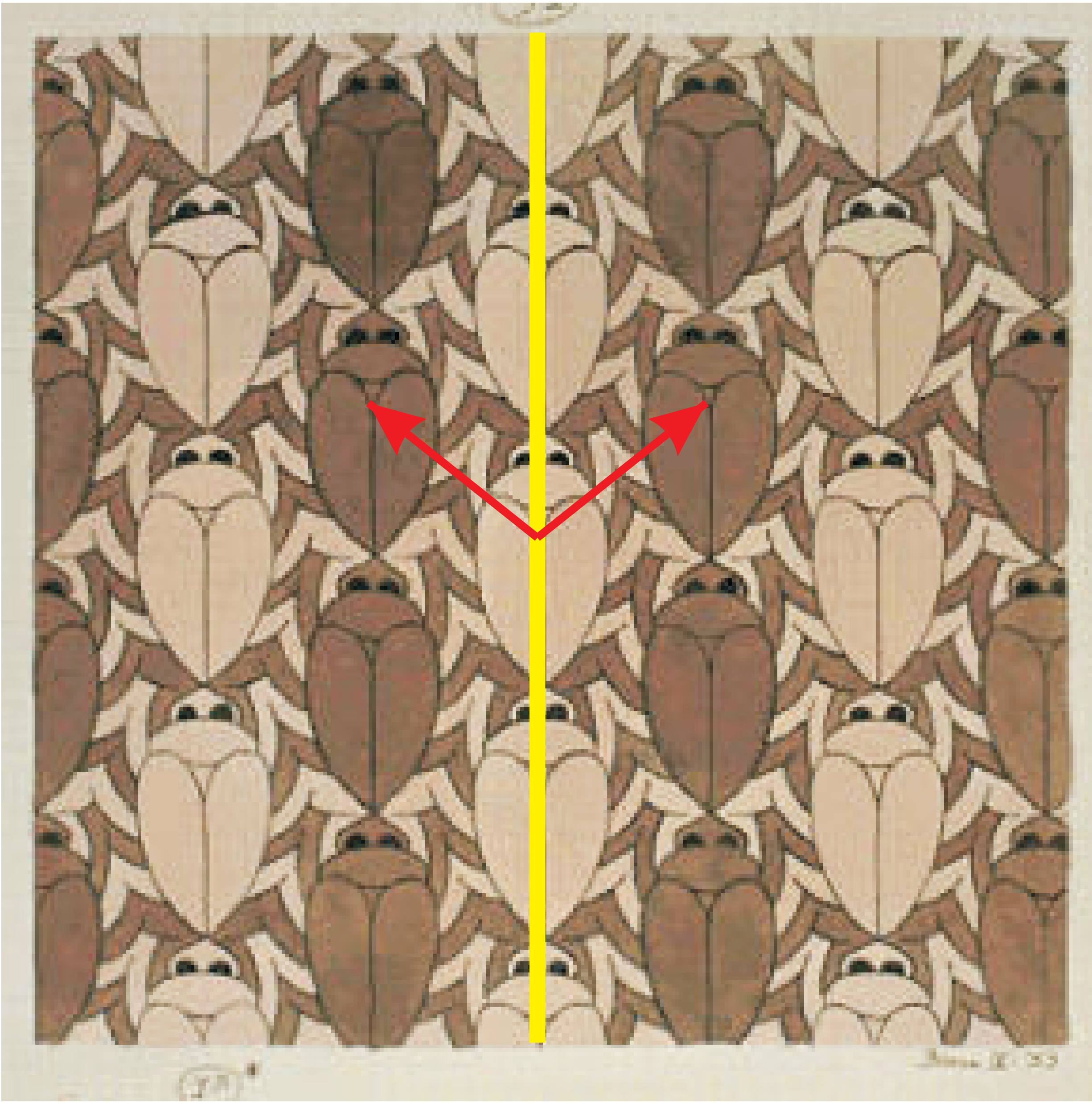
$$W_1^1 = \langle \tau_1, \tau_2, \sigma_m \rangle.$$

Notice a glide reflection whose axis is not a reflection line.

Here is an example of a painting by Escher with symmetry group W_1^1 (ignoring colors).

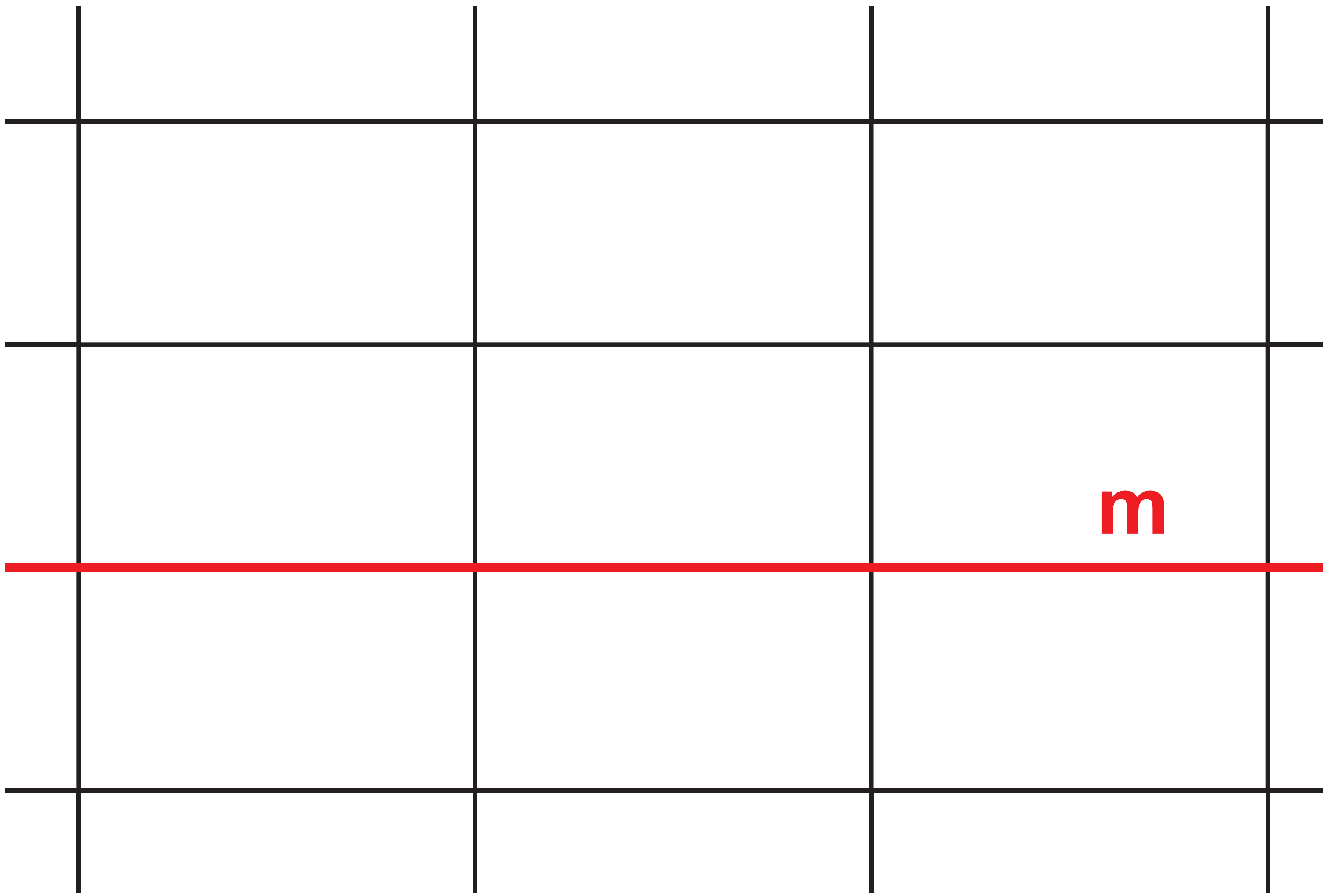


Here are two generating translations of equal length.

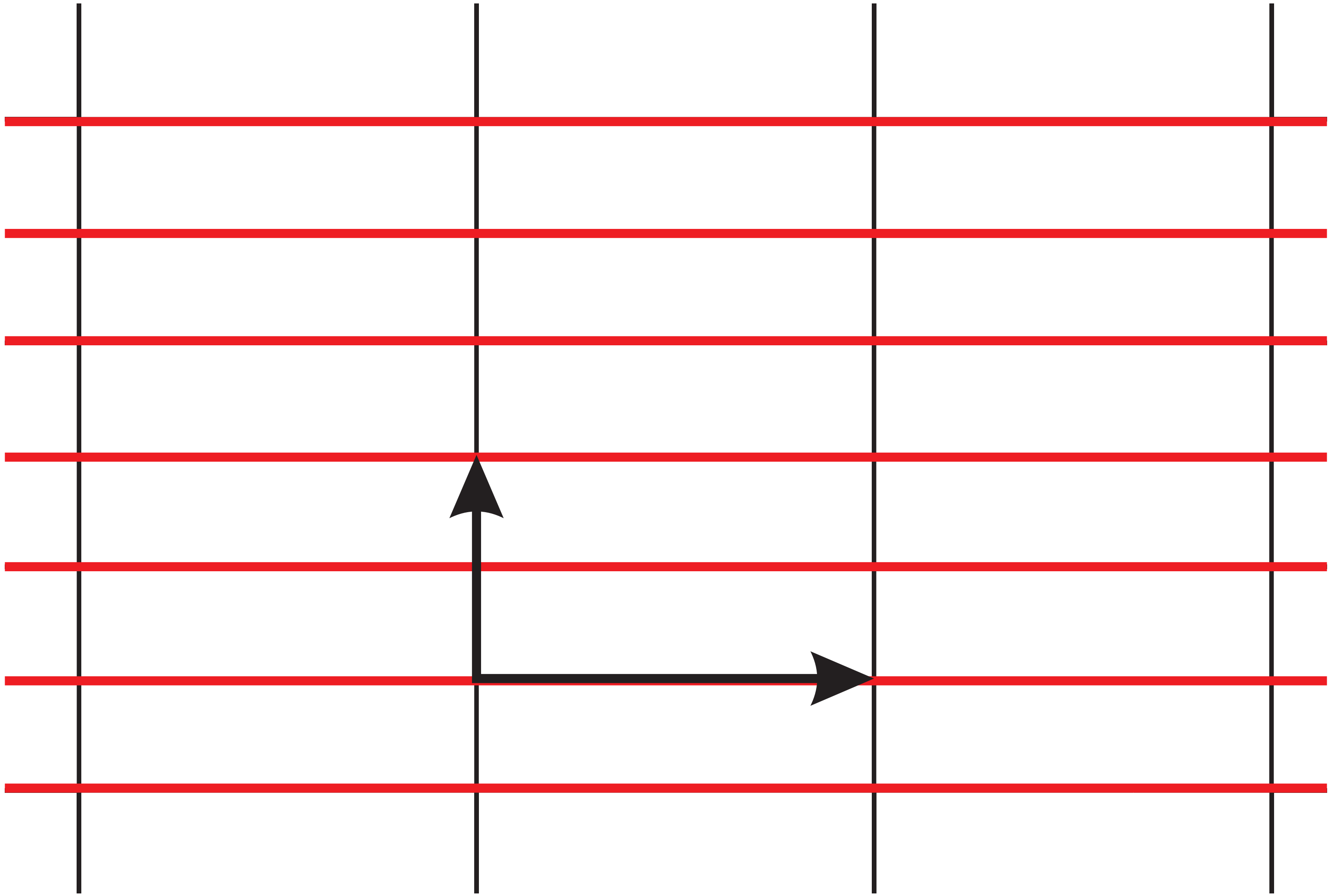


Here is a line of reflection parallel to $\tau_1\tau_2$.

Also, notice a glide reflection whose axis not a line of reflection. This separates W_1^1 from the next group W_1^2.

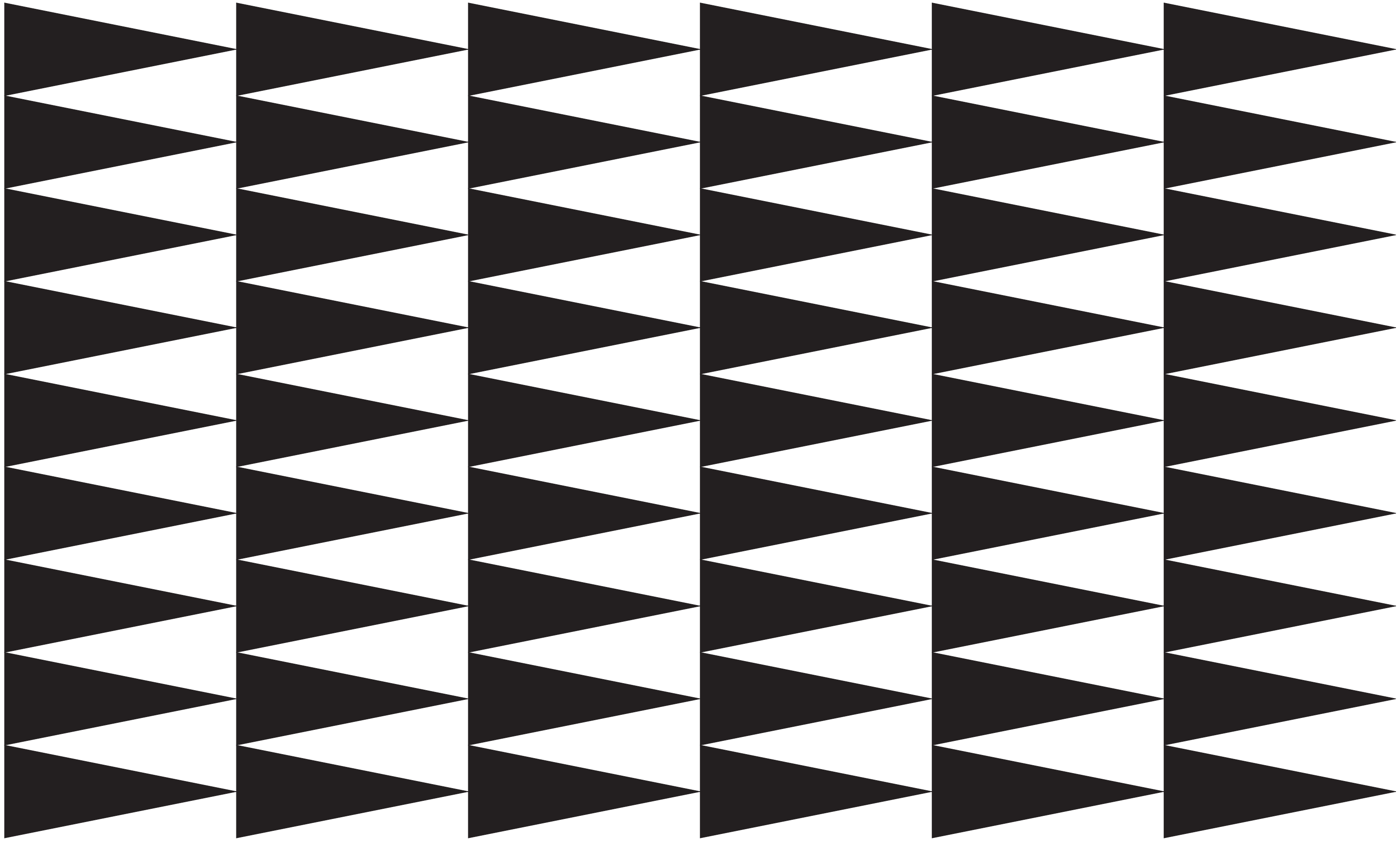


Next case (a1): $\sigma_m \in G$ and $\tau_1 \parallel m$ and $\tau_2 \perp m$. In this case the lattice is rectilinear, and m is parallel to a generator τ_1.

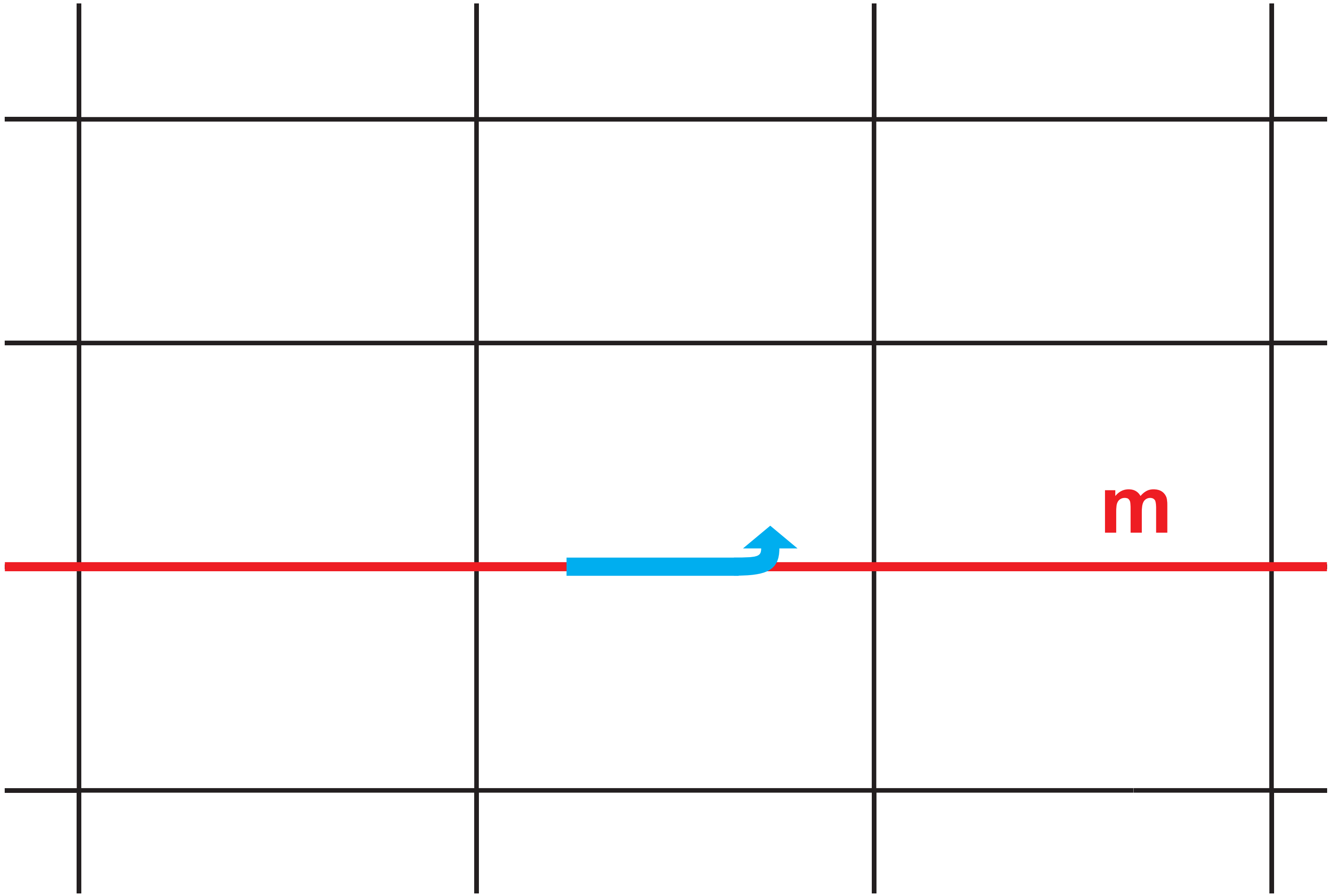


We have the symmetry group

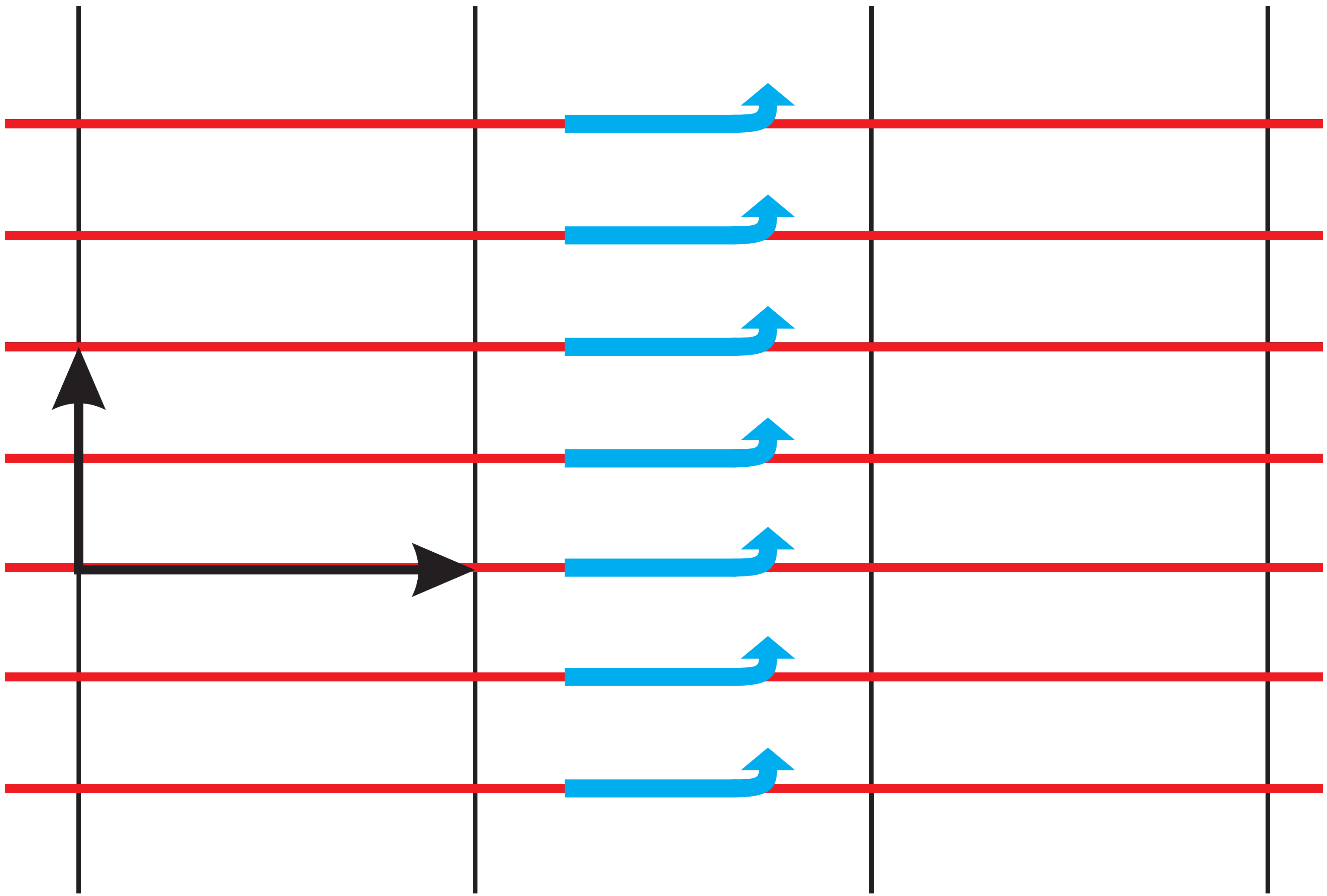
$$W_1^2 = \langle \tau_1, \tau_2, \sigma_m \rangle.$$



An example of a pattern whose symmetry group is $W_1^2 = \langle \tau_1, \tau_2, \sigma_m \rangle$. Notice that any axis of a glide reflection is also a line of symmetry.

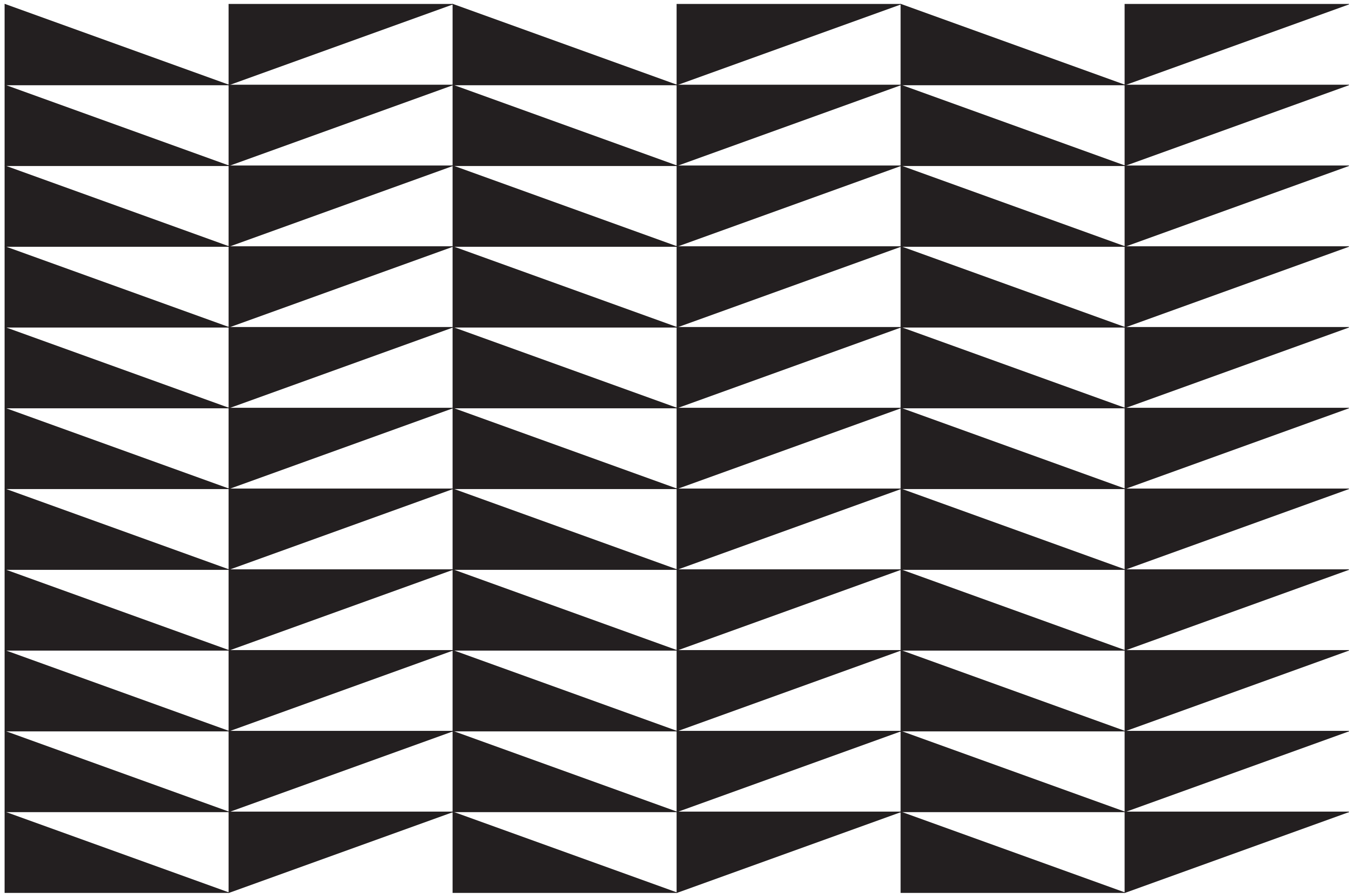


The final(!) case (b1): G does not contain a reflection but it contains a glide reflection γ with axis m. Case (1) of the lemma holds, so $\tau_1 \parallel m$ and $\tau_2 \perp m$.



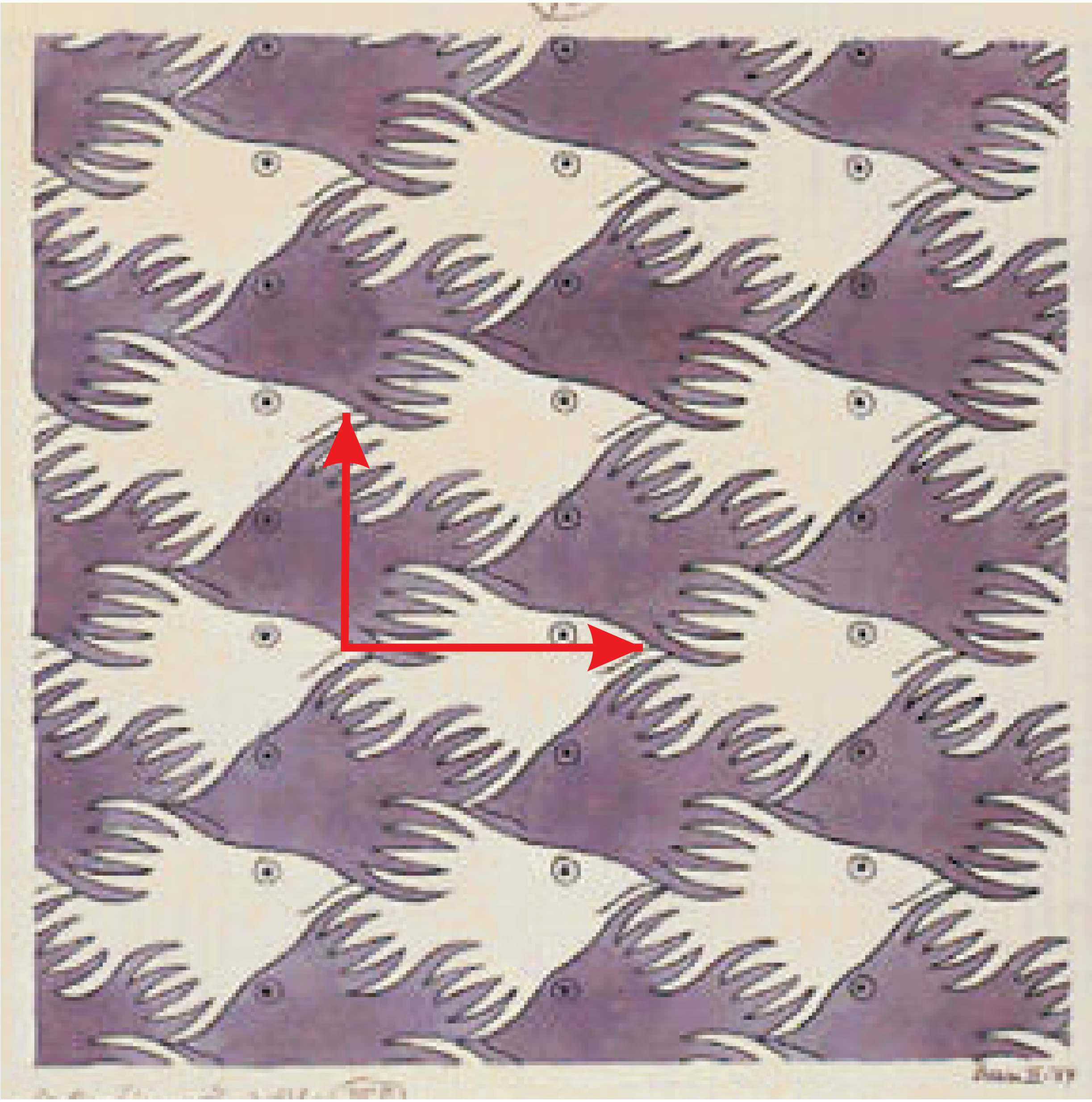
We can choose the glide so that $\gamma^2 = \tau_1$. We obtain the wallpaper group

$$W_1^3 = \langle \gamma, \tau_2 \rangle.$$

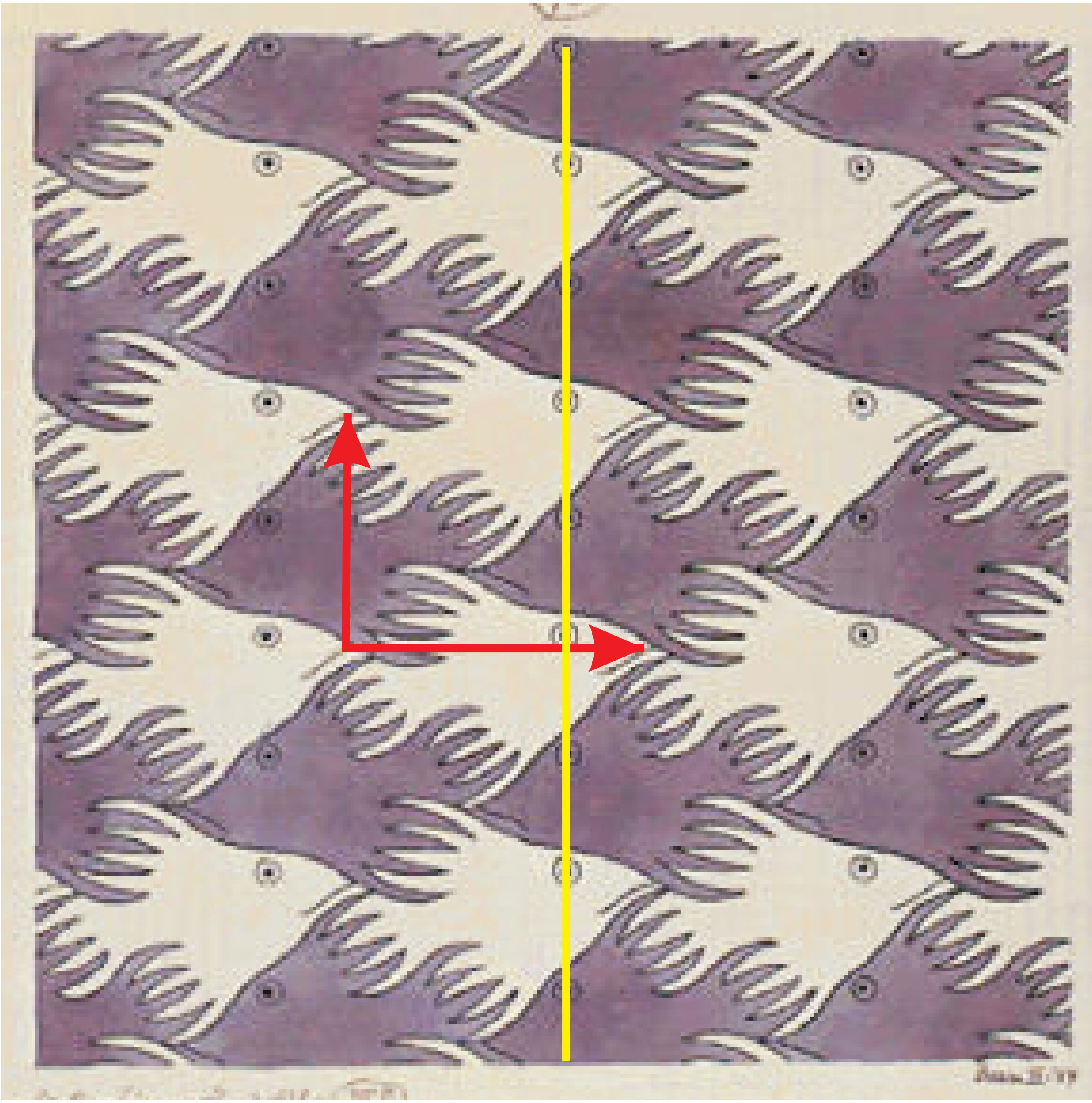


An example of a pattern whose symmetry group is $W_1^3 = \langle \gamma, \tau_2 \rangle$.

Here is an example of a painting by Escher with symmetry group W_1^3 (ignoring colors).



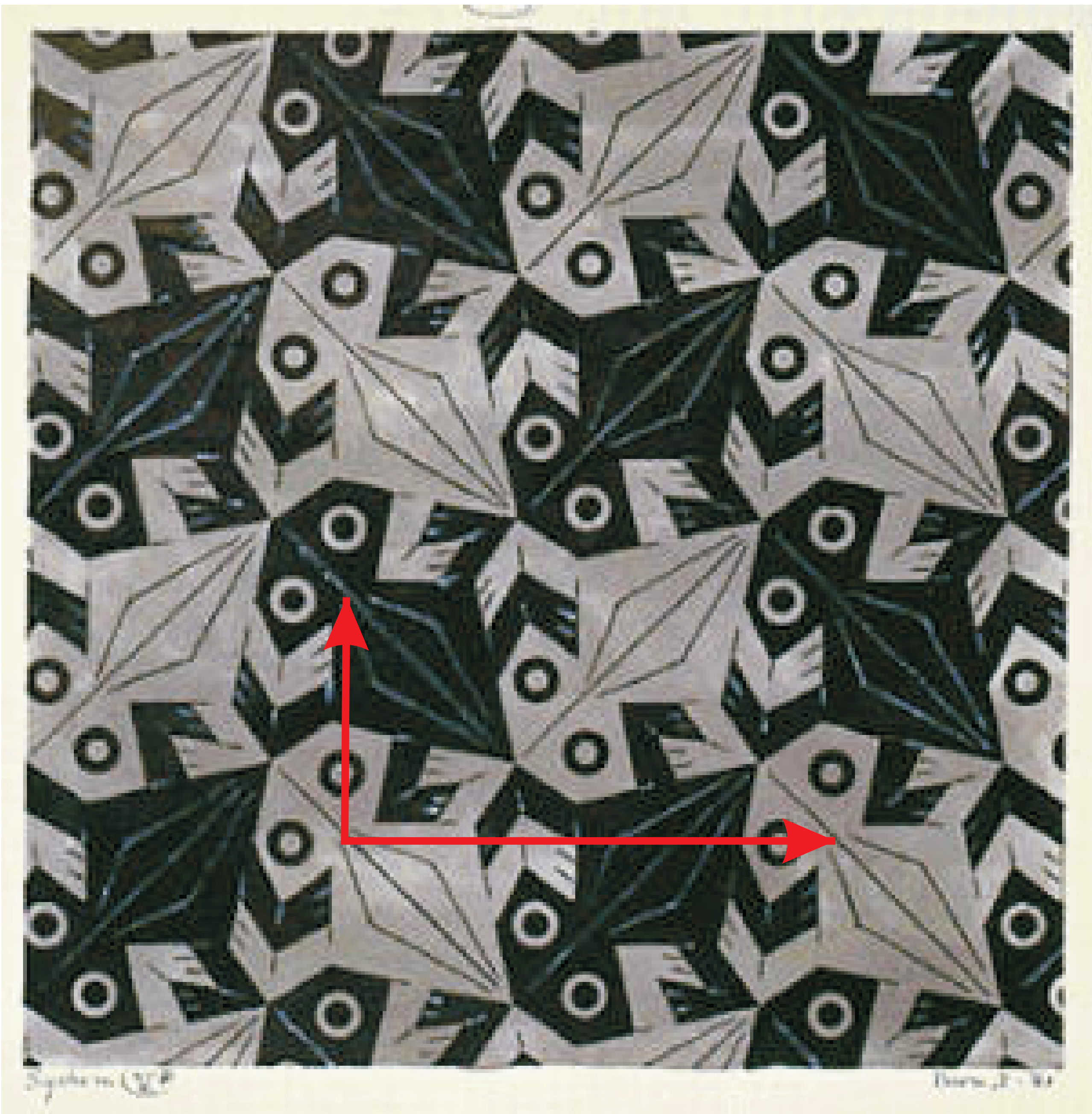
Here are perpendicular generating translations.



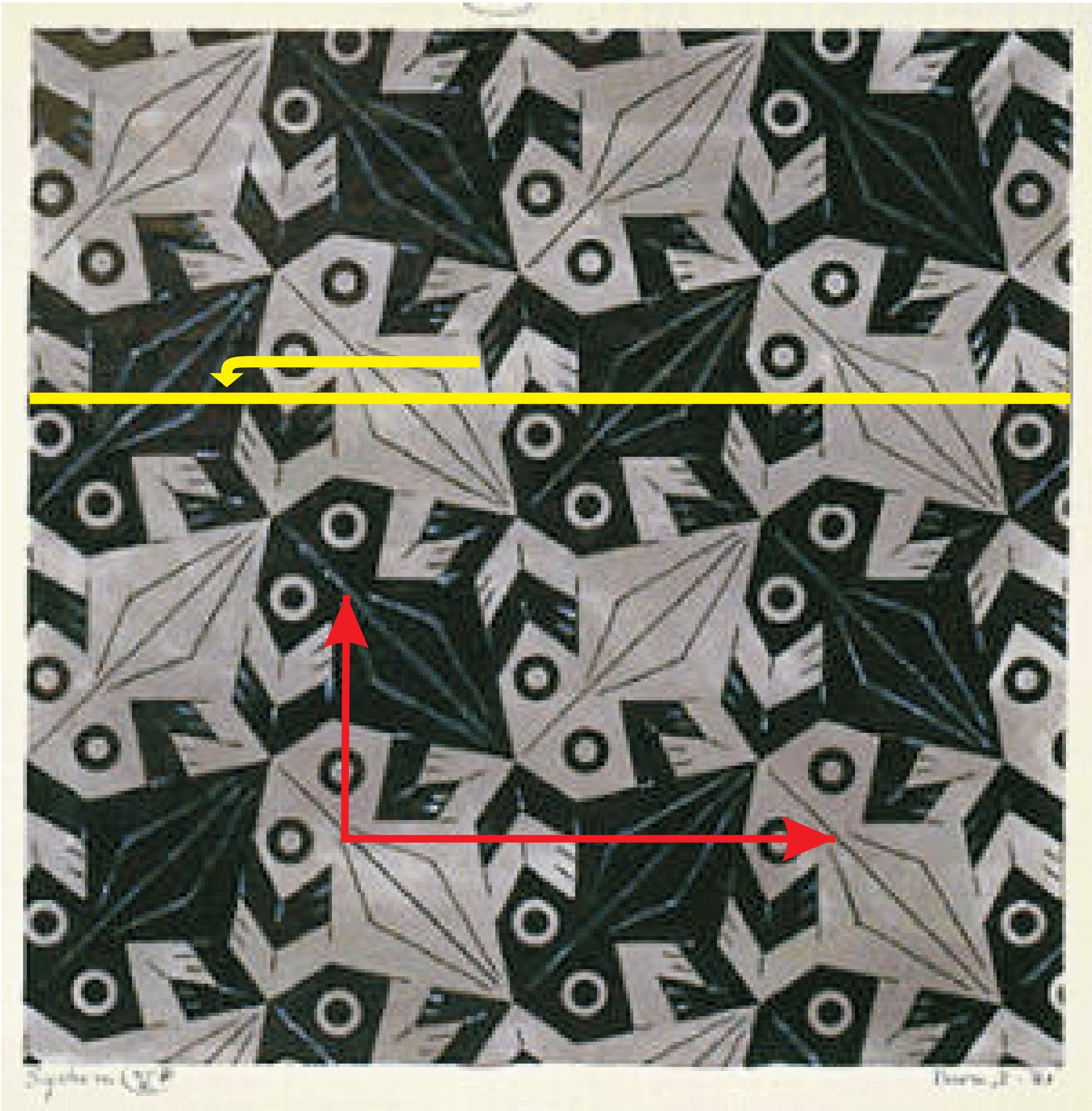
And here is an axis of a glide reflection.

Here is another, analogous example.

A third example of group W_1^3.



Perpendicular generating translations.



And a line of a glide reflection.

Yet one more example of W_1^3.

Our notation	Fejes Tóth	Crystallographic
W_6	W_6	p6
W_6^1	W_6^1	p6m
W_3	W_3	p3
W_3^1	W_3^1	p3m1
W_3^2	W_3^2	p31m
W_4	W_4	p4
W_4^1	W_4^1	p4m
W_4^2	W_4^2	p4g
W_2	W_2	p2
W_2^1	W_2^1	cmm
W_2^2	W_2^2	pmm
W_2^3	W_2^3	pmg
W_2^4	W_2^4	pgg
W_1	W_1	p1
W_1^1	W_1^1	cm
W_1^2	W_1^2	pm
W_1^3	W_1^3	pg