Cellular Automata. Homework 10 (28.3.2022)

1. Determine all conserved quantities of the three state reversible CA of Example 4 (page 19) of the notes, also present in last week's exercise number 4. More precisely, form the linear equations for $\mu(0), \mu(1)$ and $\mu(2)$ as explained in the notes, and solve the system of equations.

2. Determine all conserved quantities $\mu : S \longrightarrow \mathbb{R}$ of the one-dimensional CA that uses neighborhood $(0, 1)$, has state set $S = \{0, 1, 2, 3\}$ and the value $f(a, b)$ of the local rule f is given in the following table:

a \ b	0	1	2	3
0	0	1	1	2
1	0	1	1	2
2	1	2	2	3
3	1	2	2	3

3. Consider a radius-$\frac{1}{2}$ one-dimensional CA with a quiescent state and a local rule $f : S^2 \longrightarrow S$. Let $\mu : S \longrightarrow \mathbb{R}$ be a conserved quantity of the CA. Prove that there are functions

$$\mu_L, \mu_R : S \longrightarrow \mathbb{R}$$

(called the left and the right *flux*) such that for every $a \in S$

$$\mu(a) = \mu_L(a) + \mu_R(a)$$

and for every $a, b \in S$

$$\mu(f(a, b)) = \mu_R(a) + \mu_L(b).$$

(This means that the value $\mu(a)$ is split into the left flux $\mu_L(a)$ and the right flux $\mu_R(a)$ that flow into the two neighbors. The two incoming fluxes are the added together in every cell to obtain $\mu(f(a, b))$.) Hint: define $\mu_L(a) = \mu(f(q, a))$ and $\mu_R(a) = \mu(f(a, q))$ where q is the quiescent state.

4. The concept of conserved quantities can be generalized in various ways. One approach is to assign real values to finite patterns rather than individual cells. Then each occurrence of pattern p in configuration c contributes $\mu(p)$ to the value $\hat{\mu}(c)$.

   (a) Determine if the number of occurrences of pattern 01 is conserved in the traffic CA.

   (b) Prove that in the elementary CA with Wolfram number 50, the total number of occurrences of patterns 01100110 and 10011001 is conserved.

   (c) Prove that in the elementary CA with Wolfram number 14, the number of occurrences of pattern 01 is conserved.

5. Q2R is a two-dimensional CA that models the Ising spin dynamics. It uses state set $S = \{-1, +1\}$, representing two possible spin values. The states are updated in two rounds as follows: Imagine the cells colored black and white as in an infinite checker board. In the first round only the black cells update their states and in the second round only the white cells update their states. The update rule is the following: A cell checks the states of its four immediate neighbors. If exactly half of them are in state +1 and half of them are in state -1 then the cell flips its state. Otherwise the state remains unchanged.

   (a) Is this CA reversible ? Surjective ?

   (b) Let us define the energy between two neighboring cells as -1 if their states are the same and +1 if their states are different. Show that the number of neighboring pairs with different states is conserved. (So the total energy is preserved unchanged.)

   (c) Prove that every finite configuration is temporally periodic. (Due to symmetry, one can use either state as the quiescent state.)

6. Let X be a compact space, and suppose the topology has a base all of whose members are clopen (closed and open). Prove that a set is clopen if and only if it is a finite union of base sets.

7. Let X and Y be metric spaces, with metrics d and e, respectively. Function $f : X \longrightarrow Y$ is called uniformly continuous if for every $\varepsilon > 0$ there exists $\delta > 0$ such that

$$d(x, y) < \delta \Longrightarrow e(f(x), f(y)) < \varepsilon.$$

   (The difference to continuity is that the choice of δ is independent of x.) Prove: If X is compact then every continuous $f : X \longrightarrow Y$ is uniformly continuous.