Cellular Automata. Homework 11 (4.4.2022)

1. Let $(X, \mathcal{T})$ be a topological space. Prove the following properties of the topological closure $\overline{A}$ of $A \subseteq X$.
   (a) If $A \subseteq B$ then $\overline{A} \subseteq \overline{B}$,
   (b) $\overline{A \cup B} = \overline{A} \cup \overline{B}$,
   (c) $\overline{\overline{A}} = \overline{A}$.

2. (a) Prove that in a metric space X, every compact $A \subseteq X$ is closed and bounded. (Bounded means that A is contained in some ε-ball.)
   (b) Prove that the converse is not true: there is a metric space where a closed and bounded set is not necessarily compact.

3. Prove that a configuration is in the limit set of the traffic CA if and only if it does not contain consecutive 1's to the right of consecutive 0's, that is, no pattern

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appears in c. (Traffic CA is the elementary CA number 226.)

4. Determine the limit set of the majority CA (elementary CA number 232).

5. CA G is called *stable* if there exists $n \geq 0$ such that

$$\Omega_G = G^n(S^{\mathbb{Z}^d}),$$

where Ω_G is the limit set. Determine which of the following cellular automata are stable.
   (a) Elementary CA 128 (see Example 38, page 107).
   (b) Xor CA (=Elementary CA 102).
   (c) Majority CA. (=Elementary CA 232).

6. See the previous problem for the definition of stable CA.
   (a) Show that all surjective CA are stable, and that all eventually periodic CA are stable.
   (b) Prove that it is undecidable if a given one-dimensional CA is stable.

7. Prove that if the limit set of a CA is a subshift of finite type (SFT) then the CA is stable. (Recall that a subshift of finite type is defined by forbidding a finite number of finite patterns. In other words, SFT is the complement of $\bigcup_{\tau} \tau(C)$ for some clopen set C, where the union is over all translations.)