Cellular Automata. Homework 8 (14.3.2022)

1. Determine if there exists a one-dimensional CA $G : S^{\mathbb{Z}} \longrightarrow S^{\mathbb{Z}}$ for which the following decision problem is RE-complete: "Does a given spatially periodic initial configuration eventually become a configuration that contains some state from a fixed accepting state set $H \subseteq S$?"

2. A configuration $c \in S^{\mathbb{Z}}$ is called *recursive* if there exists an algorithm A that with input $n \in \mathbb{Z}$ returns $c(n)$. Let us call CA "universal" if the following question is RE-complete for some fixed accepting state $h \in S$: "Does a given recursive configuration c evolve into a configuration where cell 0 is in state h ?" (The recursive configuration is specified by giving an algorithm A that computes it.)

   Prove that the left shift σ is "universal" under this definition.

   [This means that this definition of "universality" is too general.]

3. Let us call a CA *totalistic* if its state set is $\{1, 2, \ldots, s\}$ and the local rule f satisfies

$$s_1 + \cdots + s_m = r_1 + \cdots + r_m \Longrightarrow f(s_1, \ldots, s_m) = f(r_1, \ldots, r_m).$$

   (In other words: the sum of the states in the neighborhood determines the next state.) Which of the following CA are totalistic (after suitably renaming the states) ?

   (a) The *xor* CA of Example 1 from the notes.
   (b) The *majority rule* of the Homework set 1, problem 6.
   (c) The *traffic* CA defined in the Homework set 5.

4. Prove that a one-dimensional totalistic CA with at least two states and neighborhood vector $N = (1, 2, \ldots, m)$ of $m \geq 2$ consecutive cells is never reversible.

In the last three problems we use the following concepts: A reversible CA G is called an *involution* if it is its own inverse, that is, if $G \circ G = id$. A reversible CA F is *time-symmetric* if there exists an involution G such that $F^{-1} = G \circ F \circ G$.

5. (a) Prove that F is time-symmetric if and only if it is the composition of two involutions.
   (b) Prove that if F is time-symmetric then also F^{-1} is time-symmetric, and F^n is time-symmetric, for all $n \in \mathbb{N}$.
   (c) Prove that if F is time-symmetric and H is reversible then $H \circ F \circ H^{-1}$ is time-symmetric.

6. Determine if the following 1D CA are time-symmetric:

   (a) The left shift σ,
   (b) The radius-0 CA with n states $1, 2, \ldots, n$ that rotates the states

$$1 \mapsto 2 \mapsto 3 \mapsto \ldots \mapsto n \mapsto 1.$$

   (c) The one-dimensional partitioned CA with state set $S = \{0,1\} \times \{0,1\} \times \{0,1\}$, neighborhood vector $N = (-1, 0, 1)$ and the local rule determined by the permutation $\pi : S \longrightarrow S$ that maps

$$(a, b, c) \mapsto (a, a \oplus b \oplus c, c).$$

   Here, "$\oplus$" denotes the modulo two addition.

7. (a) Let $G : S^{\mathbb{Z}^d} \longrightarrow S^{\mathbb{Z}^d}$ be any reversible CA with the state set S. Prove that $F = G \times G^{-1}$ is time-symmetric: The state set of F is $S \times S$. In the first component, CA G is executed, while on the second component, CA G^{-1} is executed.

   [More precisely: F maps $(c_1, c_2) \mapsto (G(c_1), G^{-1}(c_2))$, for any $c_1, c_2 \in S^{\mathbb{Z}^d}$, where we denote by (c_1, c_3) the element of $(S \times S)^{\mathbb{Z}^d}$ whose first and second layers read c_1 and c_2, respectively, that is, for all cells $\boldsymbol{n} \in \mathbb{Z}^d$: $(c_1, c_2)(\boldsymbol{n}) = (c_1(\boldsymbol{n}), c_2(\boldsymbol{n}))$.]

   (b) Prove that there exists a one-dimensional time-symmetric CA that is universal in the sense that the decision problem: "Does a given finite initial configuration evolve into a configuration in which the state of some cell belongs to A ?" is RE-complete for some fixed $A \subseteq S$.