Cellular Automata. Homework 9 (21.3.2022)

1. Let F be the one-dimensional CA with the state set $S = \{0, 1, 2, 3, 4, 5\}$, radius-$\frac{1}{2}$ neighborhood $N = (0, 1)$ and the local rule $f : S^2 \longrightarrow S$ that maps for all $x_1, x_2 \in \{0, 1, 2\}$ and $y_1, y_2 \in \{0, 1\}$
$$f(2x_1 + y_1, 2x_2 + y_2) = 3y_1 + x_2.$$
(Note that f is well defined since every element of S can be uniquely expressed in the form $2x + y$ for $x \in \{0, 1, 2\}$ and $y \in \{0, 1\}$.

   Show that F is a partitioned CA and hence reversible after renaming the state set S as a cartesian product $S_1 \times S_2$ of suitable sets S_1 and S_2.

2. Show that F defined in Problem 1 above multiplies by 3 any positive integer expressed in base-6. More precisely, show that 0-finite configurations are mapped as follows:
$$F(\ldots 000w_{n-1} \ldots w_0 000 \ldots) = \ldots 000u_n u_{n-1} \ldots u_0 000 \ldots$$
where
$$\sum_{i=0}^{n} 6^i u_i = 3\left(\sum_{i=0}^{n-1} 6^i w_i\right).$$

3. Using the CA of the previous problem, construct a one-dimensional CA over the state set $\{0, 1, 2, 3, 4, 5, \diamond\}$ that simulates the Collatz function $f : \mathbb{N} \longrightarrow \mathbb{N}$
$$f(n) = \begin{cases} n/2, & \text{if } n \text{ is even,} \\ 3n + 1, & \text{if } n \text{ is odd.} \end{cases}$$
Positive integer n is represented by the configuration
$$c_n = \ldots 000w_{m-1} \ldots w_0 \diamond 000 \ldots$$
where
$$n = \sum_{i=0}^{m-1} 6^i w_i.$$
The CA should transform in one step any c_n into (a possibly shifted version of) $c_{f(n)}$.

4. Recall the one-dimensional reversible CA of Example 4, page 19 in the notes. It has three states, neighborhood $N = (0, 1)$ and the local rule $f(a, b)$ given by the table

a \ b	1	2	3
1	1	1	2
2	2	2	1
3	3	3	3

   (a) Find all left and right stairs when radius $r = 1$ is used.

   (b) Show that the CA can be defined using one-dimensional Margolus neighborhood, that is, as a composition of two permutations on overlapping partitionings of the configuration into segments of length two.

   (c) Show that the 2-block presentation of $G \circ \sigma$ is partitioned (when the states are renamed appropriately). Give the partitioning $S_1 \times S_2$ used, and the permutation π.

5. (a) Show that it is undecidable if a given two-dimensional CA G that is an involution (i.e., $G^{-1} = G$) has any fixed points.
   (b) Show that it is undecidable if a given partitioned two-dimensional CA G with the von Neumann neighborhood has any fixed points.

6. Determine how the HPP lattice gas (defined in Section 4.2 of the notes) evolves from a finite initial configuration. More precisely, show that if the particles initially fit inside a $w \times h$ box then after $\max\{w, h\}$ steps no more particle collisions happen, and all particles continue on straight trajectories.

7. Prove that there is an algorithm to determine for given finite configurations c_1 and c_2 whether c_1 evolves into c_2 under the BBM cellular automaton (defined in Section 4.2 of the notes). (So BBM is not "computationally universal" on finite configurations.)