Cellular Automata. Homework 12 (Tuesday 12.4.2022)

1. Let G be the majority CA, and let

$$U = \{c \in \{0,1\}^{\mathbb{Z}} \mid c(0) = c(1) \text{ or } c(0) = c(-1)\}.$$

   (a) Verify that U is clopen, non-empty and satisfies $G(U) \subseteq U$.

   (b) Find the attractor A determined by U.

   (c) Determine the basin of attraction for the attractor you found in (b).

2. A subset $X \subseteq S^{\mathbb{Z}^d}$ is called *inward* for CA G if $G(\overline{X}) \subseteq X^\circ$, where $\overline{X}$ and X° are the closure and the interior of X, respectively, that is, $\overline{X}$ is the intersection of all closed supersets of X and X° is the union of all open subsets of X.

   (a) Prove that a clopen U is inward if and only if $G(U) \subseteq U$.

   (b) Prove that if X is inward then there exists an inward clopen U such that

$$G(X) \subseteq U \subseteq X$$

   (c) Prove that if

$$A = \bigcap_{n=0}^{\infty} G^n(X)$$

   for some inward set X then

$$A = \bigcap_{n=0}^{\infty} G^n(U)$$

   for some inward clopen set U.

   (Note: Attractors are usually defined using arbitrary inward sets. This exercise shows that our definition based on clopen inward sets is equivalent.)

3. Let G be a surjective CA. Prove that the attractors of G are exactly those clopen sets U that satisfy $G(U) = U$. Also prove that if U is an attractor then also its complement $S^{\mathbb{Z}^d} \setminus U$ is an attractor. (Hint: use the balanceness property to prove that in surjective CA all clopen inward sets U satisfy $G^{-1}(U) = U$.)

4. An attractor A that is a subshift (i.e. $\tau(A) = A$ for every translation τ) is called a *subshift attractor*.

   (a) Prove that if a CA has a minimal attractor A then A is a subshift attractor.

   (b) Prove that the basin of attraction for any subshift attractor is dense. (In other words, every cylinder contains a configuration that is in the basin of attraction.)

5. Determine all equicontinuity points of the given CA, and determine if the CA is sensitive to initial conditions:

   (a) The traffic CA (=elementary CA 226).

   (a) The majority CA (=elementary CA 232).

6. Suppose G is a CA whose neighborhood vector contains $\vec{0}$ and all unit coordinate vectors $(0, \dots 0, \pm 1, 0, \dots 0)$. Suppose also that G has a *spreading state* $q \in S$ with the property that $f(x_1, x_2, \dots, x_m) = q$ if some $x_i = q$.

   (a) Show that G is not sensitive.

(b) Show that G is equicontinuous if and only if it is nilpotent.

(c) Show that G has a minimal attractor, and determine that attractor.

7. Let G be the elementary CA number 108.

   (a) Show that 00 is a 2-blocking word and that 01110 is a 3-blocking word.

   (b) Show that words 1111, 0110110, 010110, 011010 and 10101 are 2-blocking.

   (c) Show that all words of length 7 are 2-blocking.

   (d) Determine all equicontinuity points of G. Is G stable ?