Isometries of $\mathbb{R}^2$

Let

$$d : \mathbb{R}^2 \times \mathbb{R}^2 \longrightarrow [0, \infty)$$

be the usual **Euclidean distance** defined by

$$d((x_1, y_1), (x_2, y_2)) = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}.$$

A plane **isometry** is a function

$$\alpha : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$$

that preserves distances:

$$\forall (x_1, y_1), (x_2, y_2) \in \mathbb{R}^2 \quad : \quad d(\alpha(x_1, y_1), \alpha(x_2, y_2)) = d((x_1, y_1), (x_2, y_2)).$$

In other words, α defines a "rigid" motion that does not change any distances.

Remark. Isometries can be studied using analytic geometry. Isometries are exactly the affine transformations

$$\begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} e \\ f \end{pmatrix}$$

whose coefficient matrix

$$M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

is **orthogonal**, meaning that

$$MM^T = I,$$

the 2×2 identity matrix.

Using this fact, all the results that we prove in the following can be proved algebraically with Cartesian coordinates. However, the geometric approach that we use, although sometimes longer, has inner "mathematical beauty" and provides insights hidden by the algebraic calculations.

Forgetting coordinates, we denote points of the plane by capital letters, so the isometry property will be written as

$$\forall P, Q \in \mathbb{R}^2 \quad : \quad d(\alpha(P), \alpha(Q)) = d(P, Q).$$

Theorem. An isometry is a bijection. Its inverse function is an isometry.

Proof.

The next observation states that isometries preserve shapes: they map every line into a line, every triangle into a triangle, and the angle between two lines remains the same. Also betweenness and midpoints are preserved.

Theorem. An isometry preserves lines, triangles, betweenness, midpoints, sizes of angles, and perpendicularity and parallelism of lines.

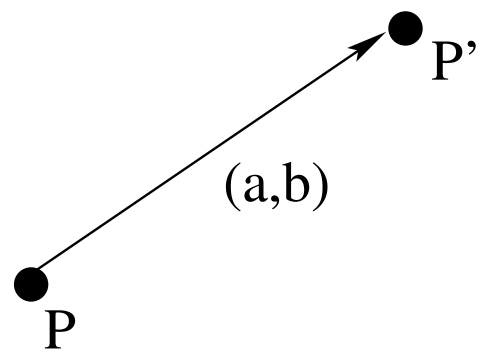
Proof.

Example. The **trivial isometry** is the identity function ι that does not move any points:

$$\iota(P) = P$$

for all $P \in \mathbb{R}^2$.

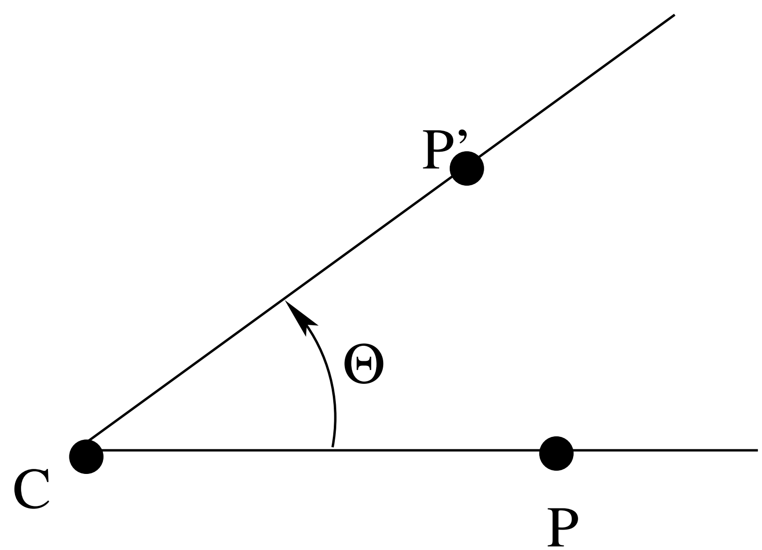
Example. Let $A = (a, b) \in \mathbb{R}^2$. The **translation** by vector $A = (a, b)$ shifts every point (x, y) into position $(x + a, y + b)$. We denote τ_A for the translation by vector A.



Every translation is an isometry. The trivial translation $\tau_{(0,0)}$ is the trivial isometry ι.

Example. Let $C \in \mathbb{R}^2$ be a point, and $\Theta \in \mathbb{R}$ an angle. The **rotation** $\rho_{C,\Theta}$ by the (directed) angle Θ about C is the isometry that

- fixes point C, and
- takes every point $P \neq C$ into the point P' where $d(C, P) = d(C, P')$ and Θ is the directed angle from CP to CP':



In terms of analytic geometry, a point (x, y) is mapped to the point (x', y') given by the formula

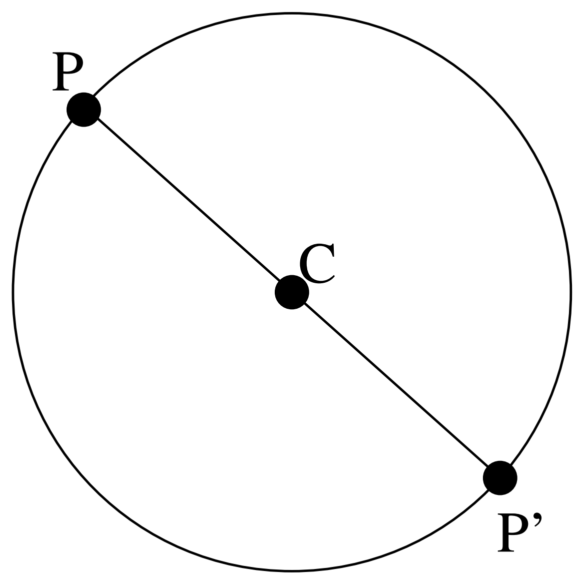
$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} \cos\Theta & -\sin\Theta \\ \sin\Theta & \cos\Theta \end{pmatrix} \begin{pmatrix} x - c_x \\ y - c_y \end{pmatrix} + \begin{pmatrix} c_x \\ c_y \end{pmatrix}$$

where $C = (c_x, c_y)$.

The trivial rotation $\rho_{C,0}$ by the angle $0°$ is the trivial isometry ι.

If $\Theta = 180°$ we get a special rotation called the **halfturn** about point C, also known as the reflection in point C.

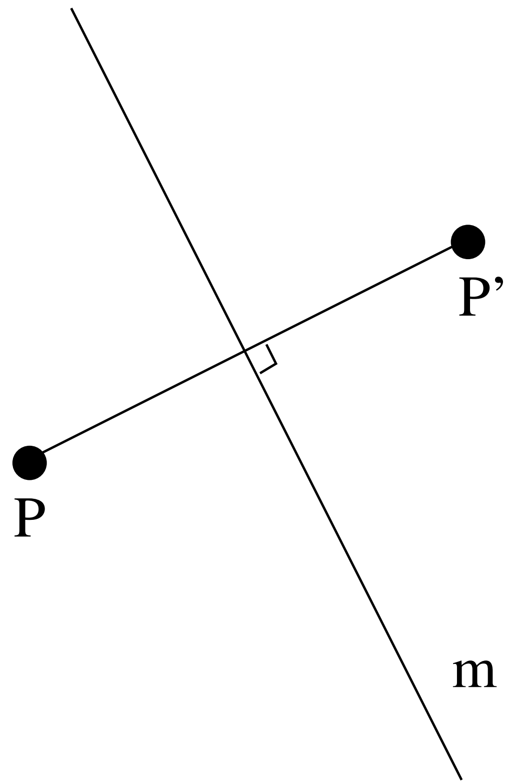
Every point P is mapped to the point P' such that the center C is the midpoint between P and P':



We introduce the special symbol σ_C for the half turn about point C. In other words,

$$\sigma_C = \rho_{C,180°}.$$

Example. Let $m \subseteq \mathbb{R}^2$ be a line. The **reflection** σ_m in line m does not move the points of line m, but any point P outside line m is moved to the point P' such that line m is the perpendicular bisector of segment PP':



Clearly a reflection σ_m is its own inverse:

$$\sigma_m^{-1} = \sigma_m.$$

Isometries that are their own inverses are called **involutions**.

Example. A **Glide reflection** is a composition of a translation and a reflection in line m that is parallel with the direction of the translation.

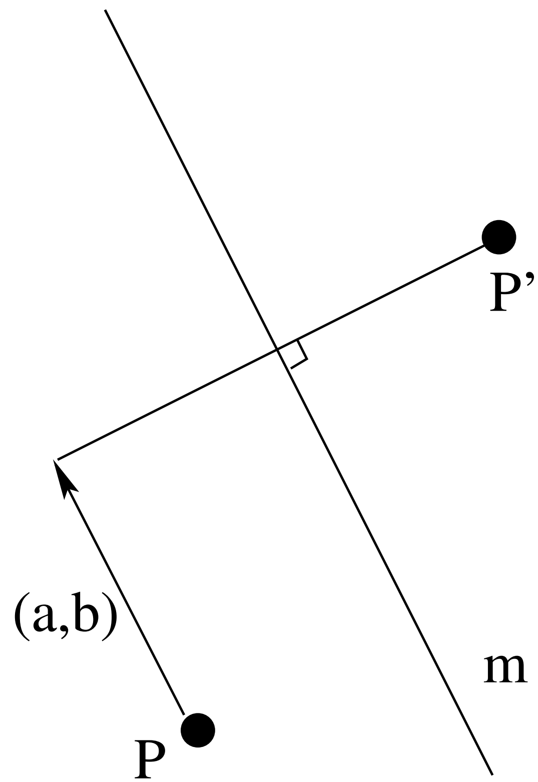
Let $A = (a, b) \in \mathbb{R}^2$ a vector of translation, and let m be a line parallel to A. That is: if $(a, b) \neq (0, 0)$ then

$$m = \{(c, d) + t(a, b) \mid t \in \mathbb{R}\}$$

where (c, d) is some point of the line, and if $(a, b) = (0, 0)$ then m can be any line.

The glide reflection $\gamma_{m,A}$ reflects the points in line m and then translates them by vector A. In this particular case it does not matter in which order the two operations are performed; we may as well translate first and reflect later:

$$\gamma_{m,A} = \sigma_m \circ \tau_A = \tau_A \circ \sigma_m.$$



Line m is called the **axis** of the glide reflection. Notice that glide reflections with trivial translation vectors $A = (0,0)$ are exactly the reflections.

Our four examples **exhaust all possibilities** (this will be proved later): translations, rotations, reflections and glide reflections are the only isometries of the plane.

And, since reflection is a special type of glide reflection, we can say that all isometries are translations, rotations or glide reflections.

The composition

$$\alpha \circ \beta$$

of two functions α and β is the function that first applies β to a point, and then applies α to the result:

$$(\alpha \circ \beta)(x) = \alpha(\beta(x)).$$

If α and β are isometries then also their **composition $\alpha \circ \beta$ is an isometry**.

Proof.

Function composition $\circ$ is an associative operation, and since the identity function ι and the inverses of all isometries are also isometries, we have the following theorem:

Theorem. The set of plane isometries is a **group $\mathcal{I}$** under the operation of composition.

We frequently drop the group operation sign "$\circ$" and simply write $\alpha\beta$ for $\alpha \circ \beta$. We then say that $\alpha\beta$ is the **product** of α and β.

We also do not need to use parentheses in products as

$$\alpha(\beta\gamma) = (\alpha\beta)\gamma.$$

We simply write this as $\alpha\beta\gamma$.

However, the group of isometries is **not commutative** (=abelian) as in most cases $\alpha\beta \neq \beta\alpha$.

An element $\alpha \in \mathcal{I}$ is called an **involution** if $\alpha^2 = \iota$.

Examples of involutions include all **reflections** in lines, all **half turns** and the **trivial isometry** ι. No other involutions exist.

Next we try to understand the structure of the group $\mathcal{I}$. We learn to form the products of different isometries, show that reflections generate all isometries, and prove that our examples exhaust all possibilities.

Review the following terms of group theory:

- **generator set** (=set of group elements such that every element of the group is a product of generators and their inverses),
- **cyclic group** (=a group that is generated by one element)
- **order of a group** (=number of elements. If the group contains an infinite number of elements then the group is called infinite, otherwise it is finite.)
- **subgroup** (=a subset of the group that is closed under the group operation and the operation of taking the inverse element. A subgroup itself is a group under the same group operation)
- **cancellation laws**:

$$\alpha\beta = \alpha\gamma \implies \beta = \gamma,$$
$$\beta\alpha = \gamma\alpha \implies \beta = \gamma.$$

Fixed points

We'll prove that:

- To verify that two given isometries α and β are the same, it is sufficient to verify that they agree on some three points that are not collinear.
- Every isometry is a product of at most three reflections.

Point $P \in \mathbb{R}^2$ is a **fixed point** of isometry α if $\alpha(P) = P$. We also say that α **fixes** point P.

Lemma. Let α be an isometry and P a point such that $\alpha(P) \neq P$. Then every fixed point of α is on the perpendicular bisector between P and $\alpha(P)$.

Proof.

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Proof.

Theorem. Let α be an isometry.

1. If α fixes three non-collinear points, then $\alpha = \iota$.
2. If α fixes two points then $\alpha = \iota$ or α is a reflection.
3. If α fixes some point then α is a product of at most two reflections.
4. Every isometry is a product of at most three reflections.

Proof.

Corollary. If α and β are two isometries such that $\alpha(P) = \beta(P)$, $\alpha(Q) = \beta(Q)$ and $\alpha(R) = \beta(R)$, and points P, Q and R are not collinear, then $\alpha = \beta$.

Proof.

The proofs provide a simple method of finding the reflections when we know the images

$$\begin{aligned} P_0 &= \alpha(P), \\ Q_0 &= \alpha(Q), \text{ and} \\ R_0 &= \alpha(R) \end{aligned}$$

of three non-collinear points P, Q and R.

We simply find reflections that **match the points one-by-one.**

Corollary. (This is Lemma 2.4 in the notes). If an isometry fixes two distinct points P and Q, then it fixes every point of the line m that contains P and Q.

Proof.