Symmetries of a set of points

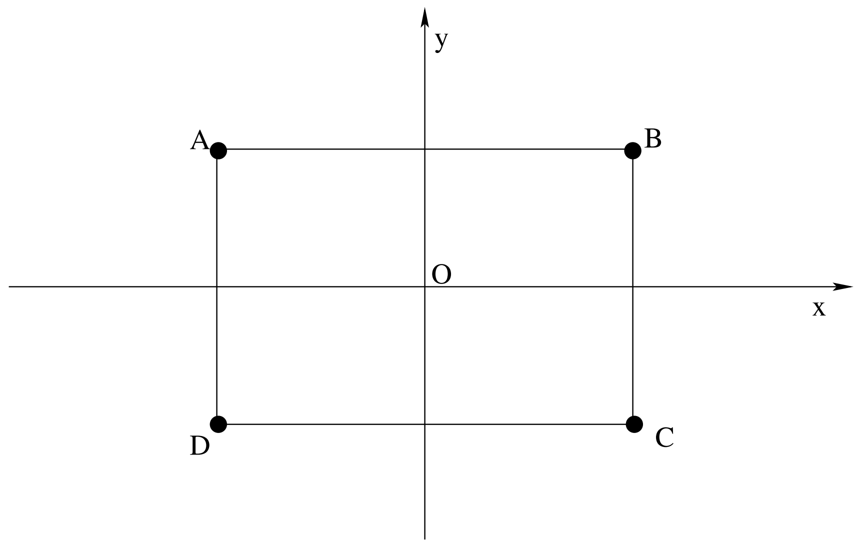
Isometry α is **symmetry** of a set $s \subseteq \mathbb{R}^2$ if $\alpha(s) = s$.

Theorem. For any $s \subseteq \mathbb{R}^2$ the symmetries of s form a subgroup of $\mathcal{I}$.

The set of symmetries of s is the **symmetry group** of s. Notice that $\mathcal{I}$ itself is the symmetry group of $s = \mathbb{R}^2$.

Proof.

Example. Let s be a rectangle $ABCD$ that is not a square. Let us position s in such a way that its center is at the origin $(0,0)$, and its sides are parallel to the x- and y-axes:

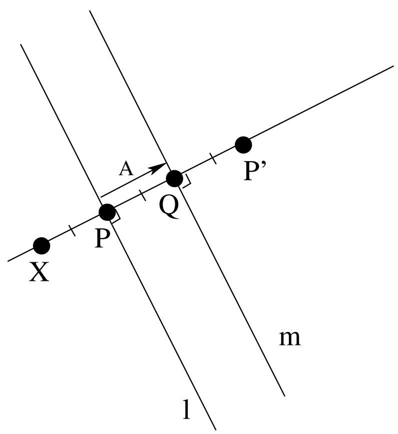


Let us find symmetries of s.

Products of two reflections

Theorem. Let m, ℓ be parallel lines. Then $\sigma_m \sigma_\ell$ is the translation τ_{2A} where A is the vector from ℓ to m that is perpendicular to ℓ and m.

Conversely, every translation is a product of two reflections in parallel lines, both perpendicular to the direction of the translation. One of the lines can be chosen freely (as long as it is perpendicular to to the translation), after which the other line is uniquely determined.



Proof.

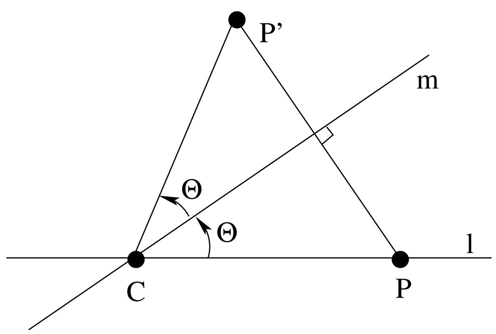
Corollary. The product of three reflections in three parallel lines is a reflection in a single parallel line.

Proof.

Theorem. Let m, ℓ be non-parallel lines, intersecting each other at point C. Then $\sigma_m \sigma_\ell$ is the rotation $\rho_{C,2\theta}$ where θ is the angle from line ℓ to line m.

Conversely, every rotation around C is a product of two reflections in lines through point C. One of these lines can be chosen freely, after which the other line is uniquely determined.

Proof.



Corollary. Halfturn σ_C is the product of two reflections in any two perpendicular lines through C. In particular, reflections in perpendicular lines commute.

Corollary. The product of three reflections in lines through the common point C is a reflection in a line through point C.

Proof.

Products of three reflections

Lemma. The following three are equivalent for an isometry α:

1. α is a glide reflection,
2. $\alpha = \sigma_P \sigma_l$ for some point P and line l,
3. $\alpha = \sigma_k \sigma_Q$ for some line k and point Q.

Proof.

Theorem. A product of three reflections is a glide reflection.

Proof.

Theorem. A product of three reflections is a glide reflection.

Proof.

Corollary. Every plane isometry is a translation, a rotation or a glide reflection.

Proof. We know that every isometry is a product of at most three reflections.

- Products of three reflections are glide reflections.
- Products of two reflections are translations and rotations.
- Single reflections are also glide reflections.
- "The product of zero reflections" is the trivial isometry which is also a translation and a rotation.

Parity

All isometries are products of reflections. An isometry is

- **even** if it is a product of an even number of reflections,
- **odd** if it is a product of an odd number of reflections.

Let us show that every isometry is even or odd but not both: products of even and odd numbers of reflections can not be identical.

Theorem. A product of four reflections is a product of two reflections.

In the proof we use the following lemma twice:

Lemma. If m and l are two lines and P is a point, then there are lines p and q such that $\sigma_m \sigma_l = \sigma_p \sigma_q$, and line q contains point P.

Proof of the Lemma:

Corollary. A product of an even number of reflections cannot equal a product of an odd number of reflections. Thus an isometry cannot be both even and odd.

Proof.

- Rotations and translations are the even isometries.
- Glide reflections are the odd isometries.

- Rotations and translations are the even isometries.
- Glide reflections are the odd isometries.

Even isometries form a subgroup of $\mathcal{I}$: composing n reflections with m reflections is a composition of $n+m$ reflections. And a sum of two even numbers is an even number.

Let us denote the group of even isometries by $\mathcal{E}$.

The set of odd isometries is not a group because a sum of two odd numbers is not odd. The set of odd isometries is the set $\gamma\mathcal{E}$, for any single odd isometry γ. In other words, it is a **coset** of the subgroup $\mathcal{E}$.