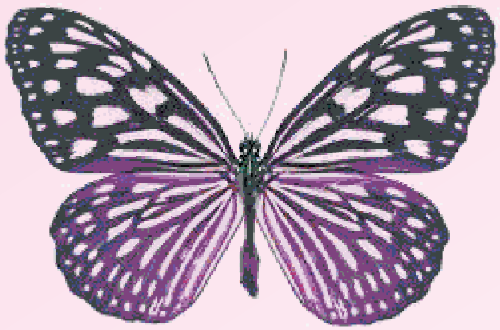
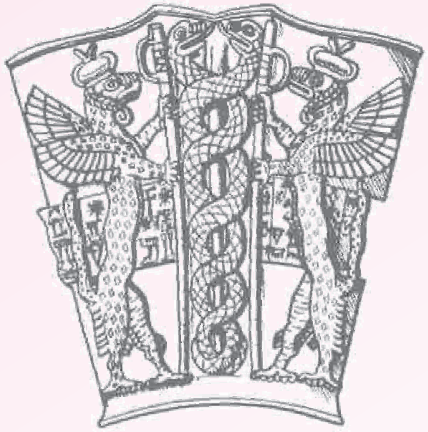
Symmetry and tilings

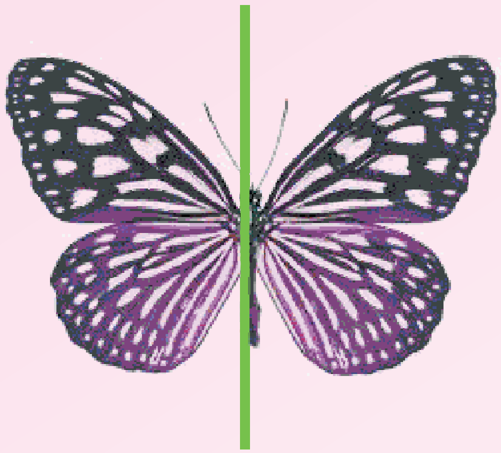
Jarkko Kari

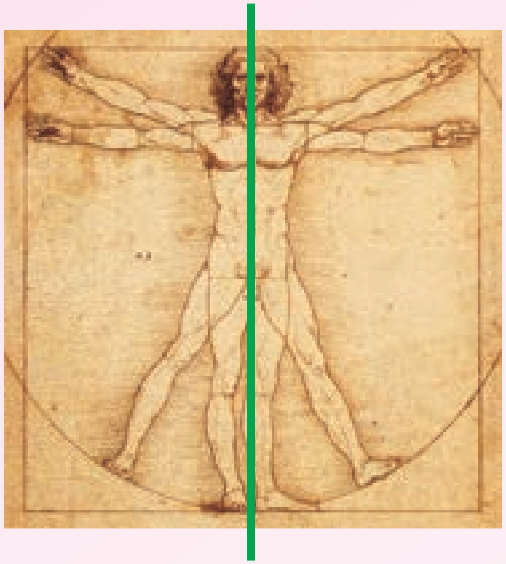
Symmetry has always been a central aesthetic element in arts. A symmetric shape look beautiful and "perfect".

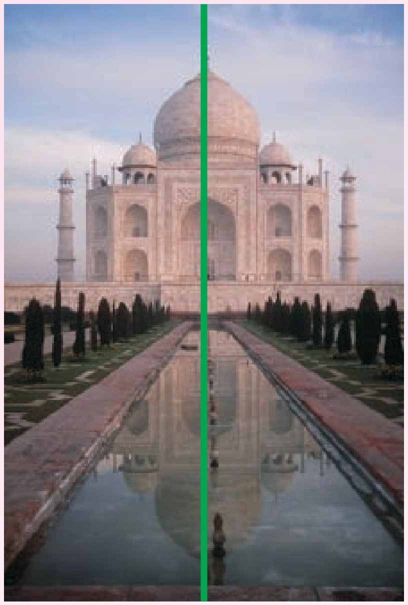


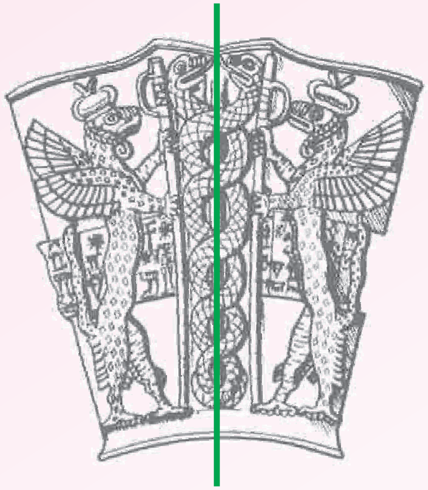


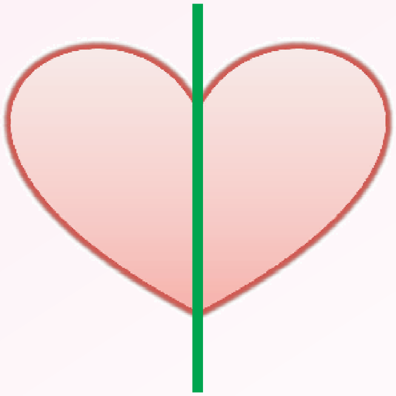
Bilateral symmetry is the simplest kind of symmetry: a picture is its own reflection.











One speaks of symmetry in many other contexts as well. For example, **palindromes** are symmetric words:

SAIPPUAKAUPPIAS

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In physics symmetries play an especially important role.

This presentation, however, focuses on symmetries on the plane $\mathbb{R}^2$.

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More generally, a transformation typically refers to a plane **isometry.**

An isometry is a transformation that **preserves the distances** between points. It is hence a "rigid" motion: shapes and sizes of patterns remain unchanged.

There are four types of plane isometries:

- Translation
- Rotation
- Reflection
- Glide reflection

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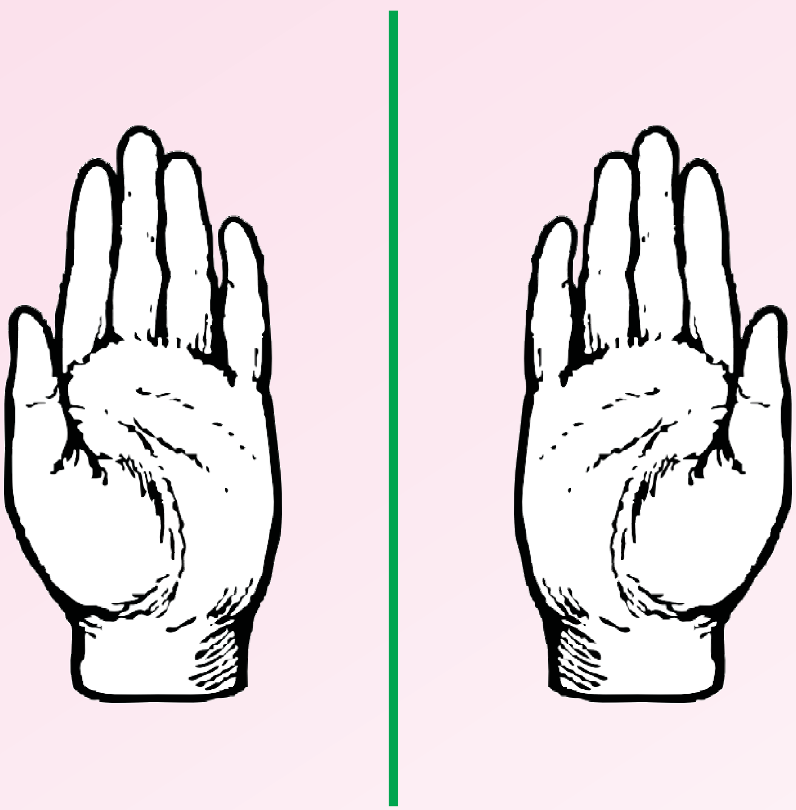
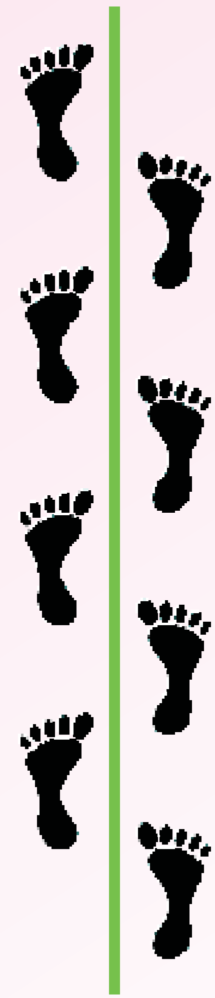
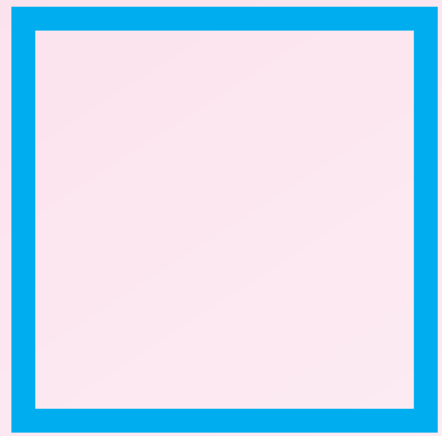
In glide reflections one first reflects and then translates along the reflection axis.

Reflection and glide reflection are **odd** isometries: patterns get "flipped over" into their mirror images.





A square, for example, has eight symmetries:



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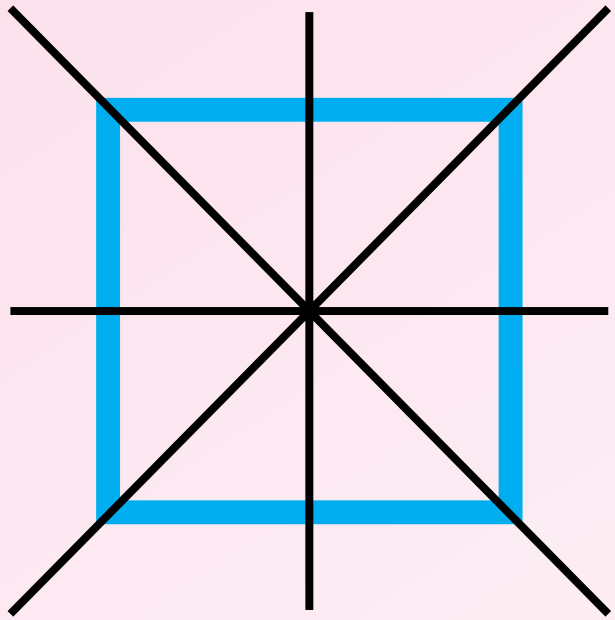

Four reflections…

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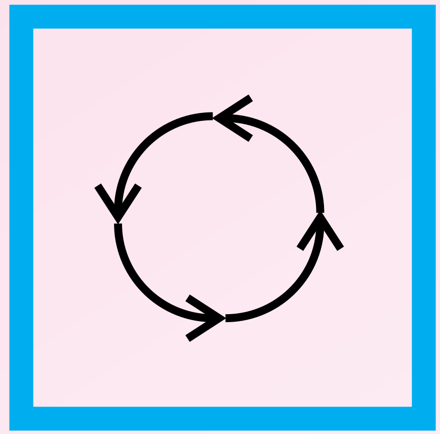
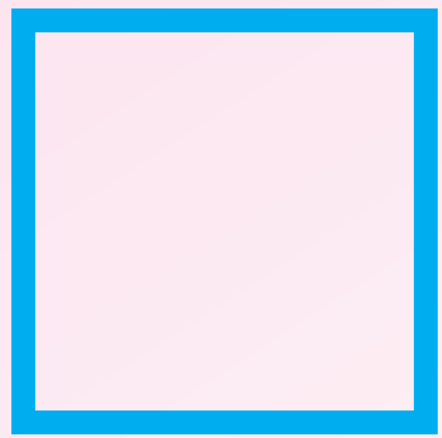


. . . and four rotations.

The inverse map of a symmetry is also a symmetry. And executing symmetries one after the other (=composition of symmetries) yields also a symmetry:



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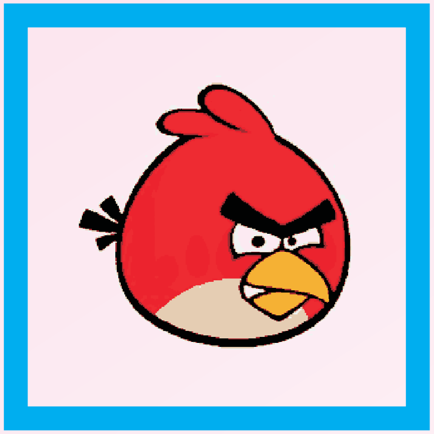

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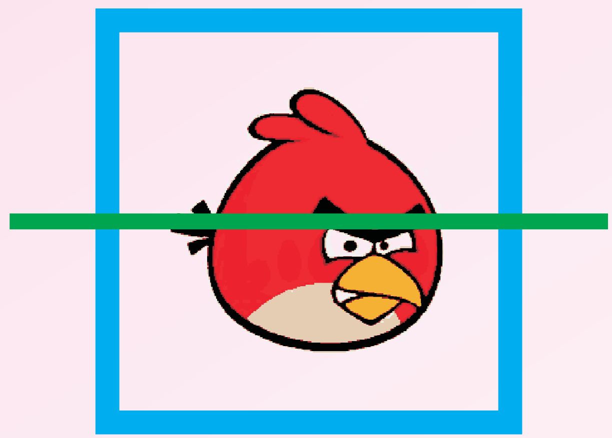

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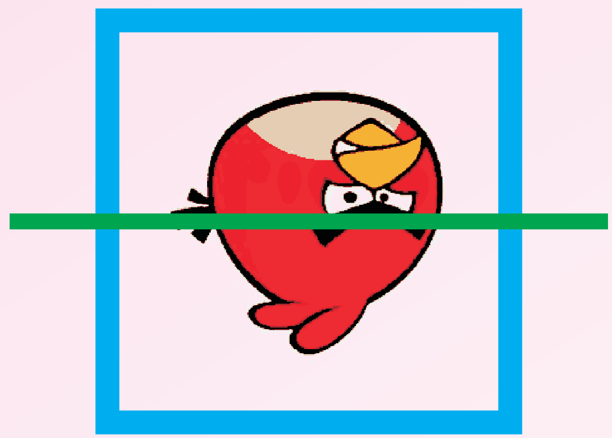

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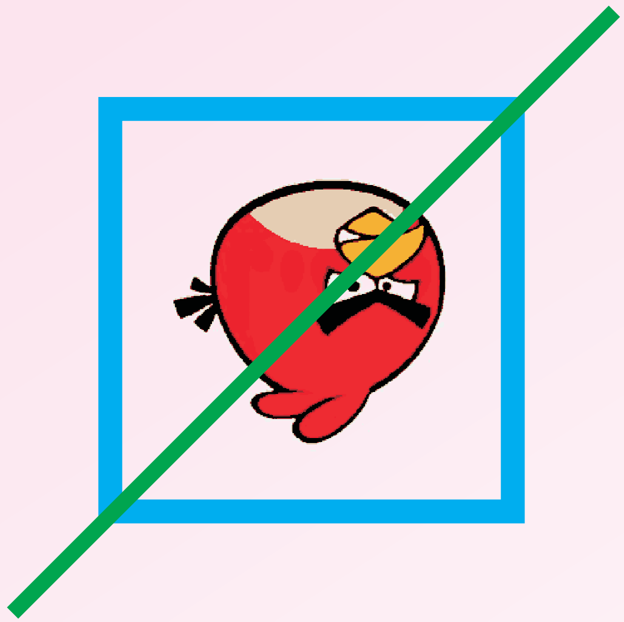

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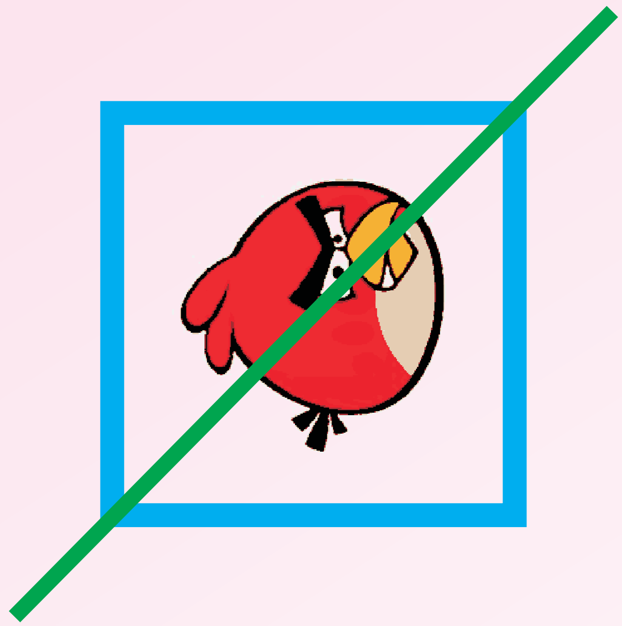

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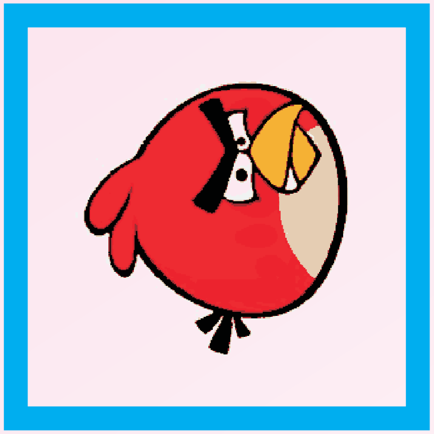

The composition is the 90° rotation.

Always: Composition of two reflections is

- **a rotation**, if the reflection axes intersect,
- **translation**, if the reflection axes are parallel.

The set of symmetries of a pattern is its **symmetry group**.
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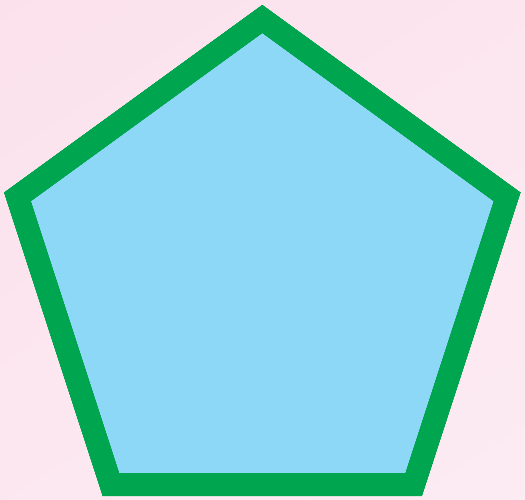
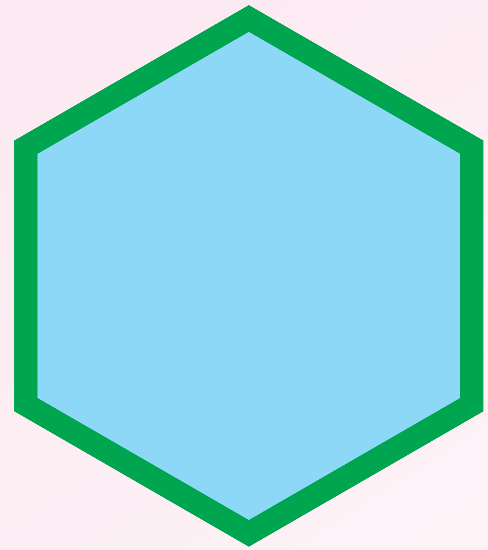
The symmetry group of a square contains eight elements.

More generally: the symmetry group of a regular n-gon (so-called **dihedral group** D_n) contains n reflections and n rotations.



D_5



D_6

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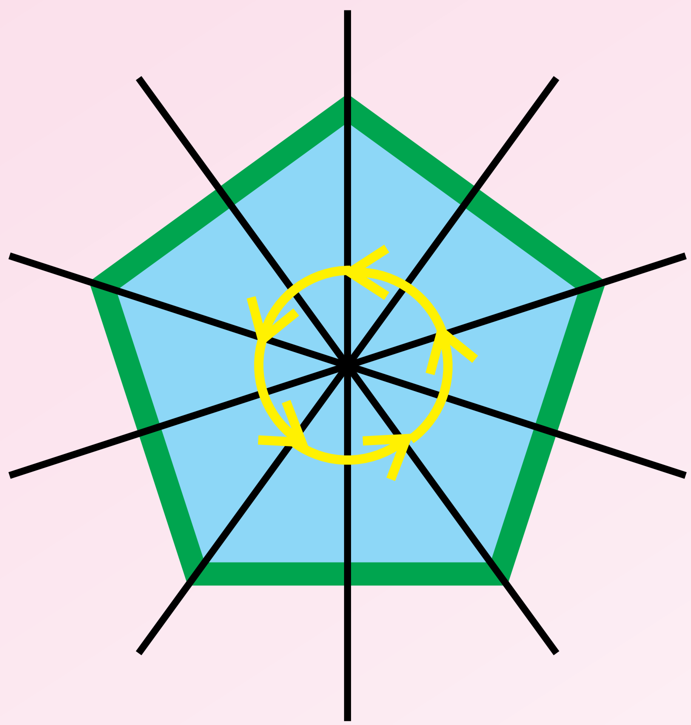

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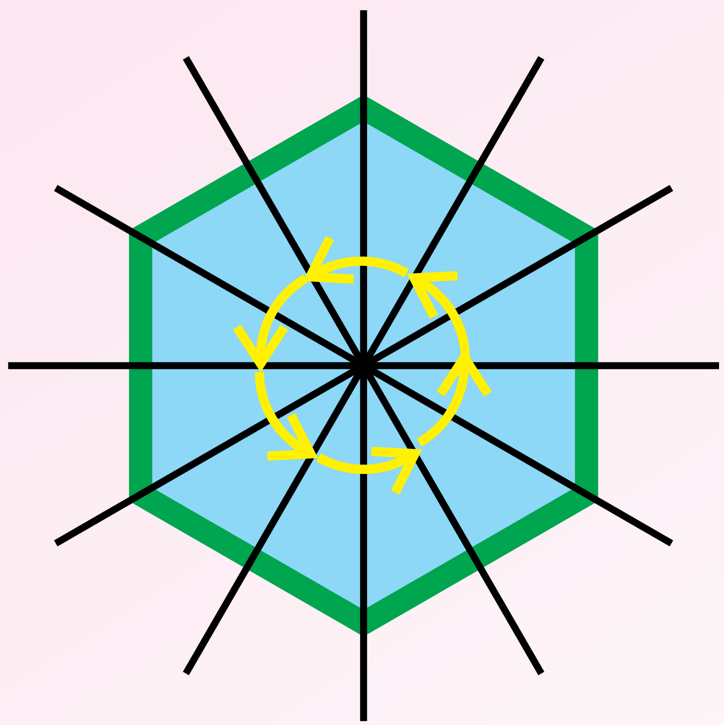

D_6

Snow flakes often have six-fold symmetry

Symmetry group D_6: 6 reflections, 6 rotations

Images *SnowCrystals.com*

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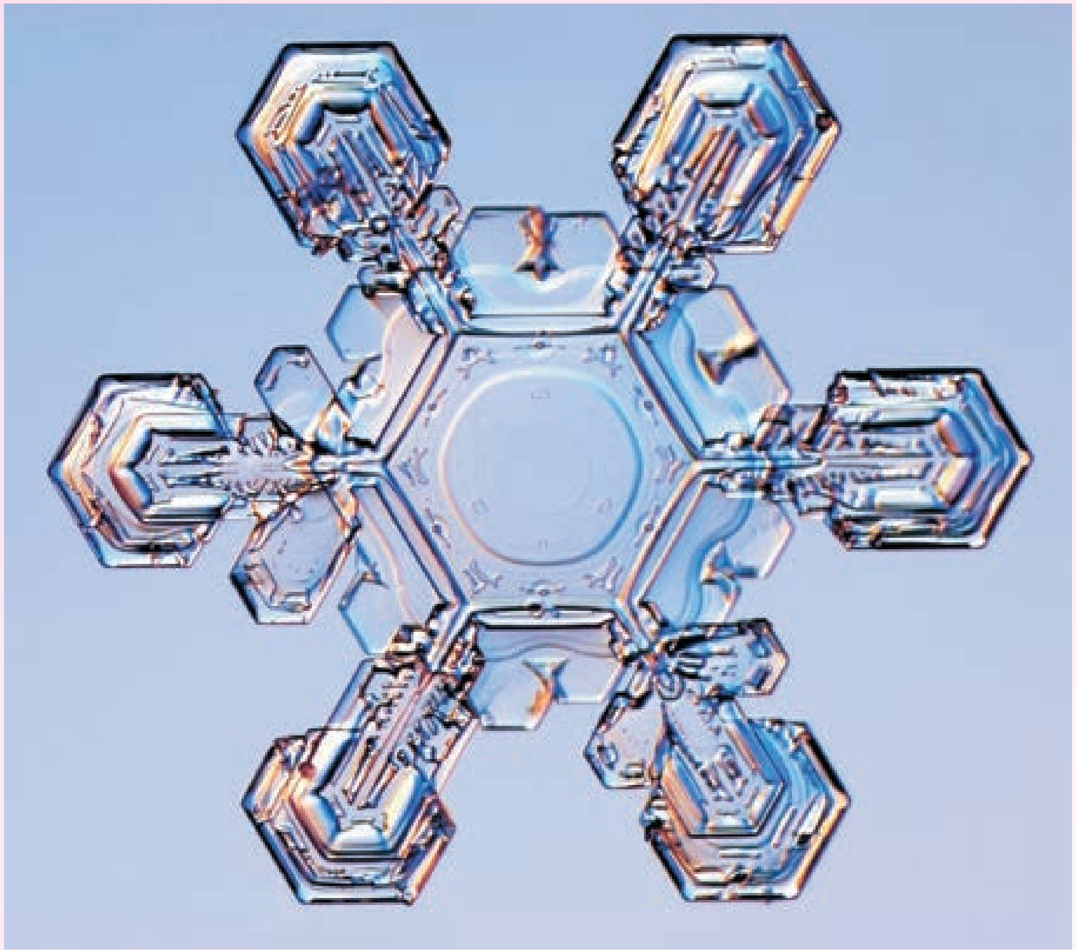

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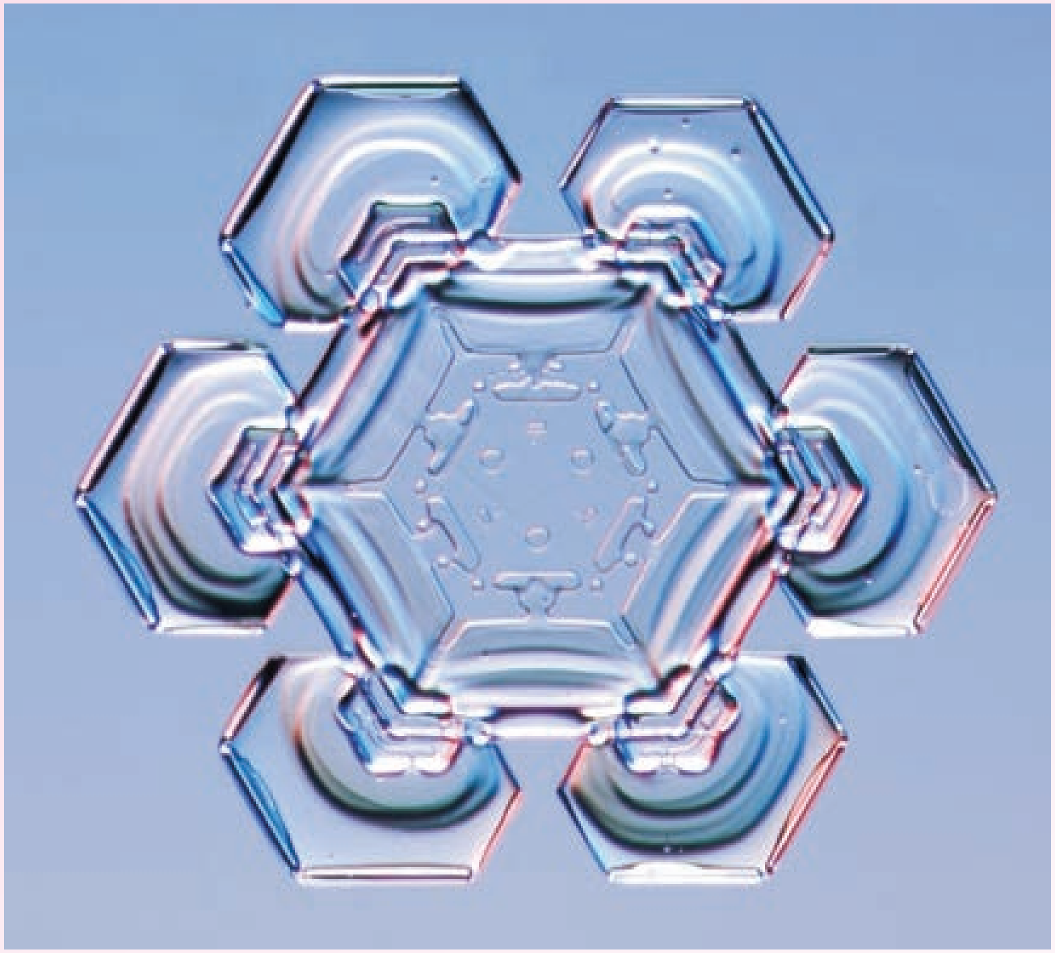

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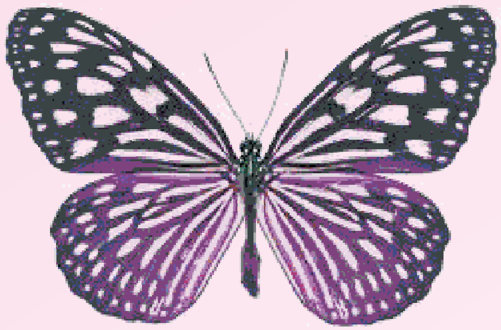
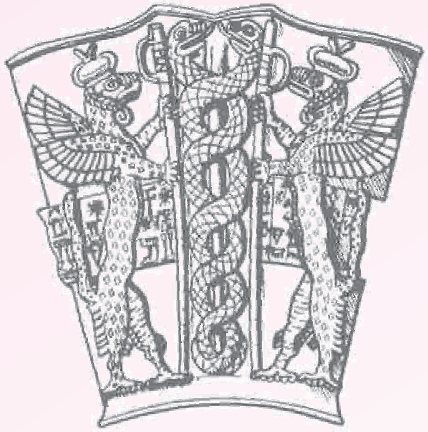
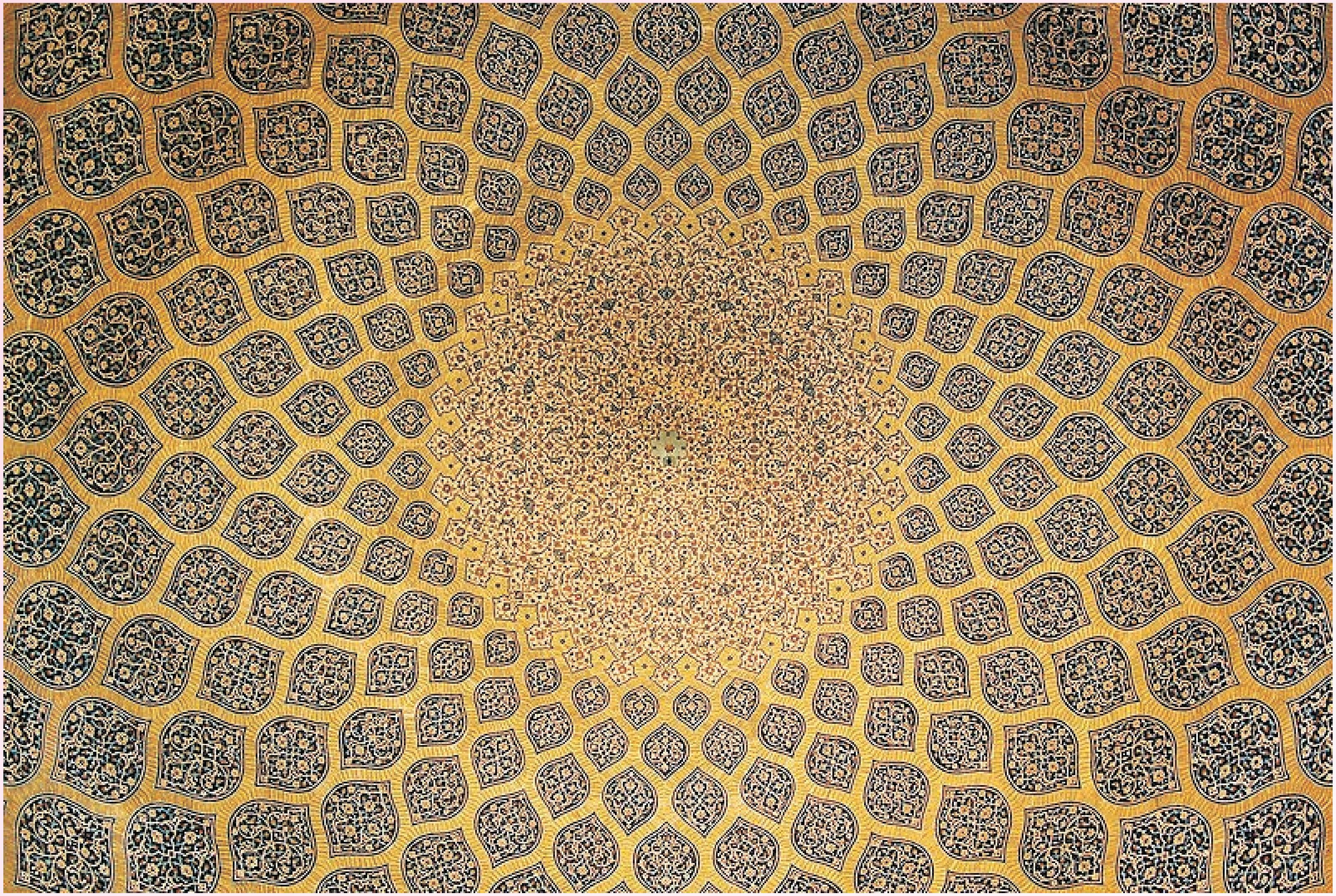
If a pattern only has bilateral symmetry then its symmetry group is D_1: One reflection and the trivial $0°$ rotation:



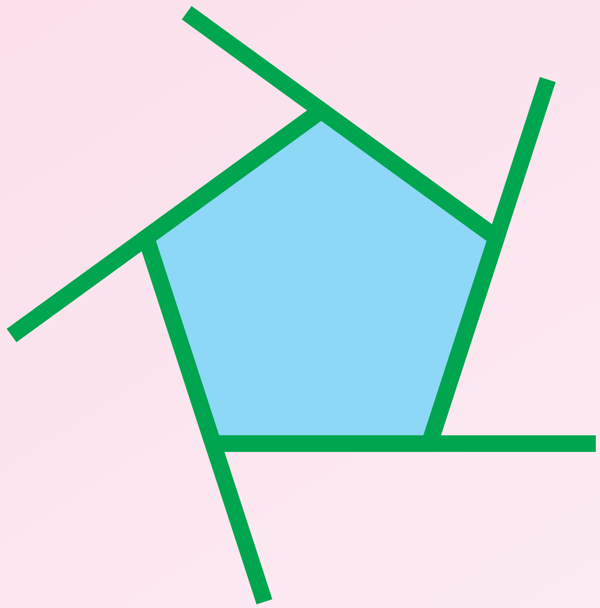


Symmetry group D_{32}:



Lotfollah mosque, Iran, early 17th century

Keeping rotations but removing reflections yields **the cyclic group** C_n



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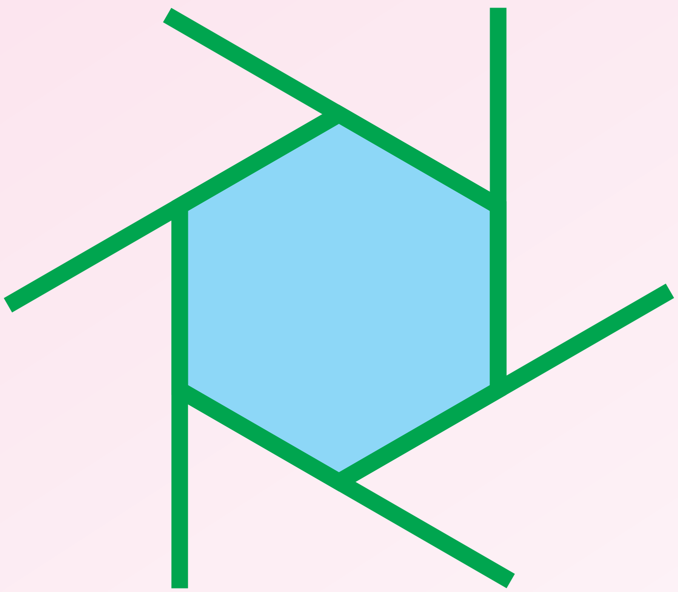

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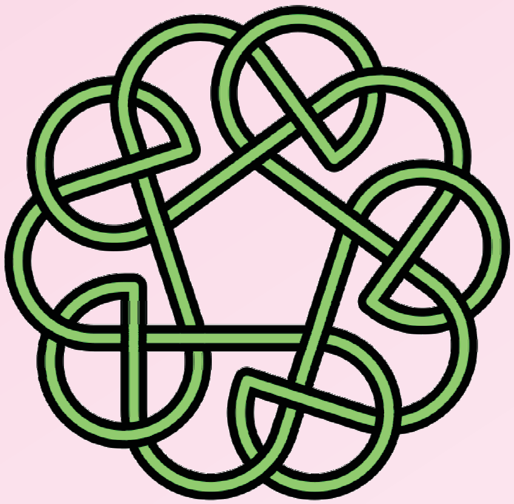
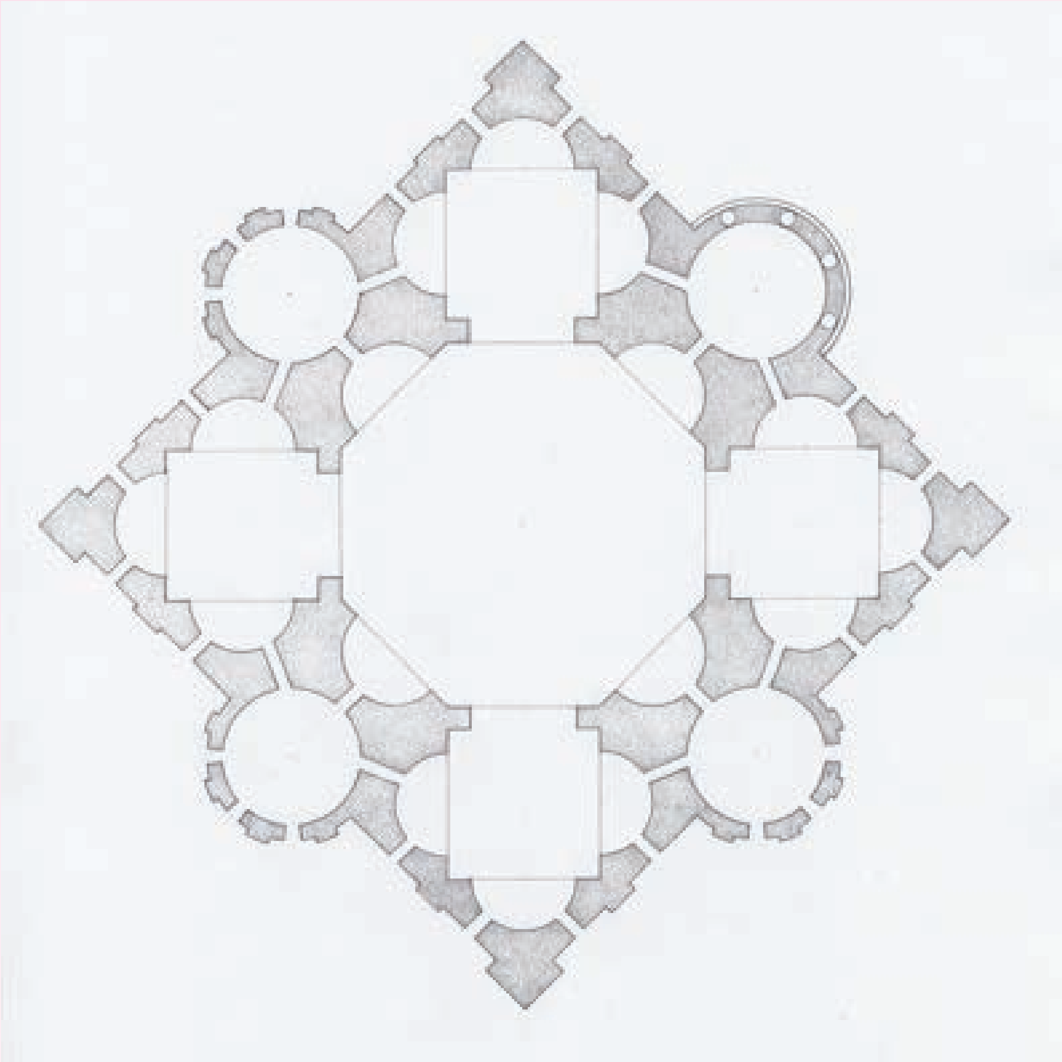




Leonardo da Vinci investigated systematically possible symmetries of floor plans of buildings.



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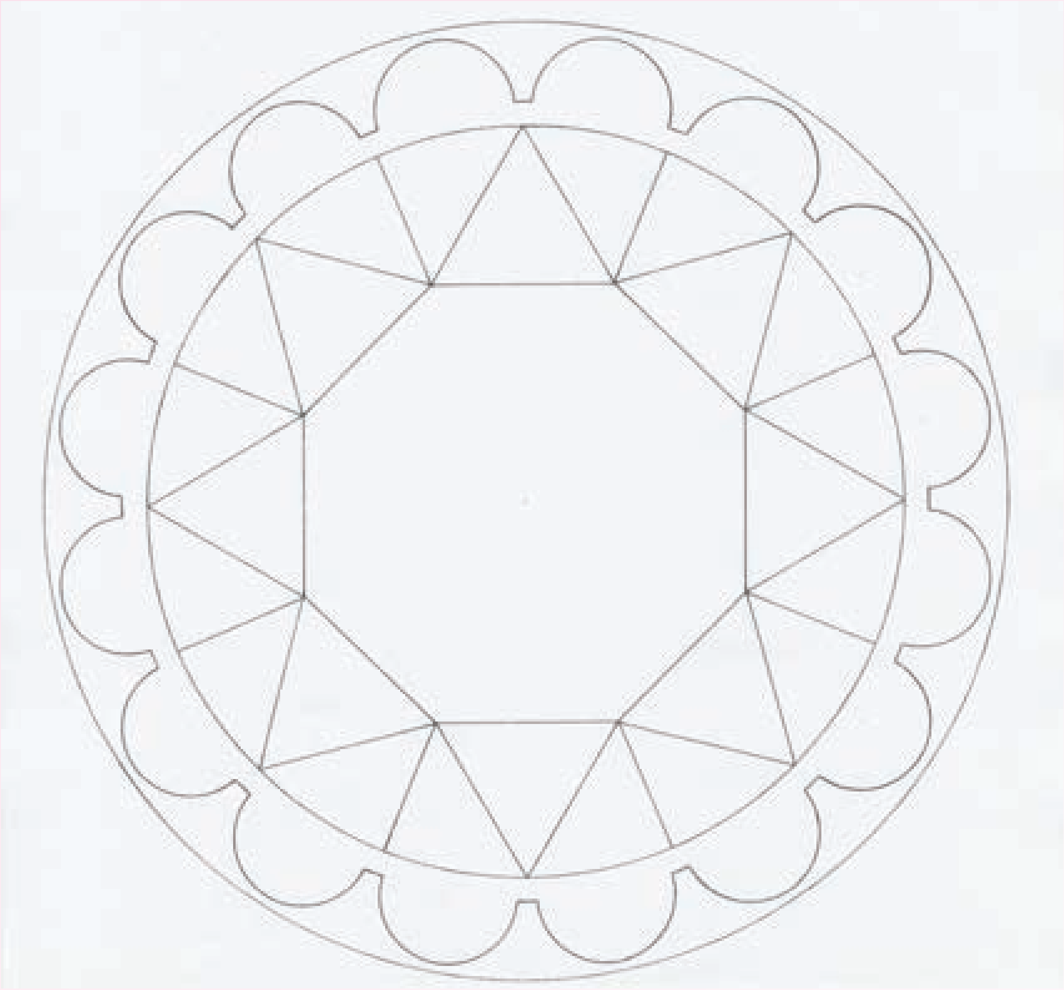

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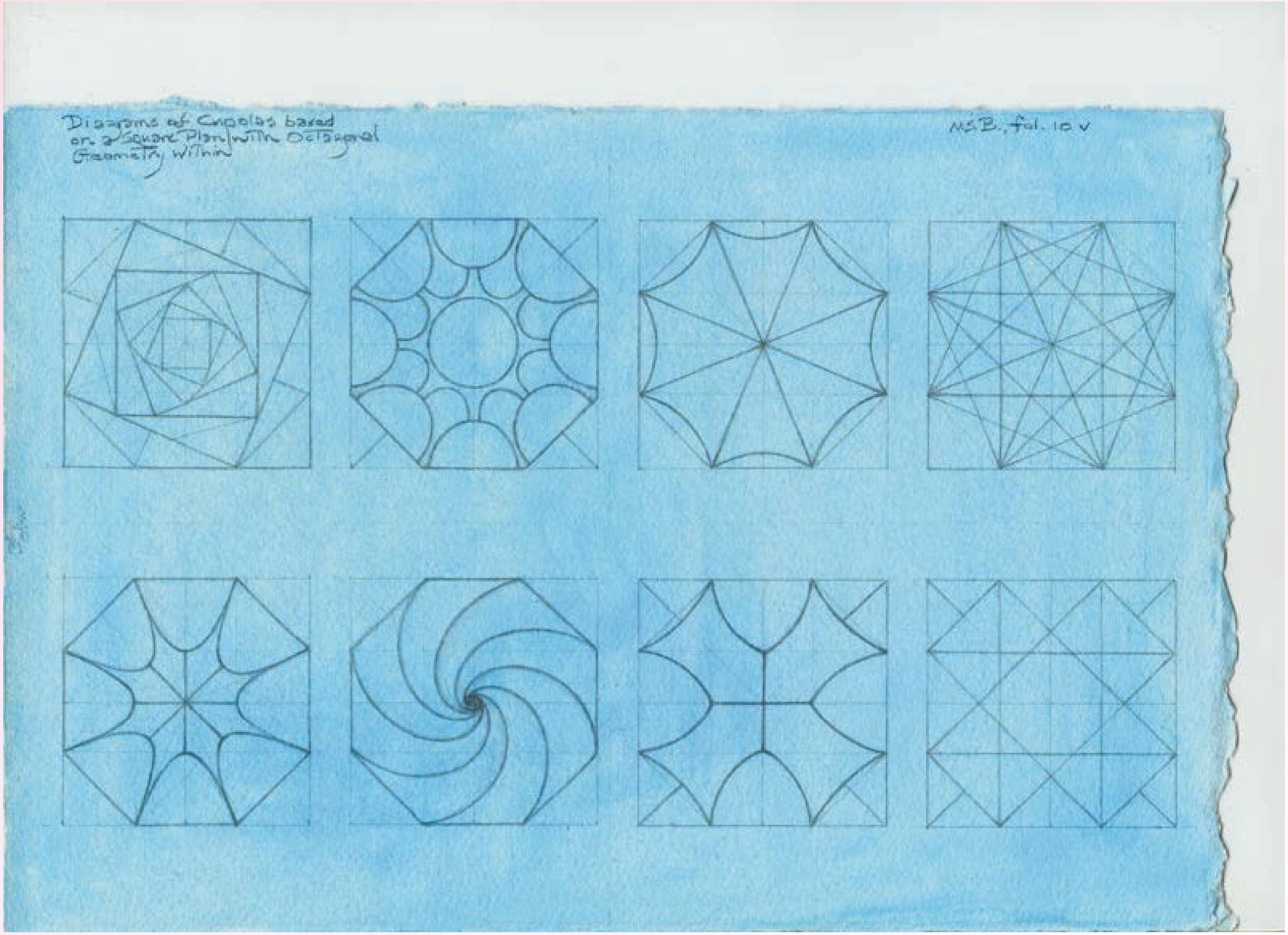

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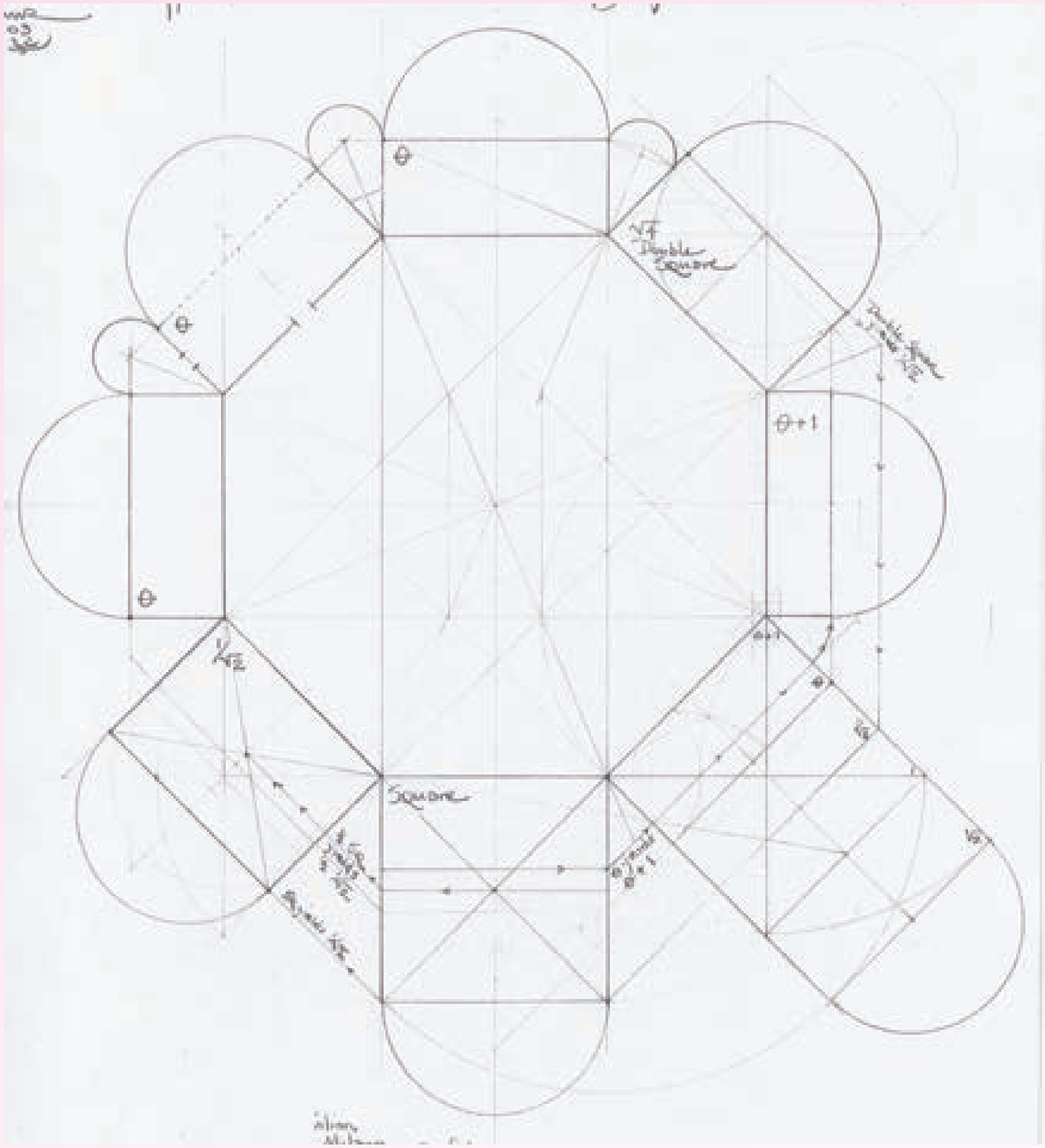

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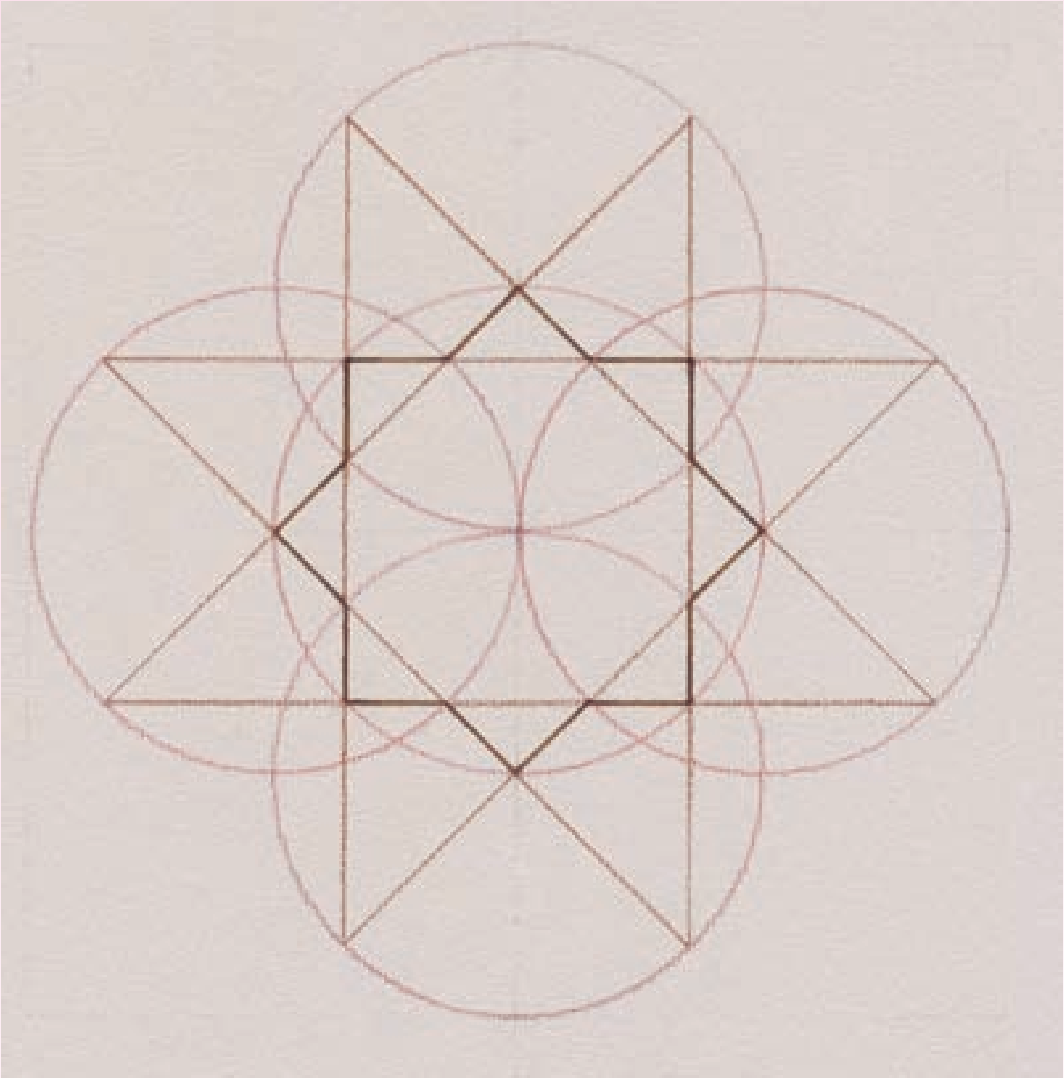


Leonardo da Vinci's theorem: The dihedral groups D_n and the cyclic groups C_n are the only finite symmetry groups.

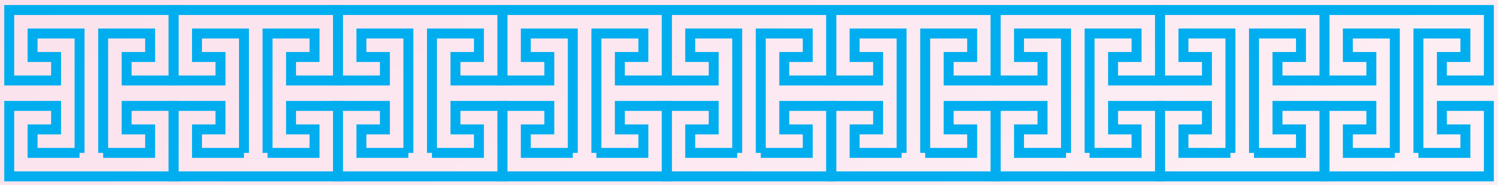
The dihedral groups and the cyclic groups together are also called **Rosette groups**.

Determining symmetries: Draw a copy of your pattern on a transparent film and find all positions of the film where the pattern perfectly lines up with itself.

Odd isometries are found by flipping the film upside down!

Frieze groups

If the translational symmetries of a pattern are multiples of a single translation, the symmetry group is one of seven **frieze groups**.



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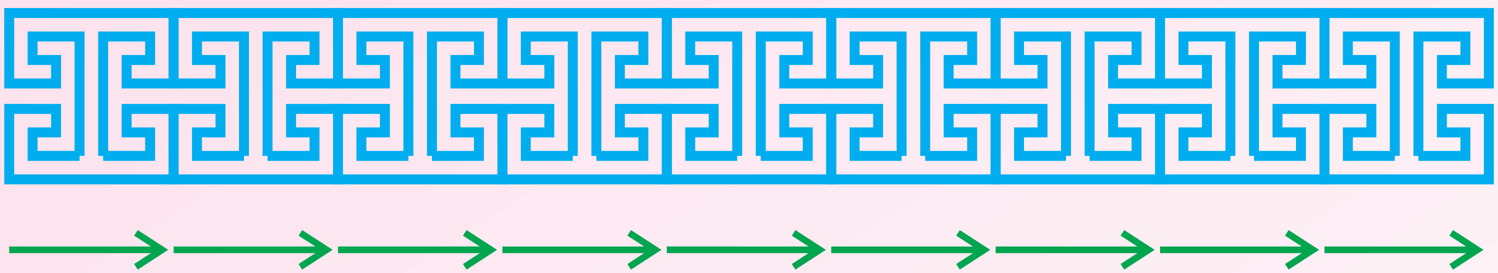

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Paigah tombs, Hyderabad, Intia

A frieze pattern may also be given as a cyclic pattern that yields an infinite strip when unfolded on the plane:

Greece, 800 – 775 BC

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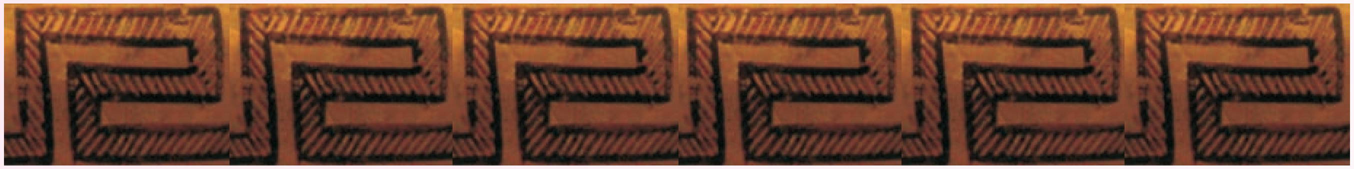

Greece, 800 – 775 BC

- **Half turn** is the only possible rotation in a Frieze group,
- reflections can only be with **horizontal and vertical axes**, and
- a glide reflection can only be **horizontal**.

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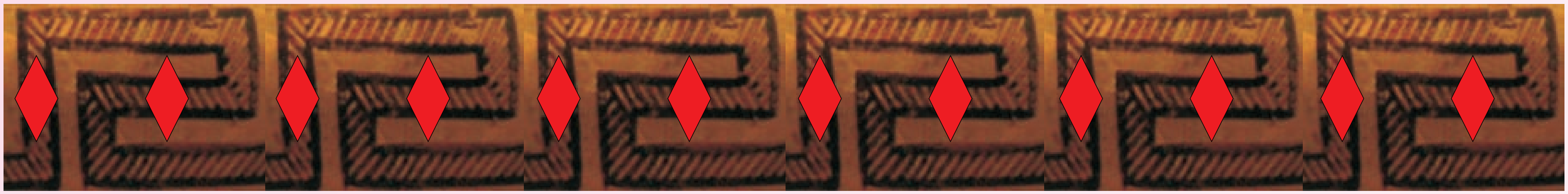

The distances between consecutive centers of half turns is half of the translational period.

This one, on the other had, has all possible types of frieze symmetries:

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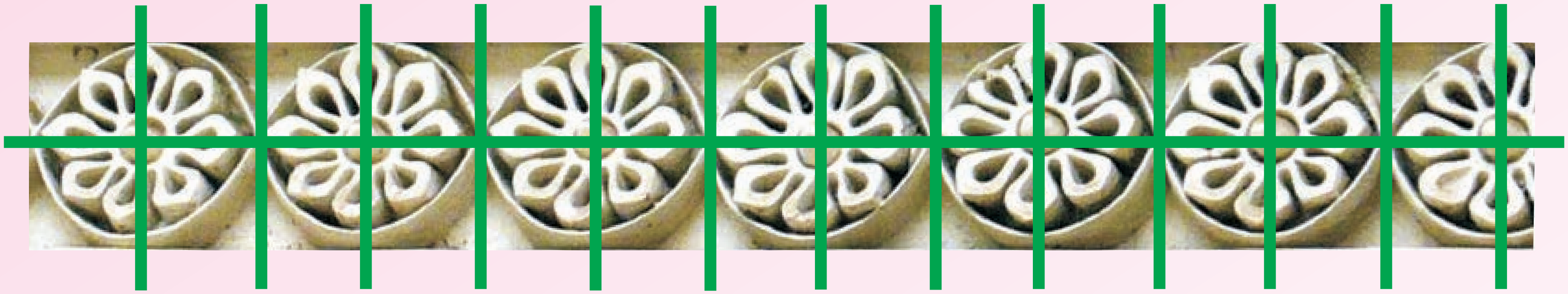
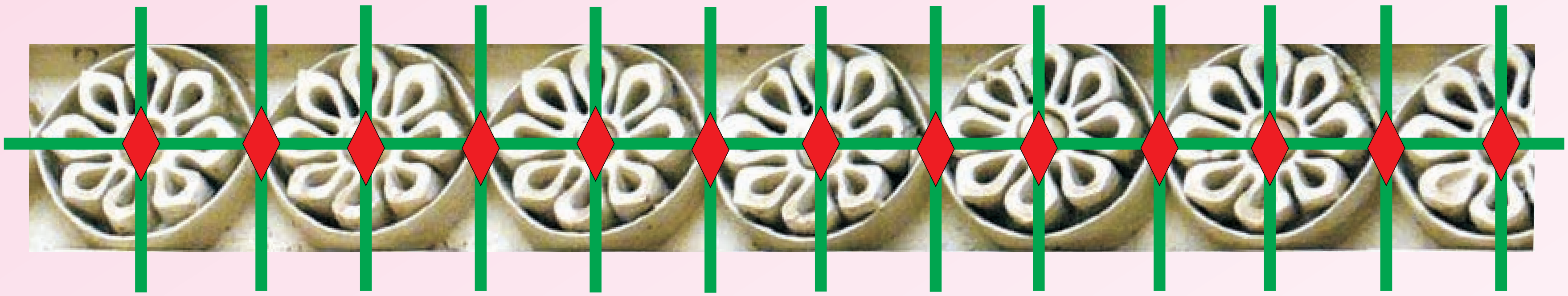


A reflection with the horizontal axis, and reflections on vertical axes. The distances between consecutive vertical reflection axes are precisely half of the shortest translation.

This one, on the other had, has all possible types of frieze symmetries:

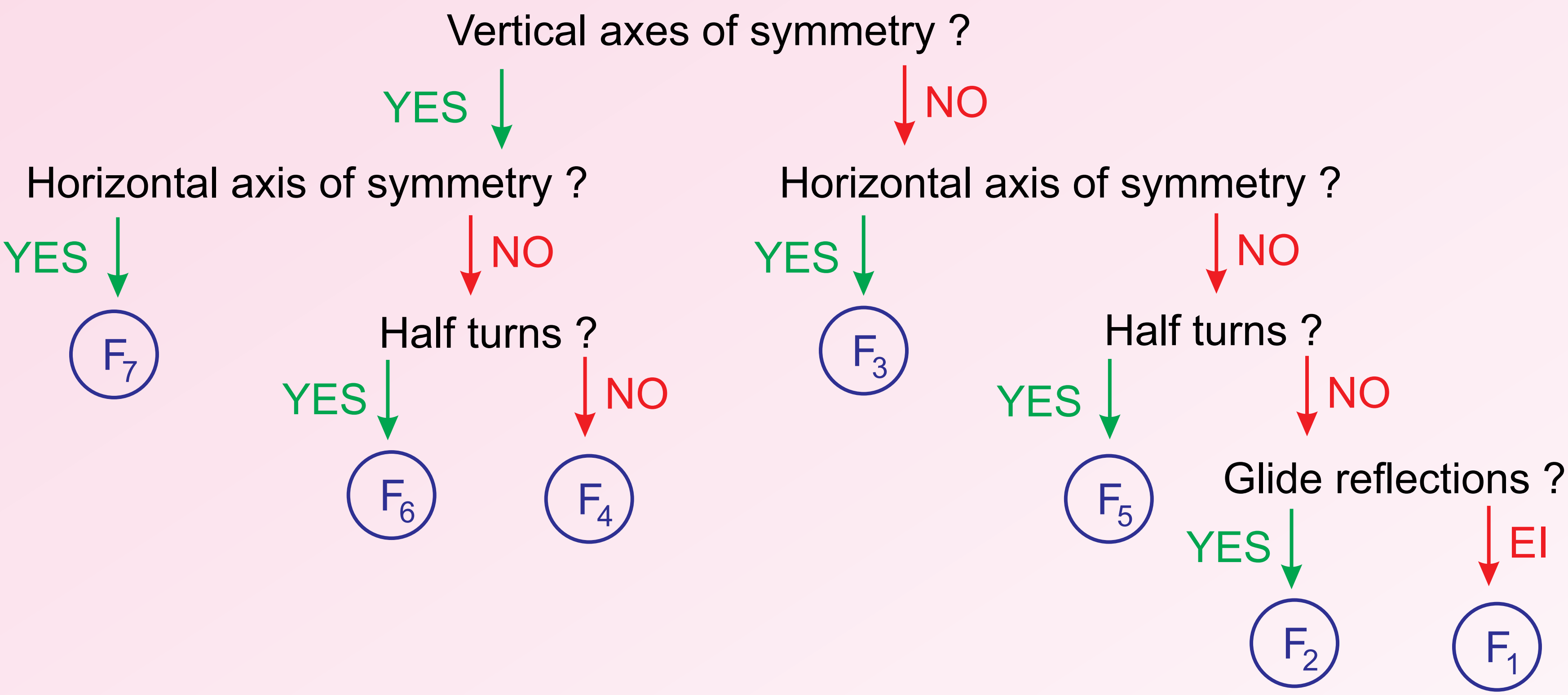


Half turns around the intersection points of the reflection axes. There are also horizontal glide reflections.

There are **seven** different frieze groups.

(If two friezes have the same symmetry group they can be scaled, rotated and moved on top of each other so that precisely the same isometries are symmetries to both of them.)

A flow chart to determine a frieze group



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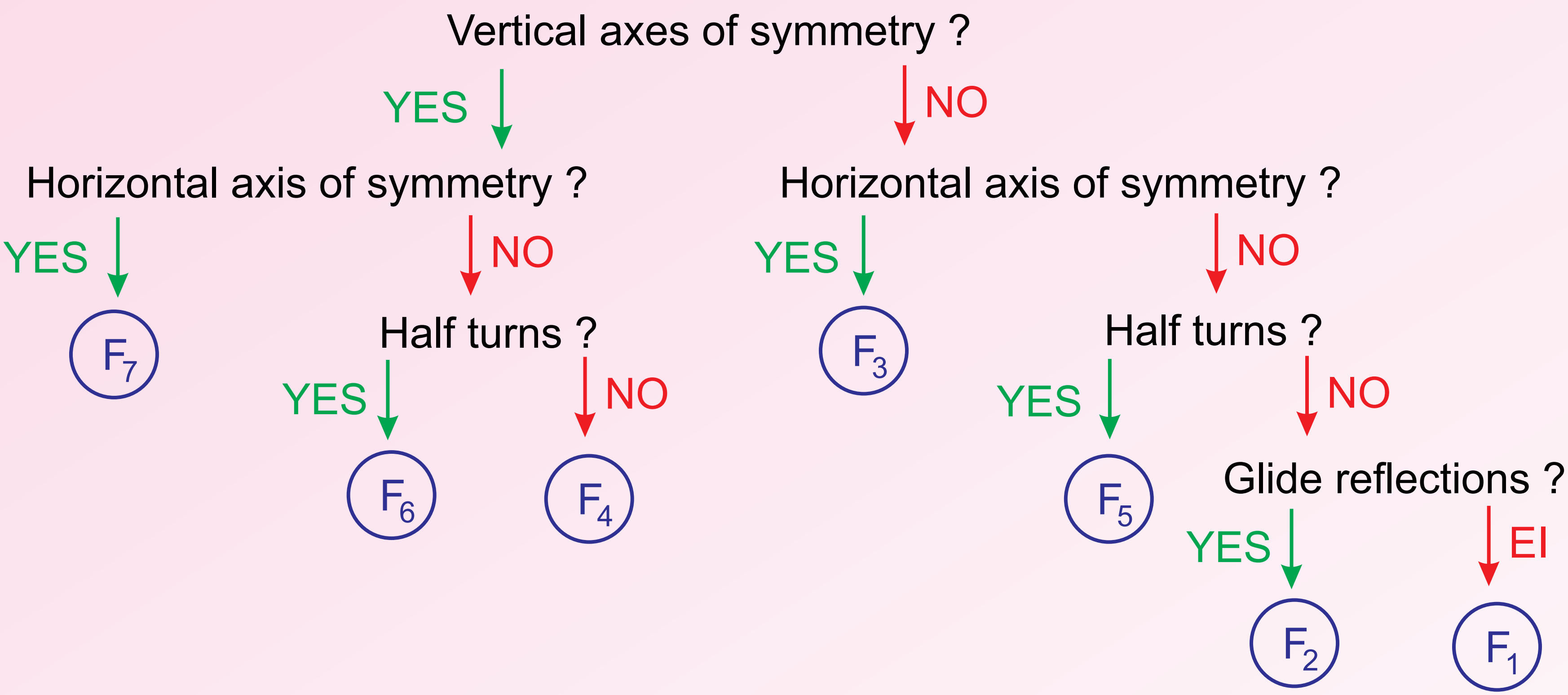

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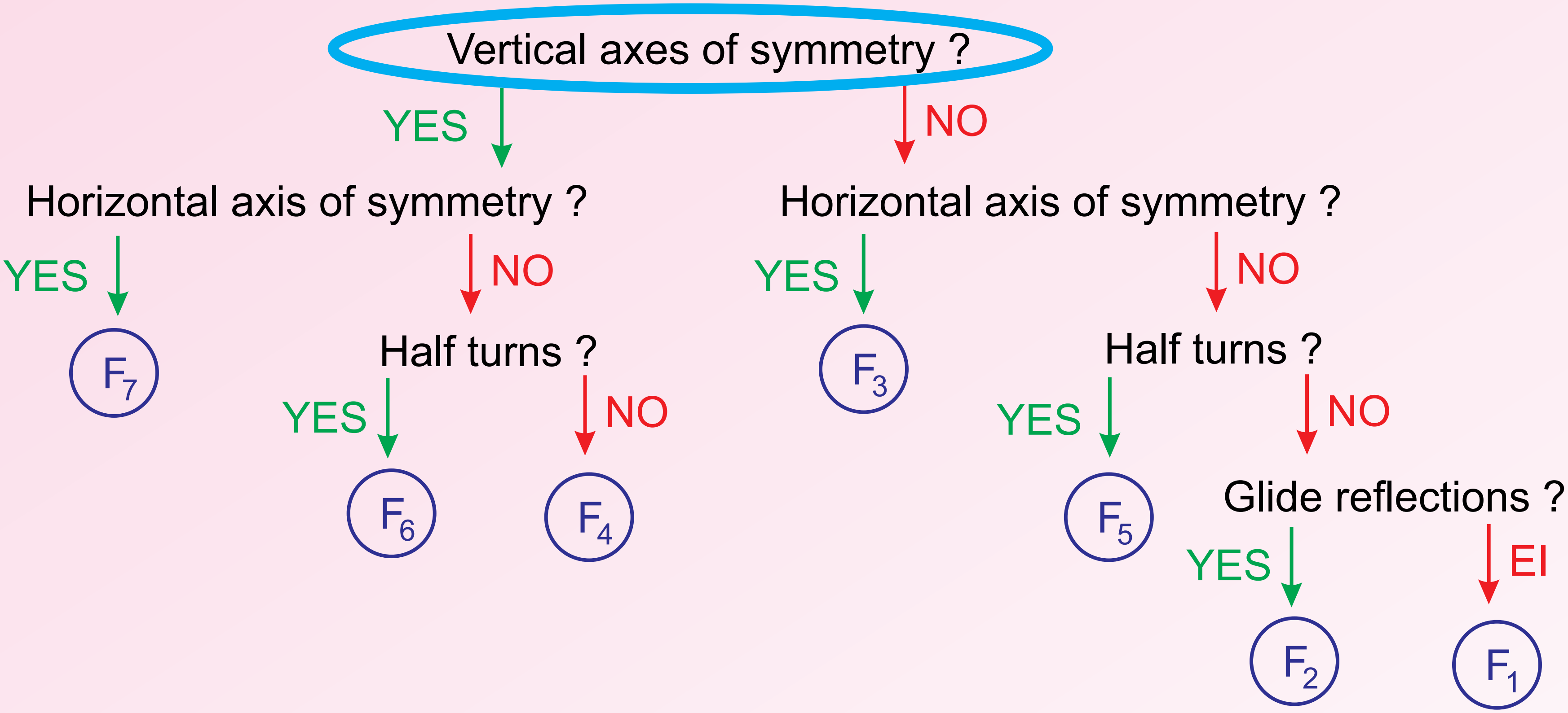

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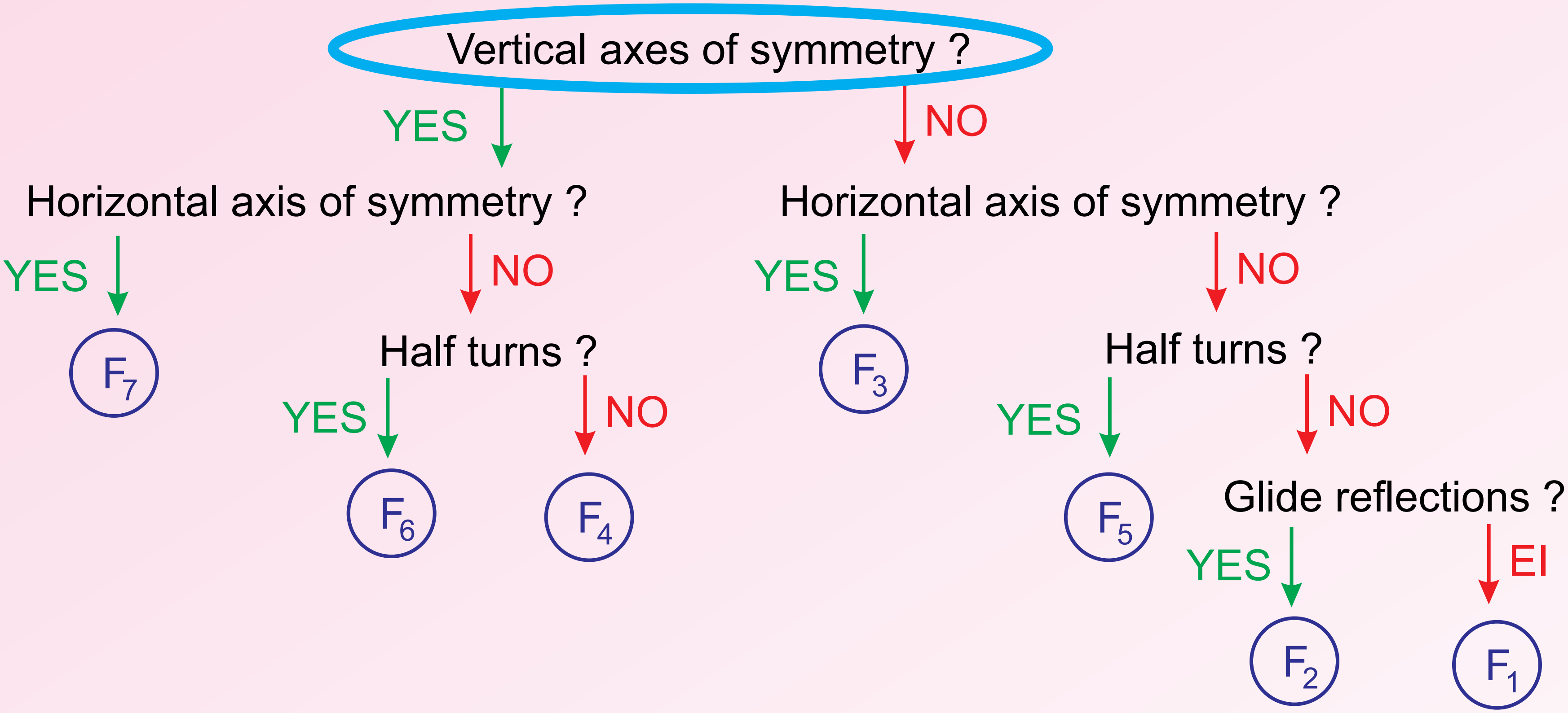

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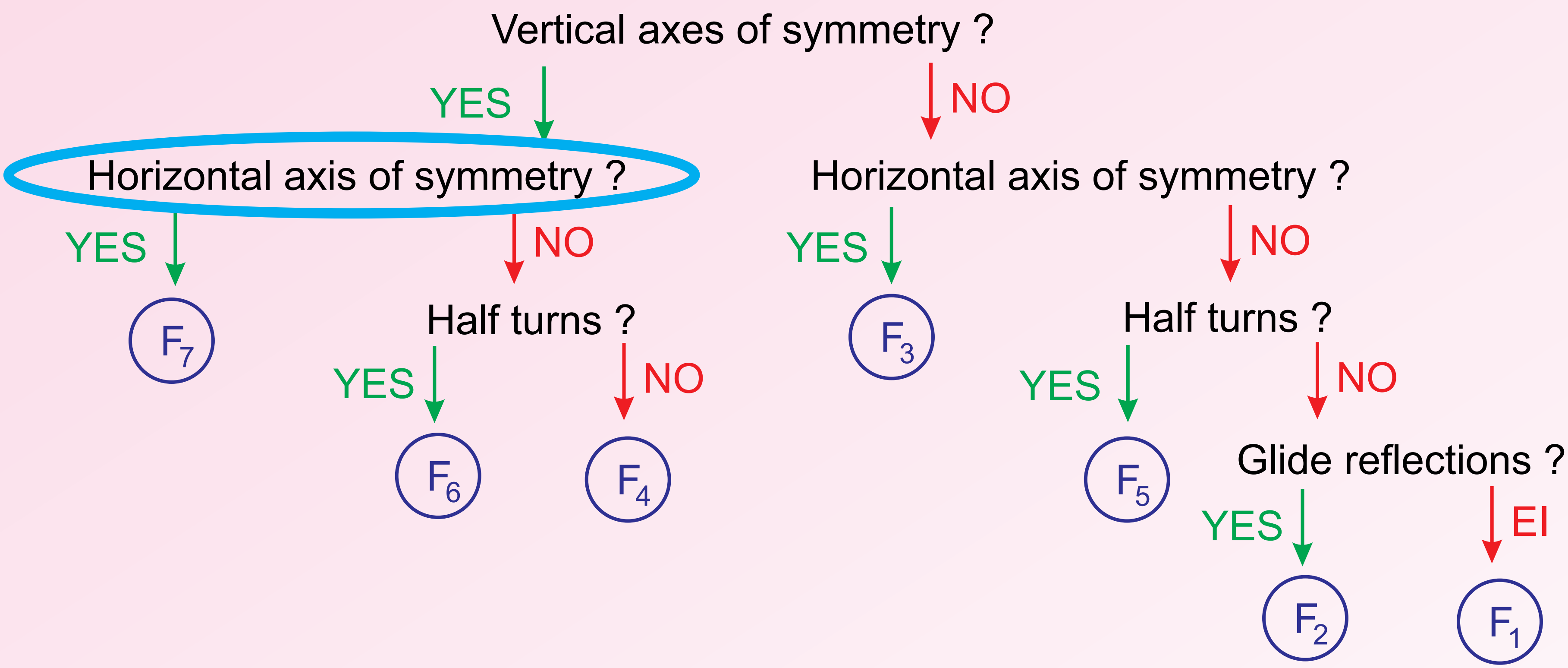

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Vertical axes of symmetry ?

- YES → Horizontal axis of symmetry ?
  - YES → F_7
  - NO → Half turns ?
    - YES → F_6
    - NO → F_4
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  - YES → F_3
  - NO → Half turns ?
    - YES → F_5
    - NO → Glide reflections ?
      - YES → F_2
      - EI → F_1

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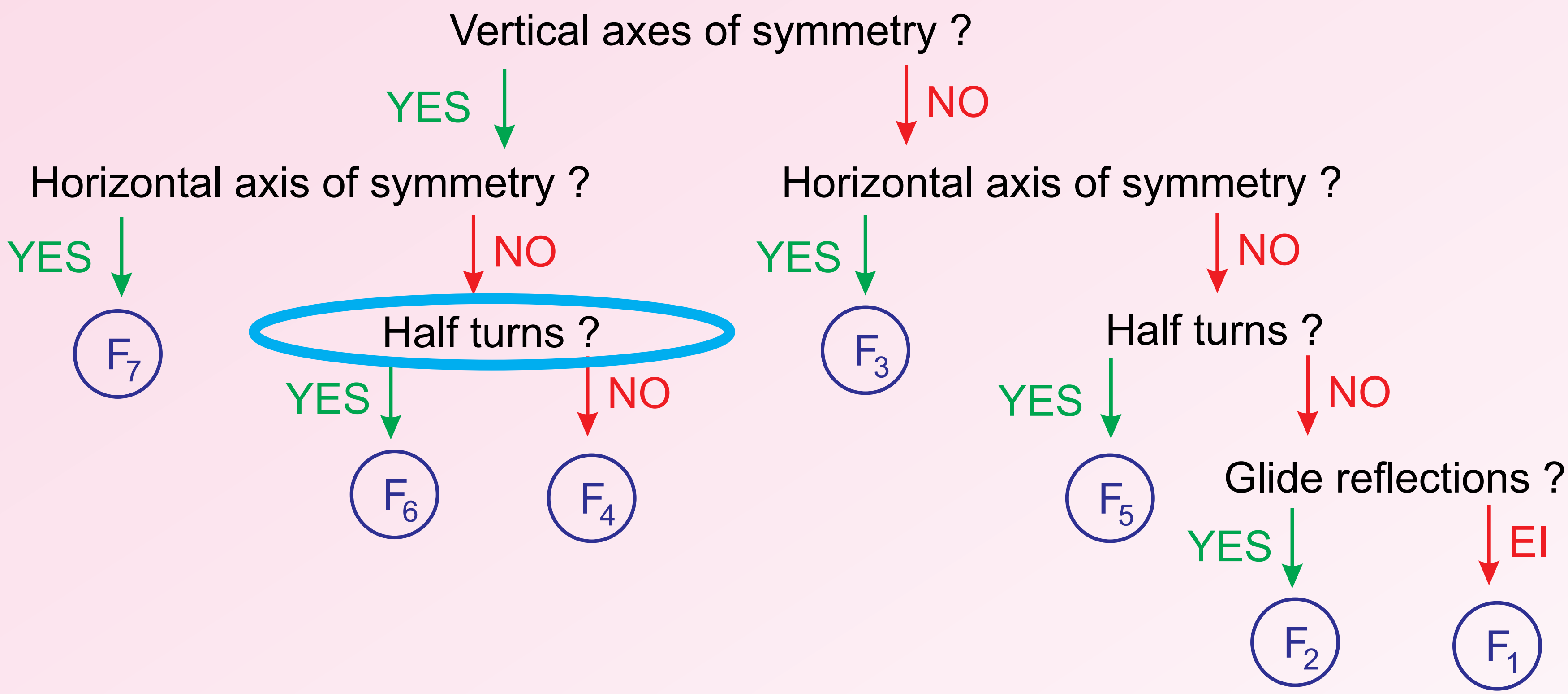

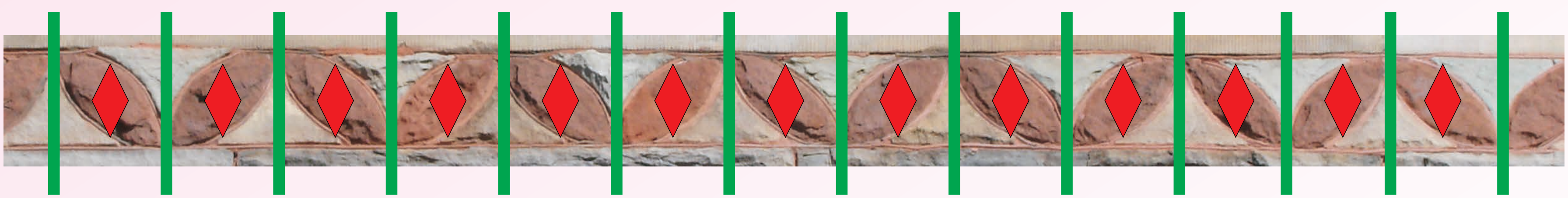

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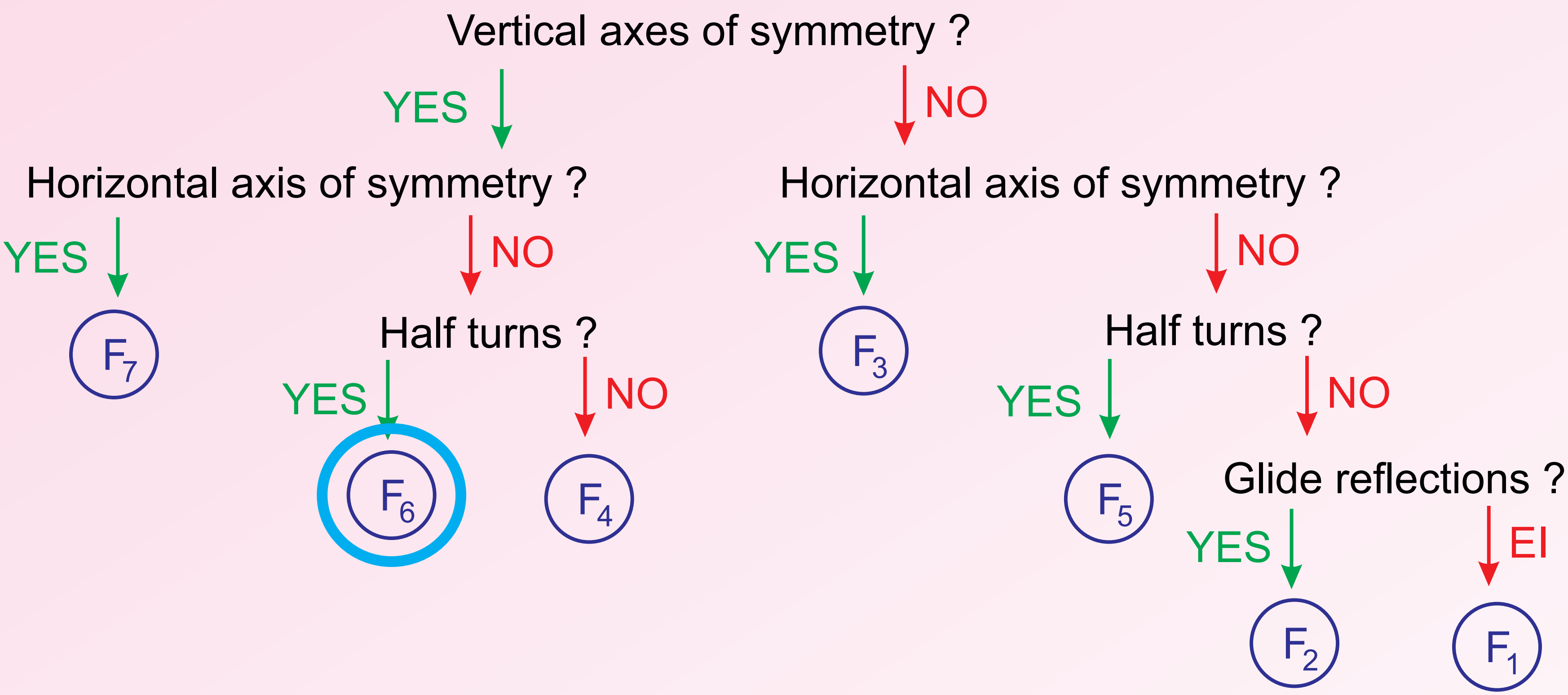

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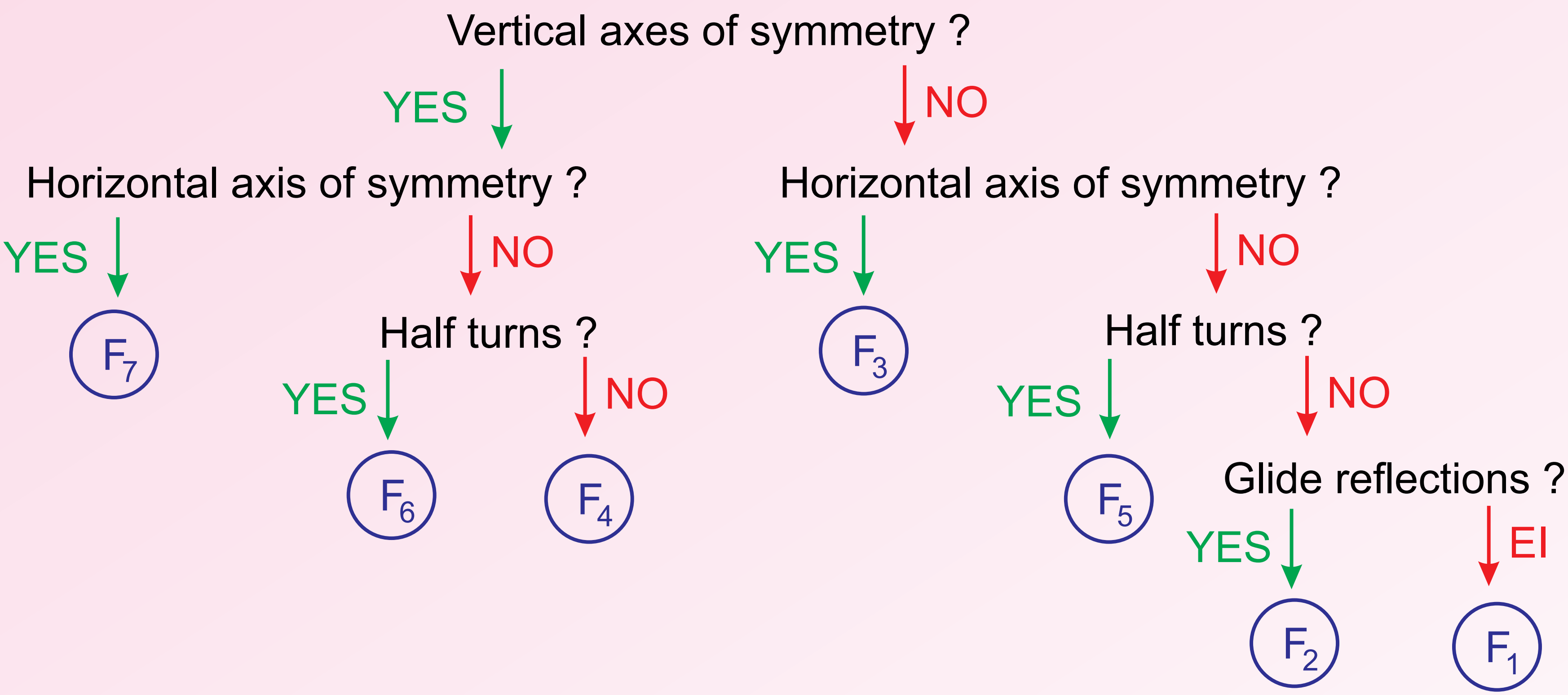

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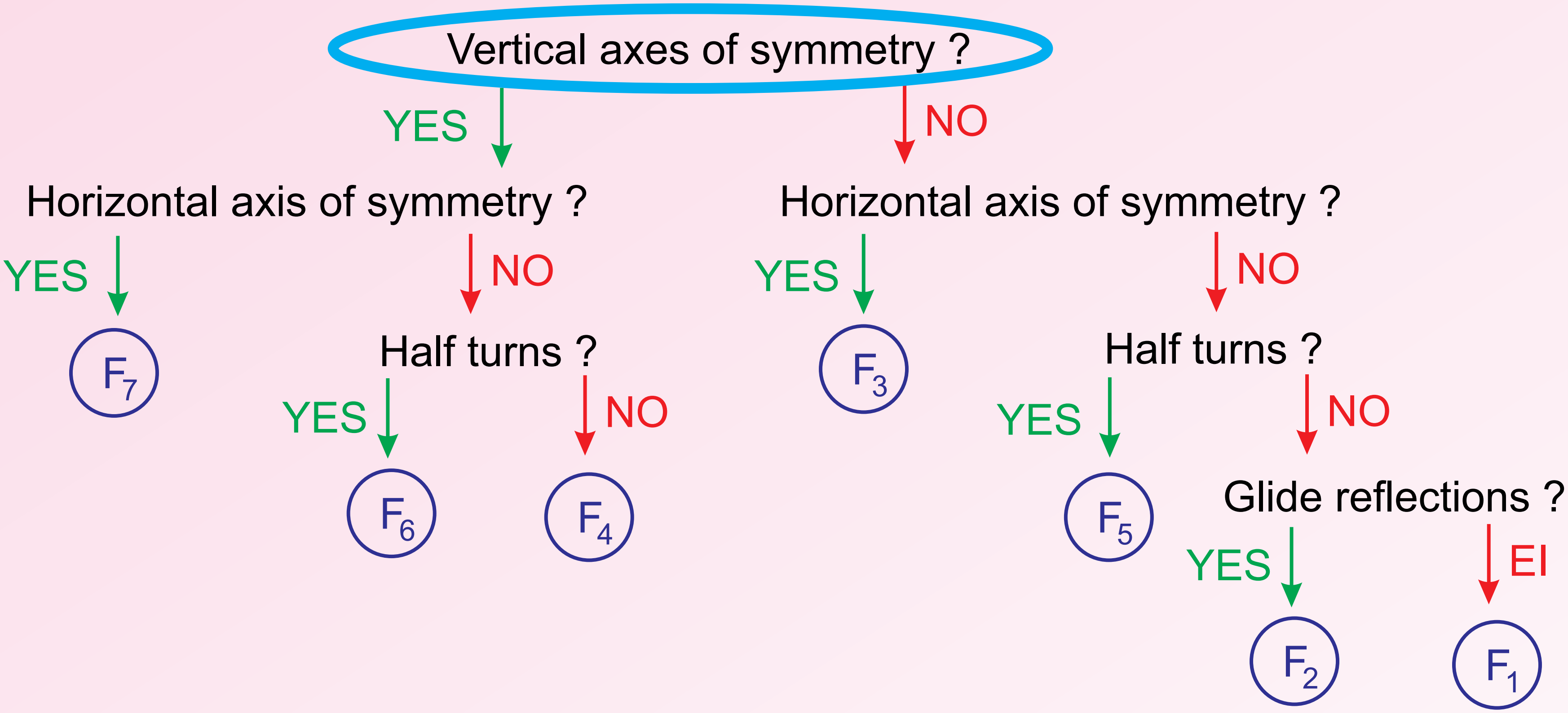

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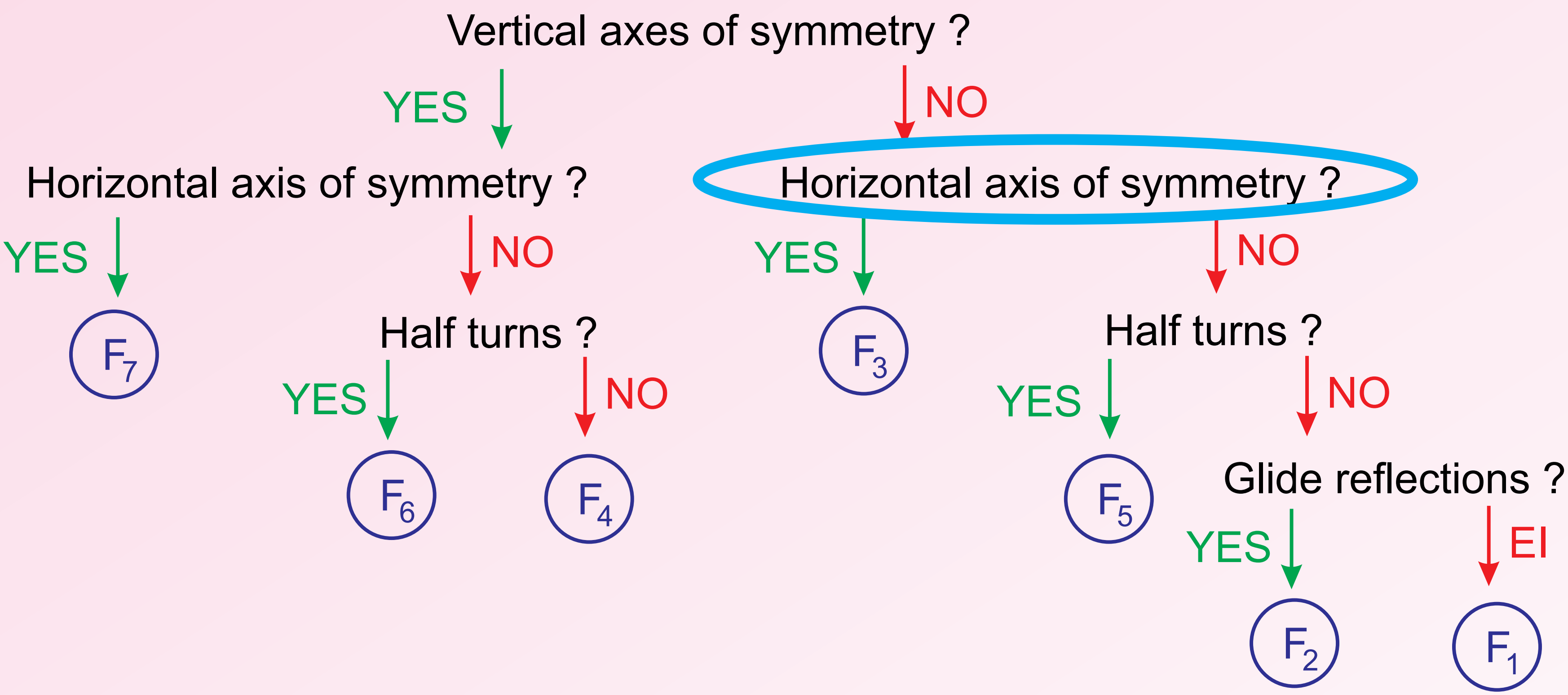

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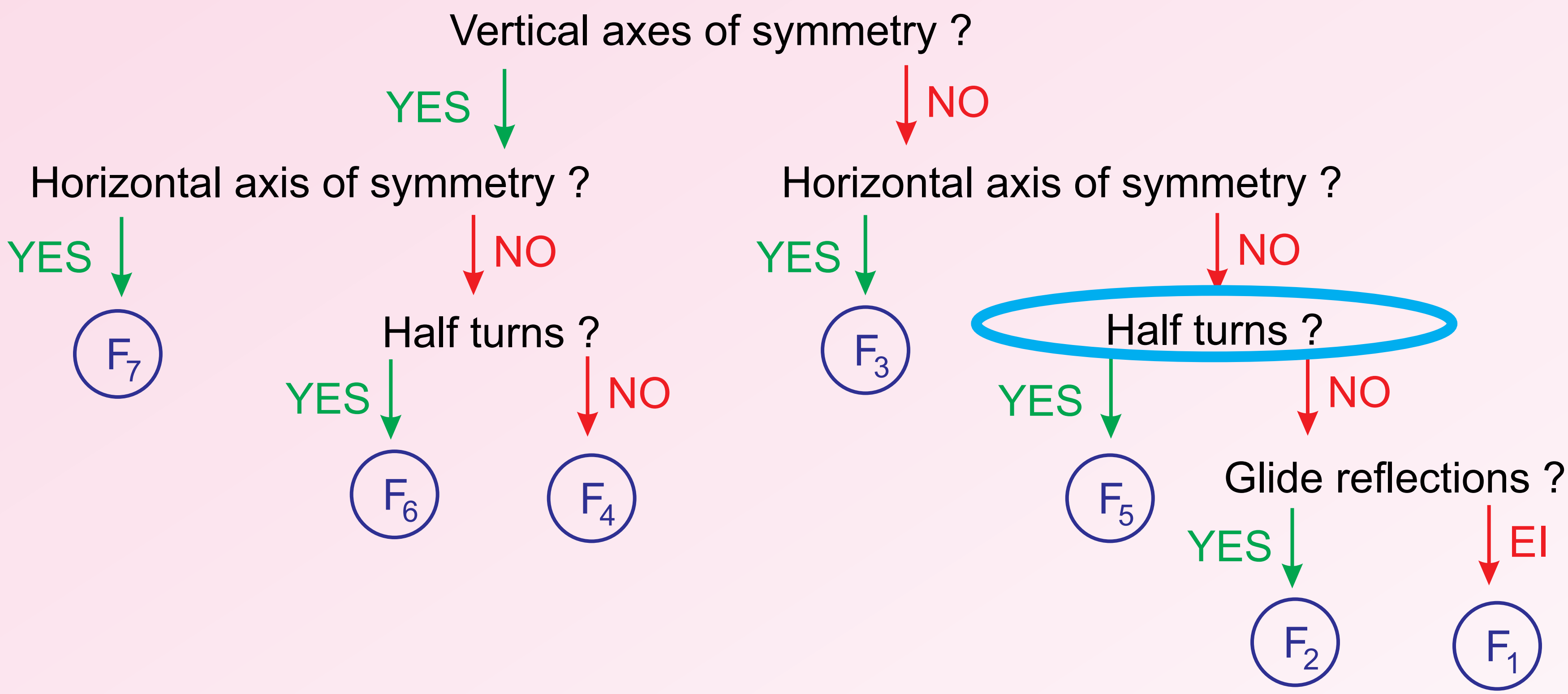

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    - YES → F_7
    - NO → Half turns ?
      - YES → F_6
      - NO → F_4
  - NO → Horizontal axis of symmetry ?
    - YES → F_3
    - NO → Half turns ?
      - YES → F_5
      - NO → Glide reflections ?
        - YES → F_2
        - EI → F_1

Wallpaper groups

If the same pattern is repeated periodically to infinity in two directions then the symmetry group is a **wallpaper group**.

Wallpaper groups

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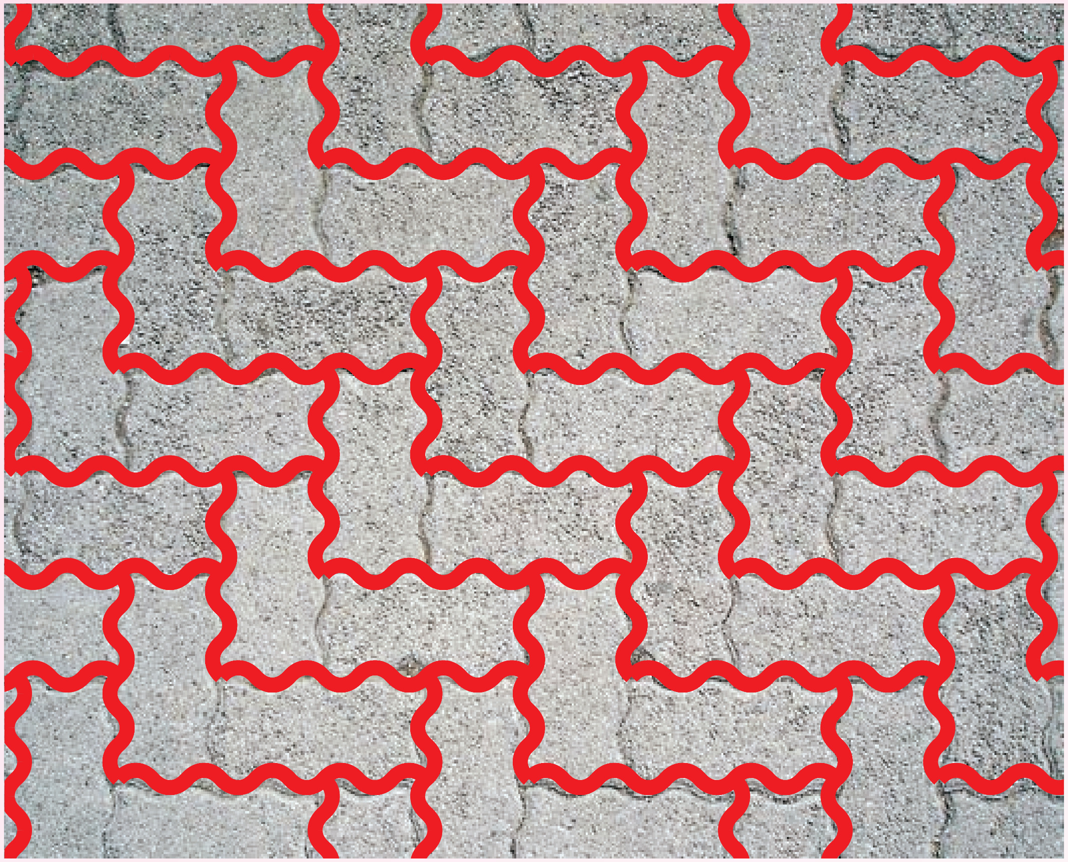

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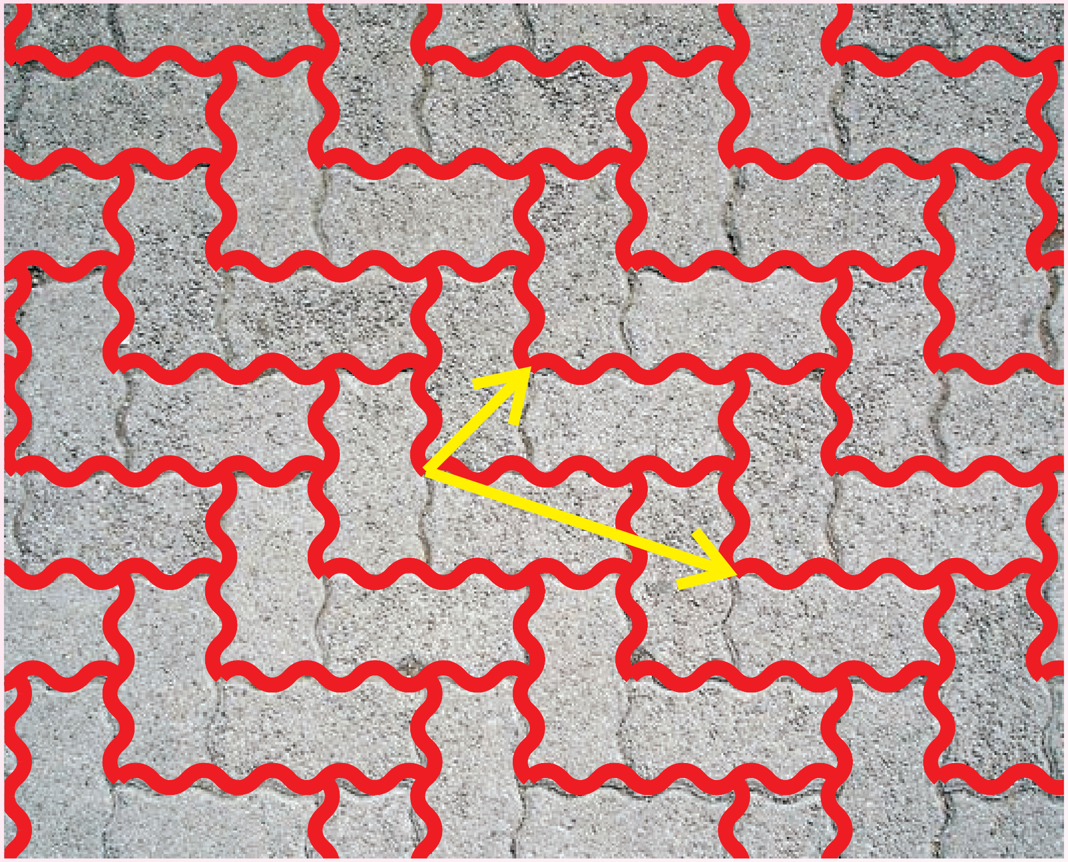

The translations of a wallpaper group are thus generated by two translations in different directions.

Wallpaper groups

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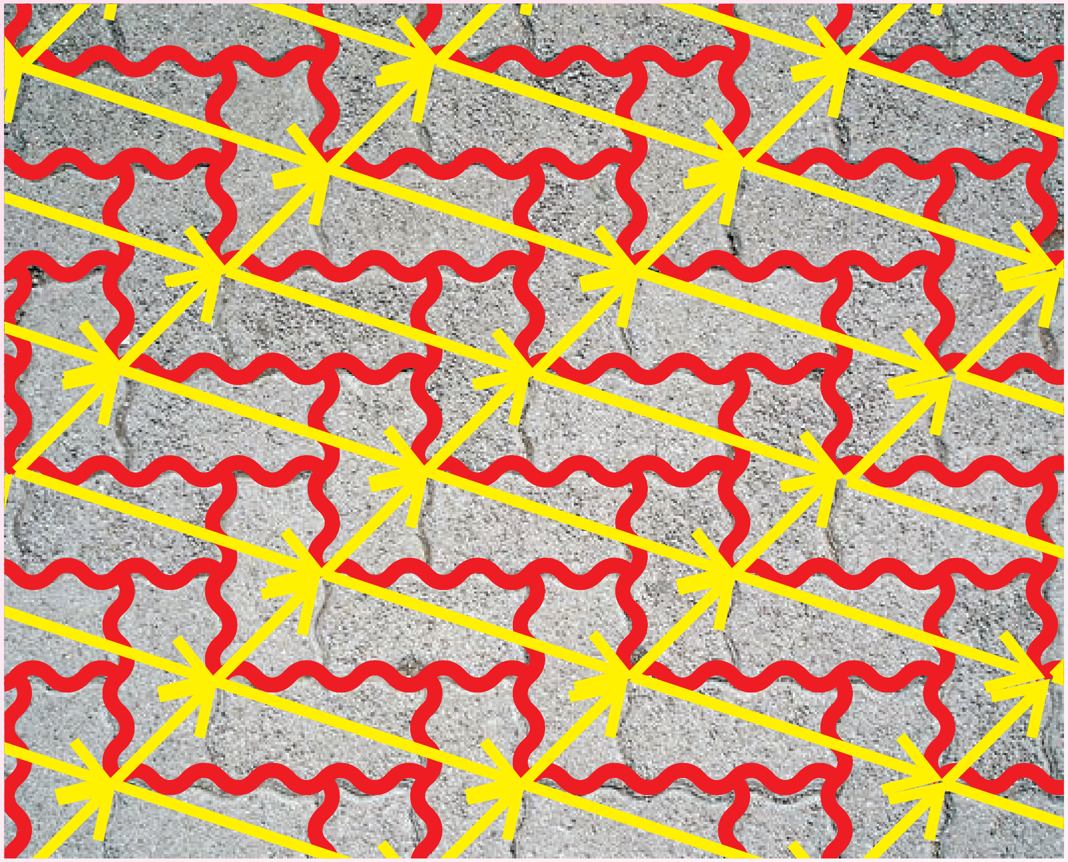
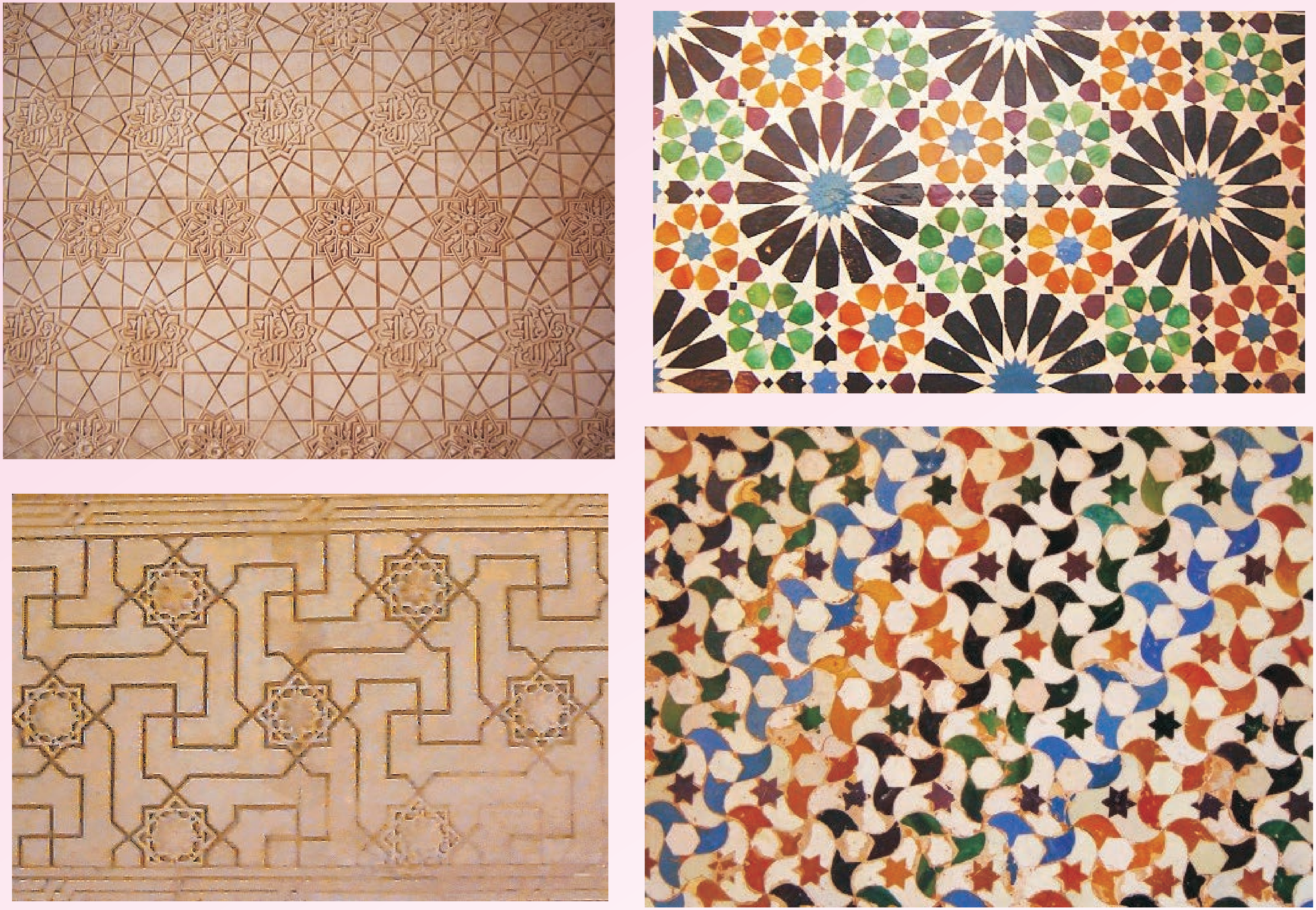


The translations of a wallpaper group are thus generated by two translations in different directions.



Alhambra, Granada, Spain

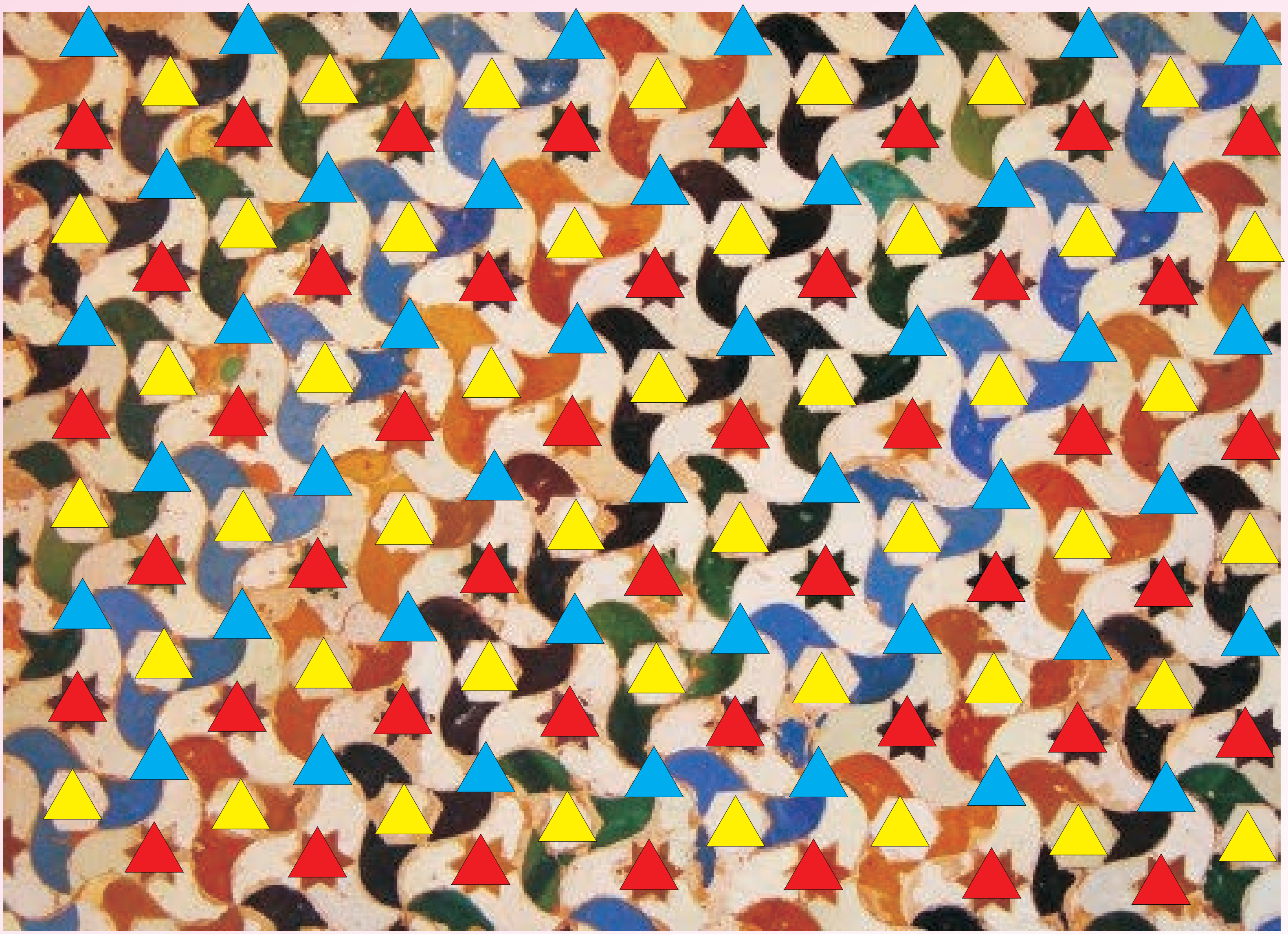
Threefold rotational symmetries:

Threefold rotational symmetries:

Threefold rotational symmetries:

Threefold rotational symmetries:

Threefold rotational symmetries:







M.C. Escher

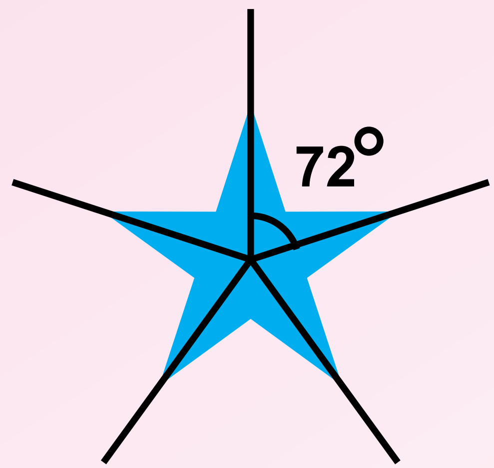
Fourfold rotational symmetries:

Fourfold rotational symmetries:

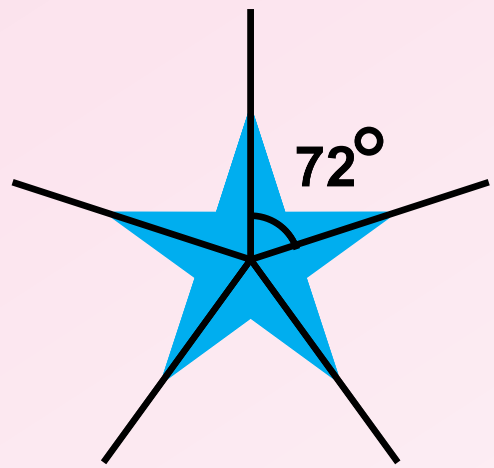
Fourfold rotational symmetries:

Red diamonds indicate centers of half turns.

We saw examples of wallpaper groups with 3- and 4-fold rotational symmetries. What about a wallpaper group containing a 5-fold rotation ?



We saw examples of wallpaper groups with 3- and 4-fold rotational symmetries. What about a wallpaper group containing a 5-fold rotation ?



Not possible: A fivefold rotation is not compatible with the translations of the group.

Crystallographic restriction: A wallpaper group may only contain 2-, 3-, 4- and 6-fold rotations. In particular, a fivefold rotation is not possible.

A sixfold rotation is possible:

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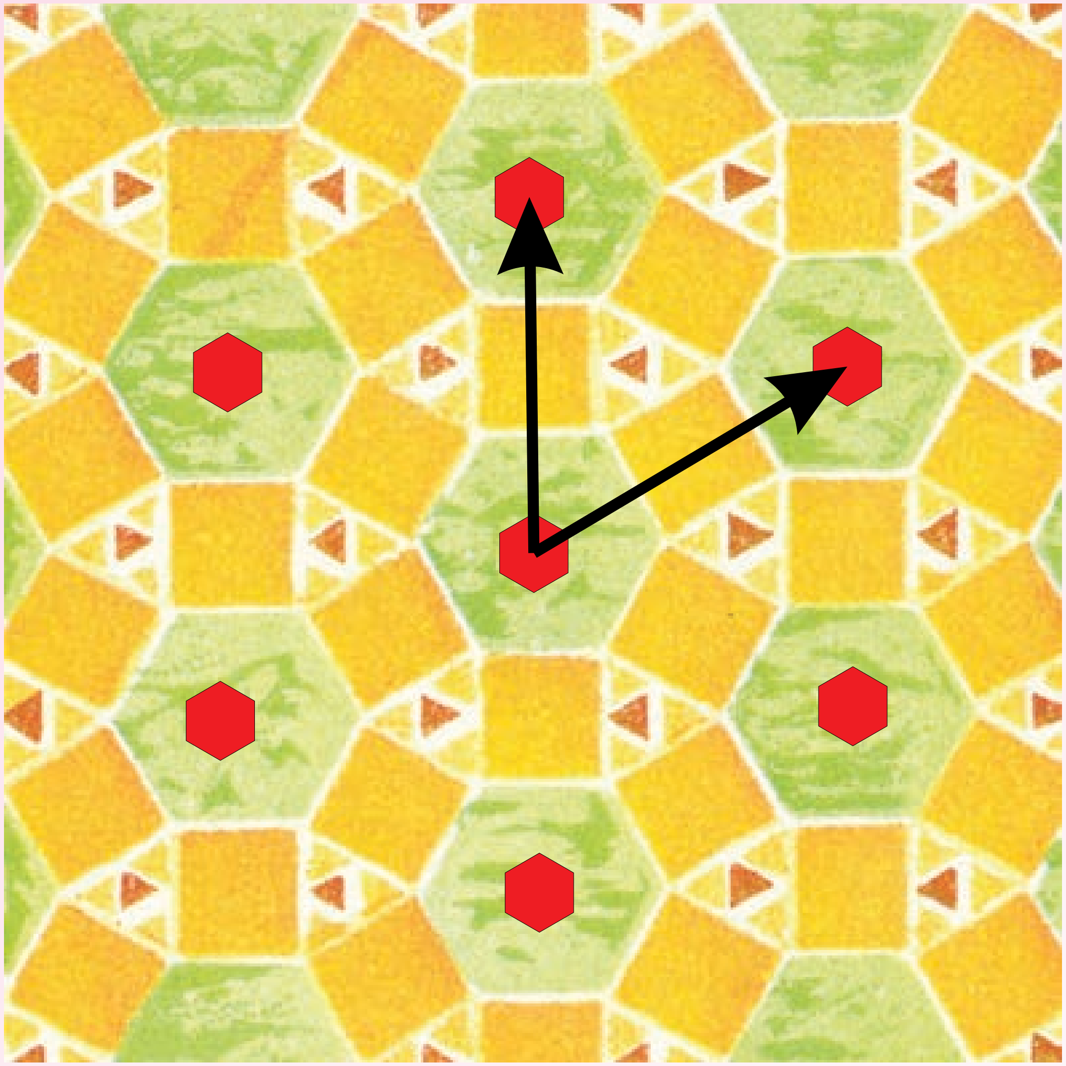
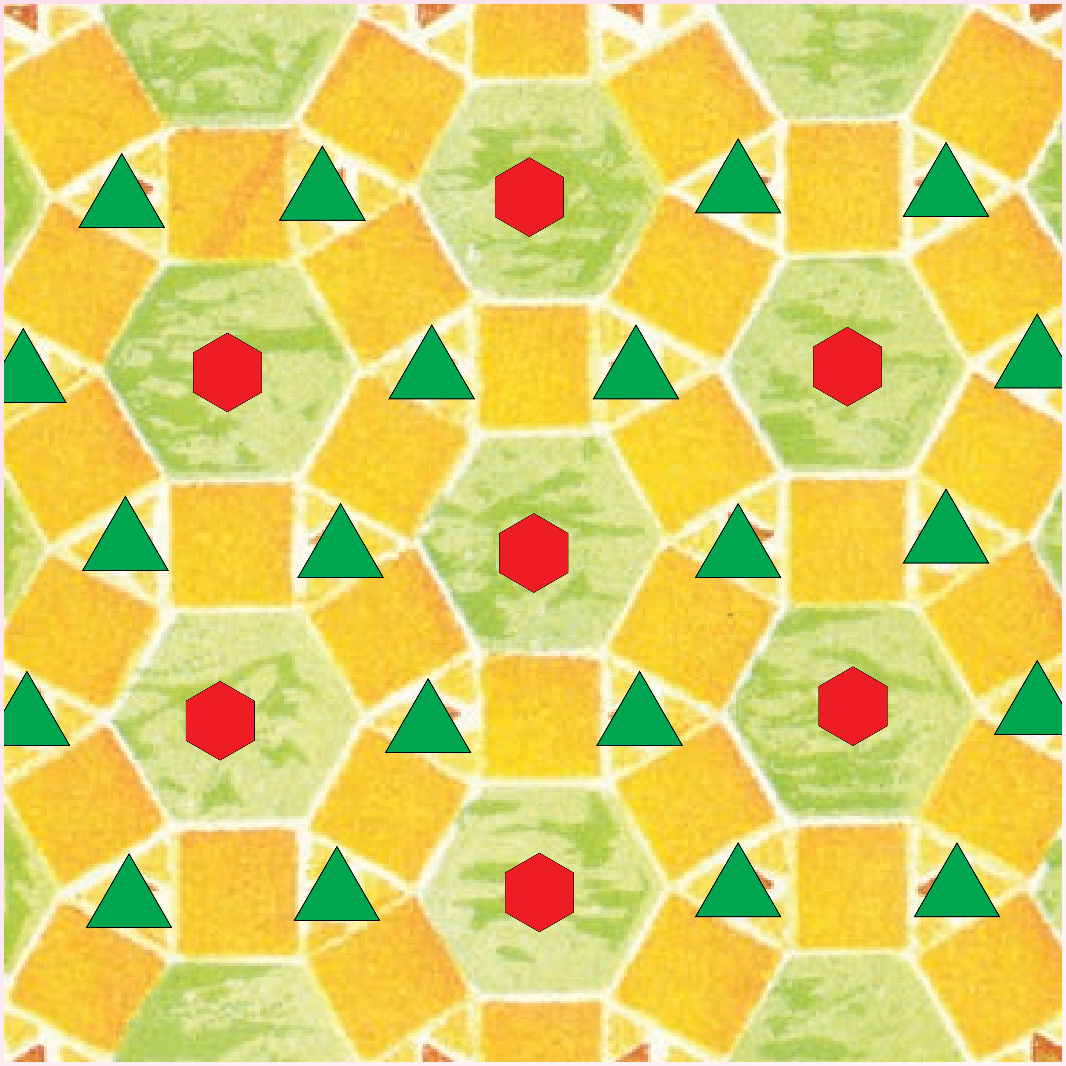
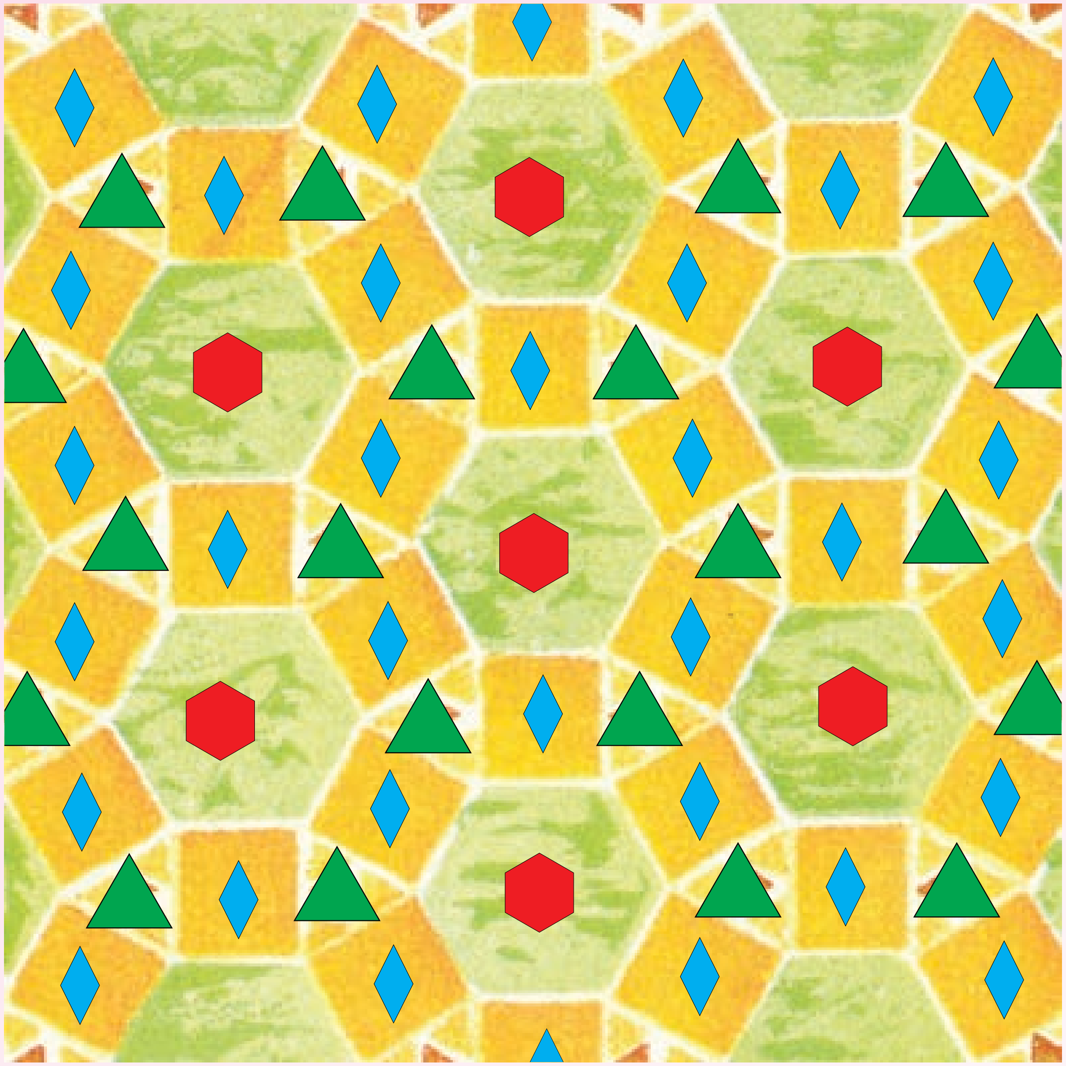


A sixfold rotation is possible:



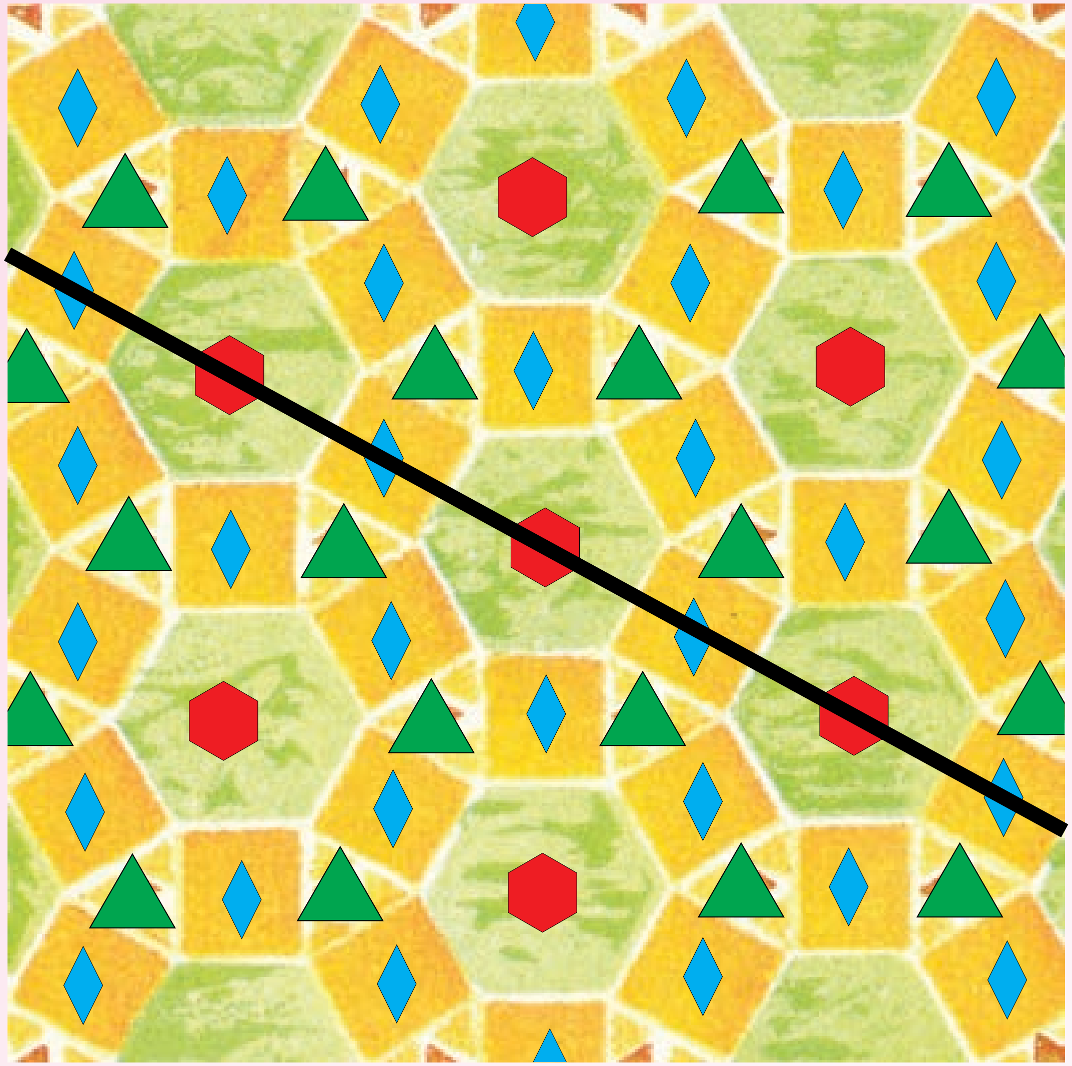
The sixfold rotations generate automatically also threefold rotations. . .

A sixfold rotation is possible:



. . . and half turns.

A sixfold rotation is possible:

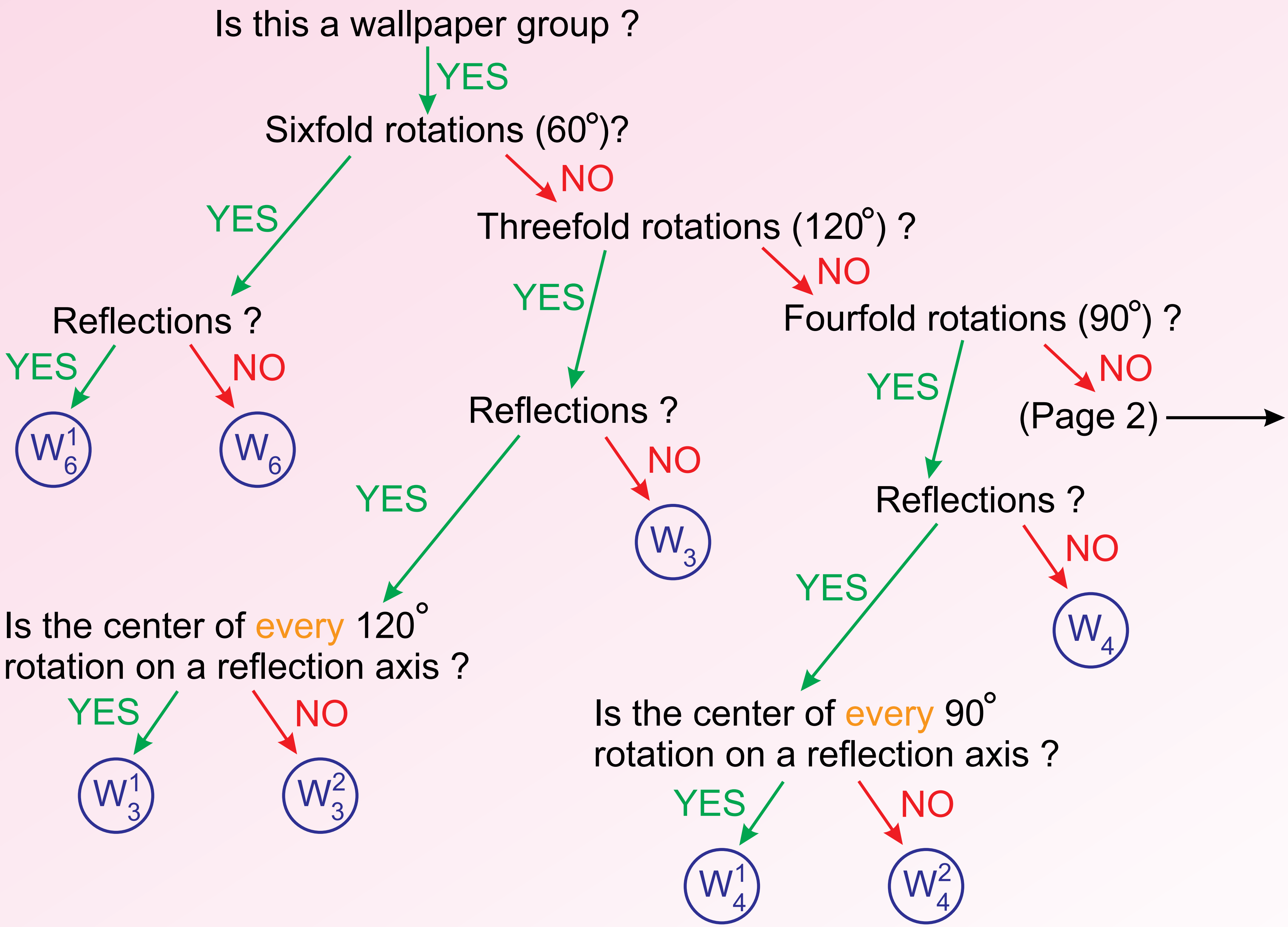


This sample also has reflections.

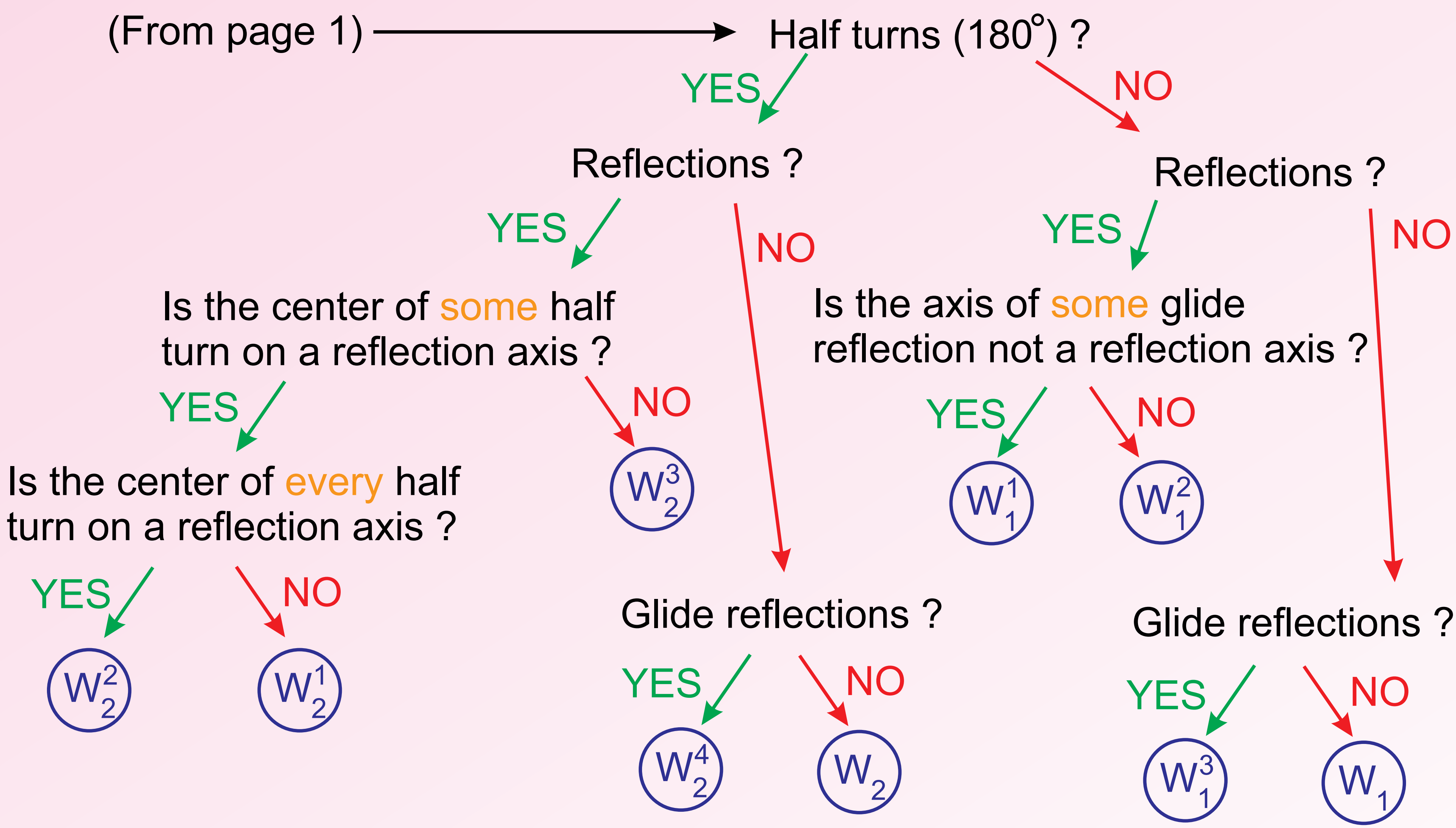
There exist **17** different wallpaper groups.

(If two wallpapers have the same symmetry group they can be scaled, rotated, skewed and moved on top of each other so that precisely the same isometries are symmetries to both of them.)

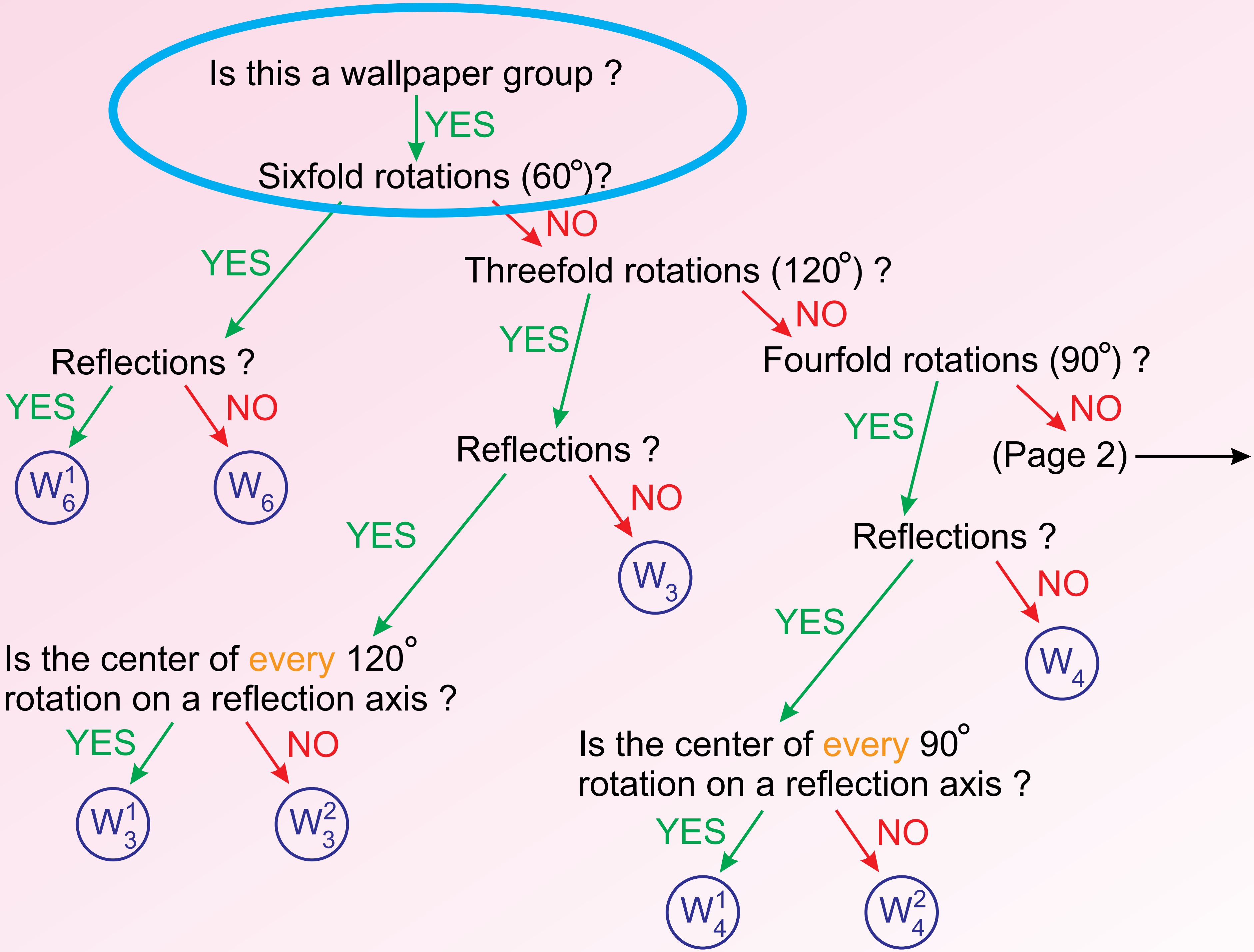
A flow chart to determine a wallpaper group (page 1)

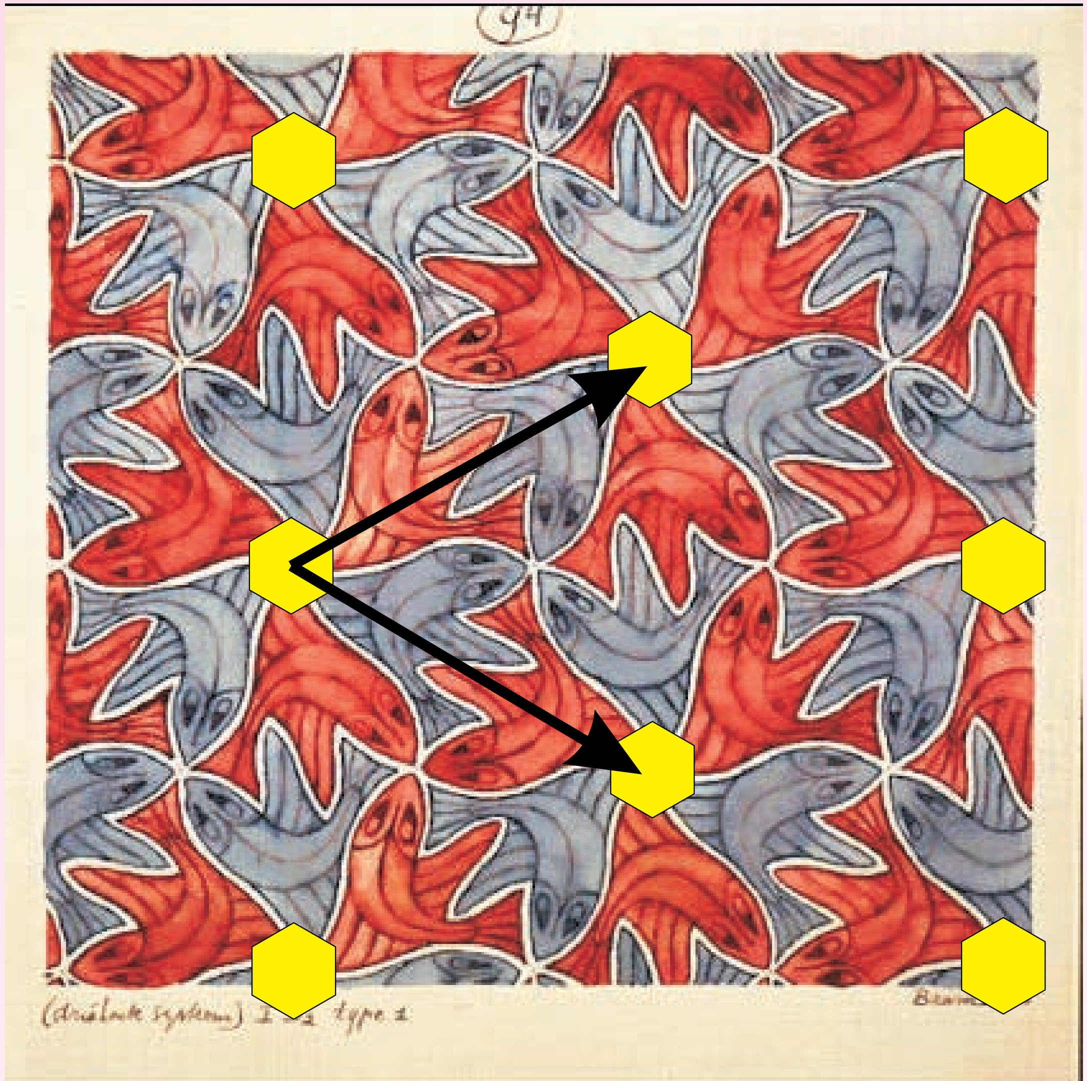


A flow chart to determine a wallpaper group (page 2)

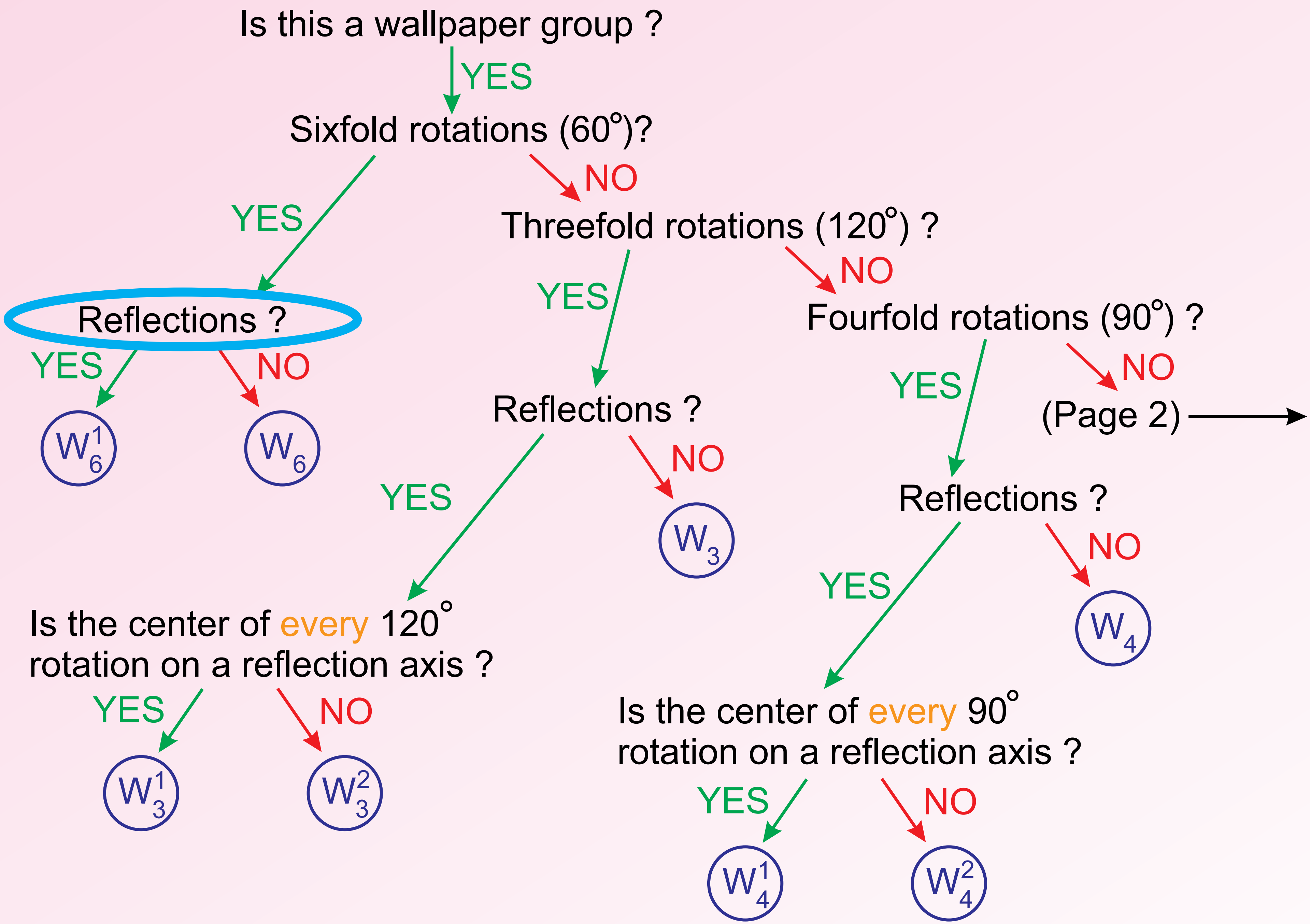


Let us determine the symmetry group of this Escher painting. Let us first ignore the colors.

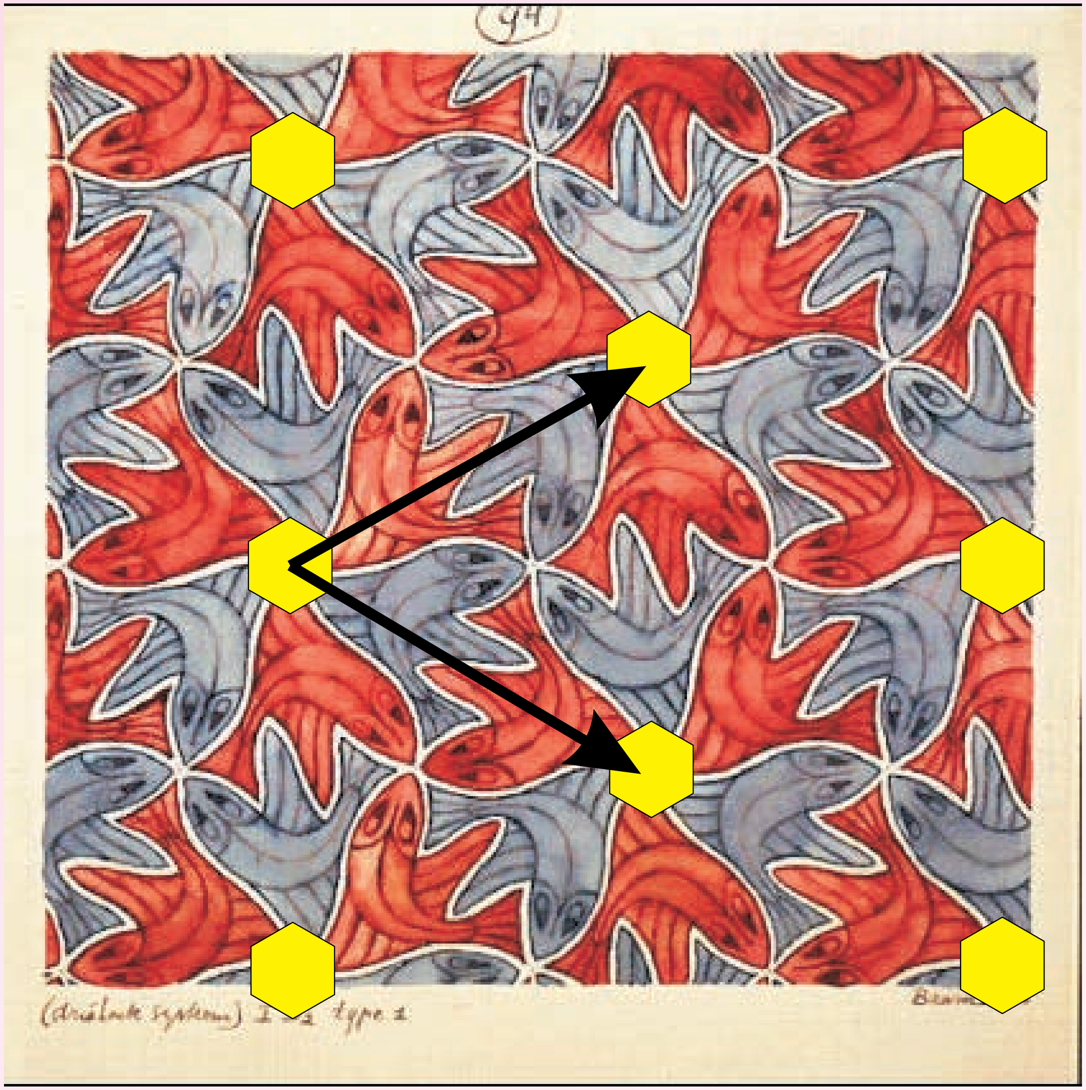

Is this a wallpaper group ?
YES
Sixfold rotations (60°)?
YES
NO
Threefold rotations (120°) ?
Reflections ?
YES
NO
W_6^1
W_6
YES
NO
Fourfold rotations (90°) ?
Reflections ?
YES
NO
W_3
Is the center of every 120° rotation on a reflection axis ?
YES
NO
W_3^1
W_3^2
YES
NO
(Page 2)
Reflections ?
YES
NO
W_4
Is the center of every 90° rotation on a reflection axis ?
YES
NO
W_4^1
W_4^2



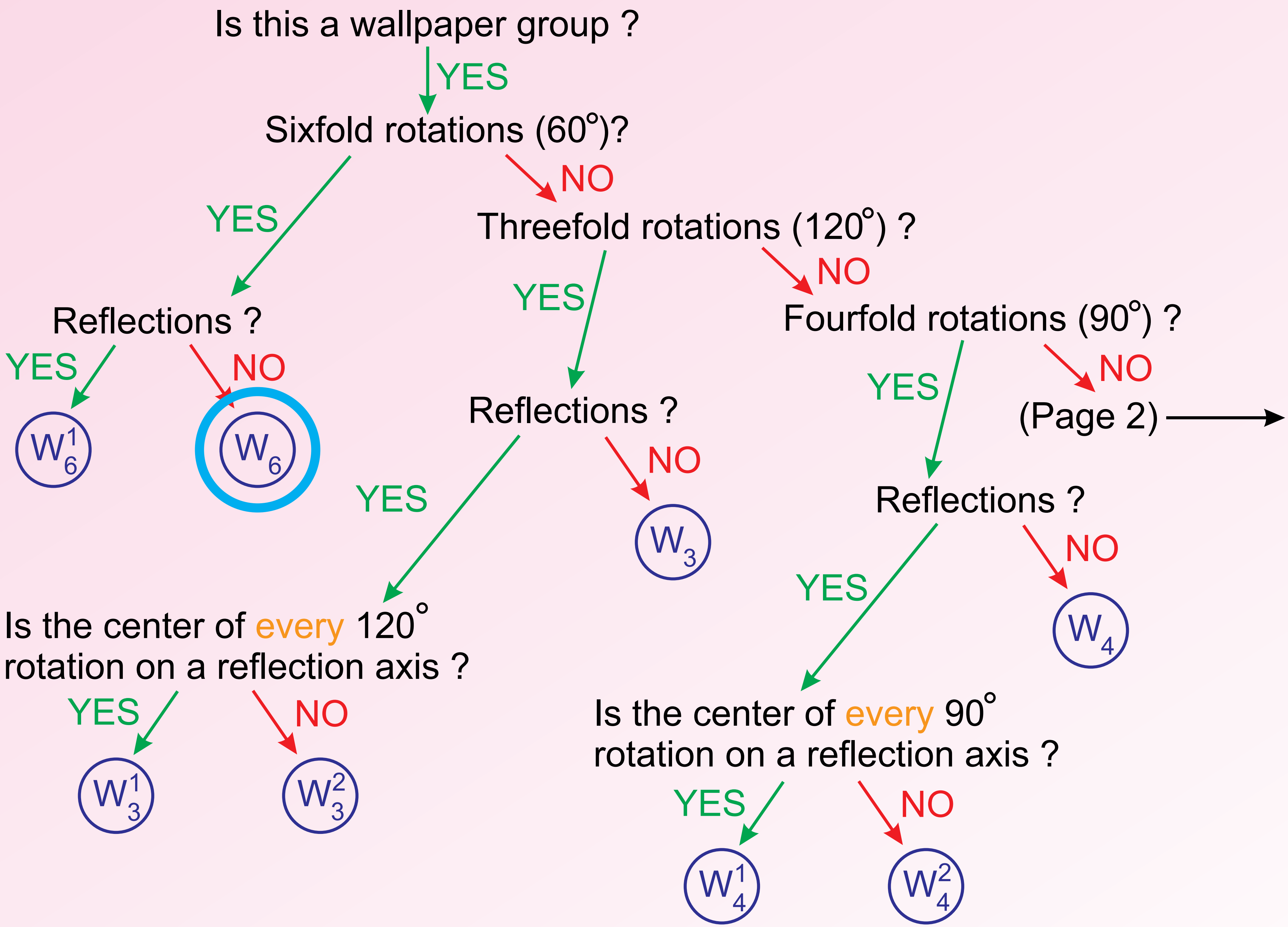
Sixfold rotations (and two generating translations).

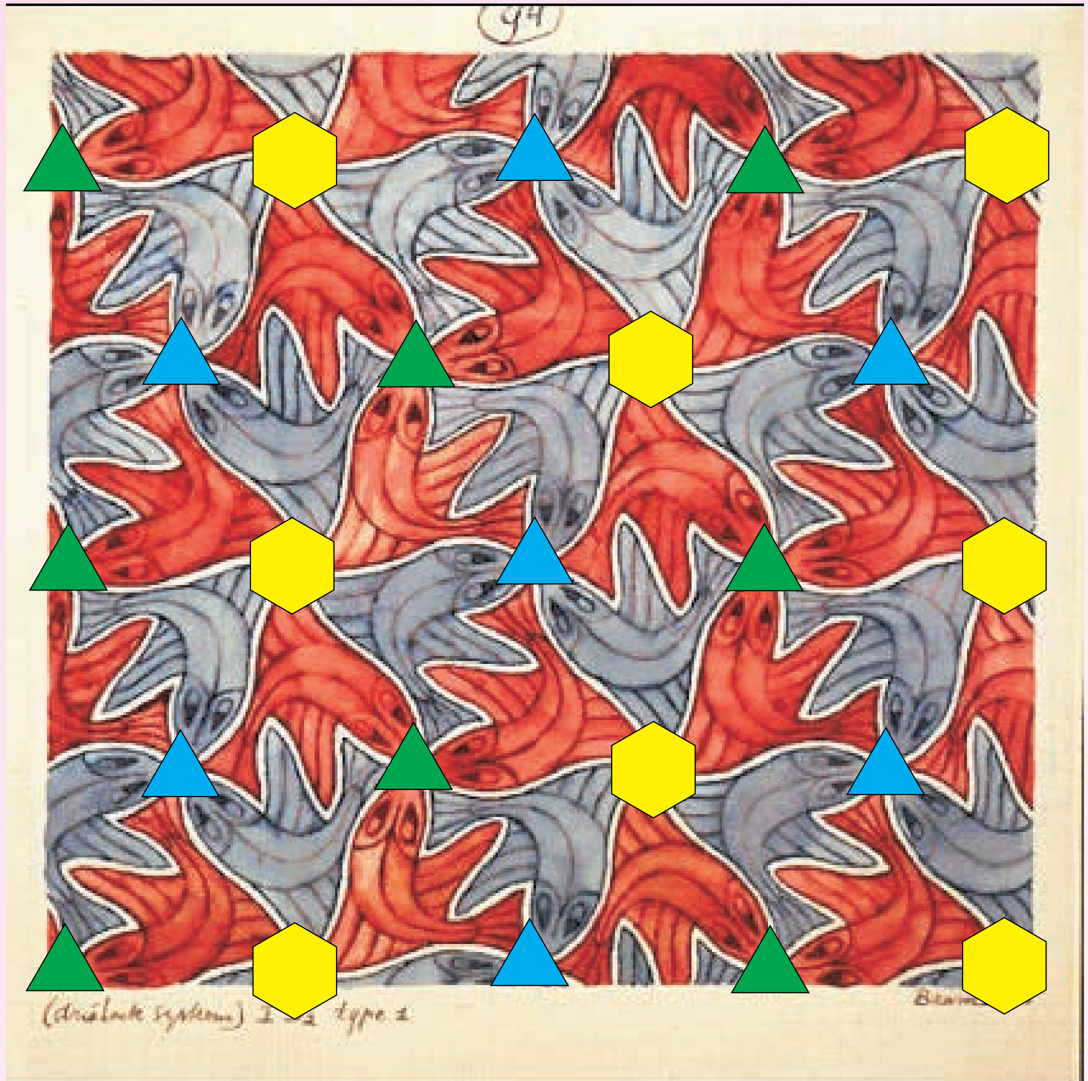


Is this a wallpaper group ?
YES
Sixfold rotations (60°)?
YES
NO
Threefold rotations (120°) ?
Reflections ?
YES
NO
W_6^1
W_6
YES
NO
Fourfold rotations (90°) ?
Reflections ?
YES
NO
W_3
Is the center of every 120° rotation on a reflection axis ?
YES
NO
W_3^1
W_3^2
YES
NO
(Page 2)
Reflections ?
YES
NO
W_4
Is the center of every 90° rotation on a reflection axis ?
YES
NO
W_4^1
W_4^2



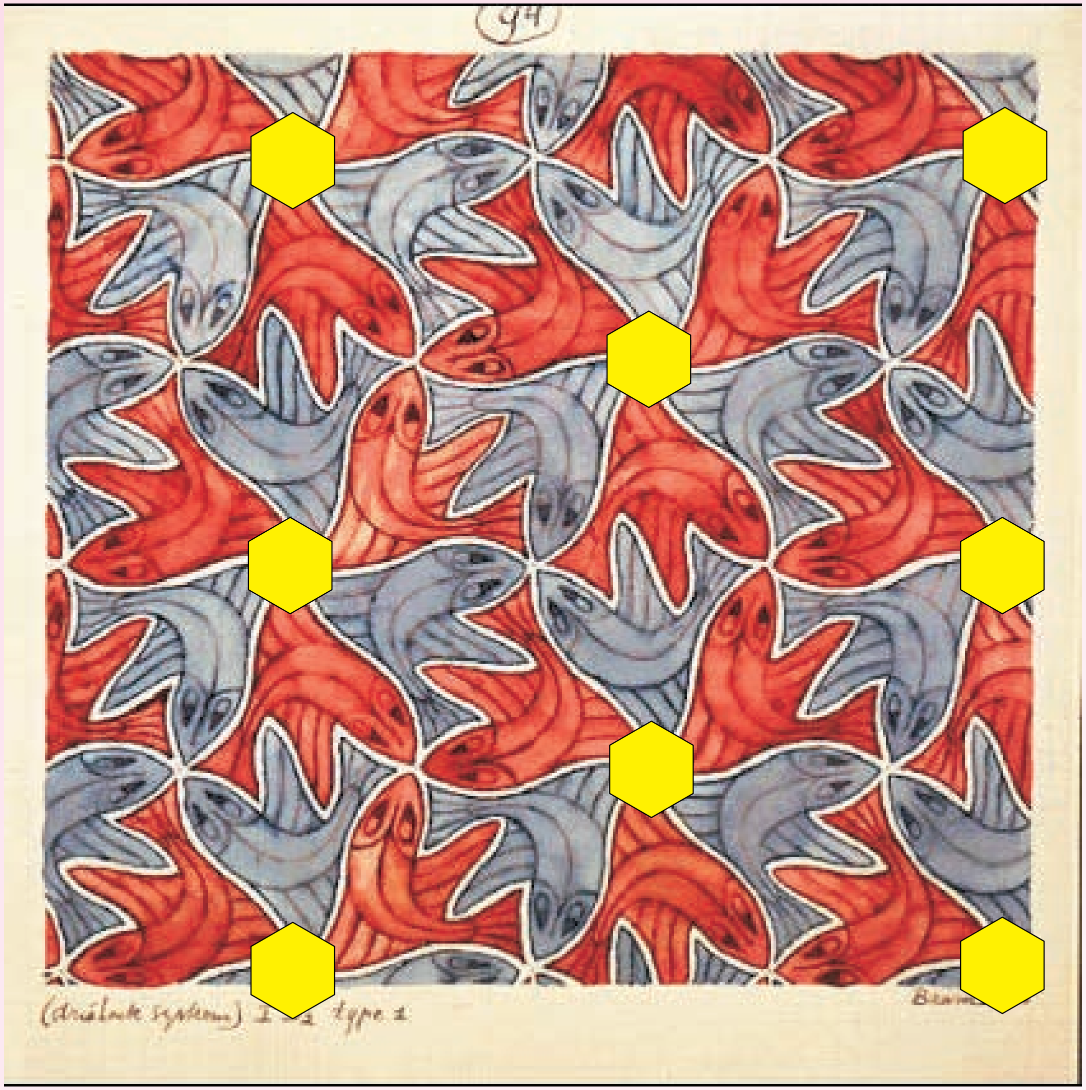
No odd symmetries: The fish are all turning clockwise.


Is this a wallpaper group ?
YES
Sixfold rotations (60°)?
YES
NO
Threefold rotations (120°) ?
Reflections ?
YES
NO
W_6^1
W_6
YES
NO
Fourfold rotations (90°) ?
Reflections ?
YES
NO
W_3
YES
NO
(Page 2)
Is the center of every 120° rotation on a reflection axis ?
YES
NO
W_3^1
W_3^2
Reflections ?
YES
NO
W_4
Is the center of every 90° rotation on a reflection axis ?
YES
NO
W_4^1
W_4^2

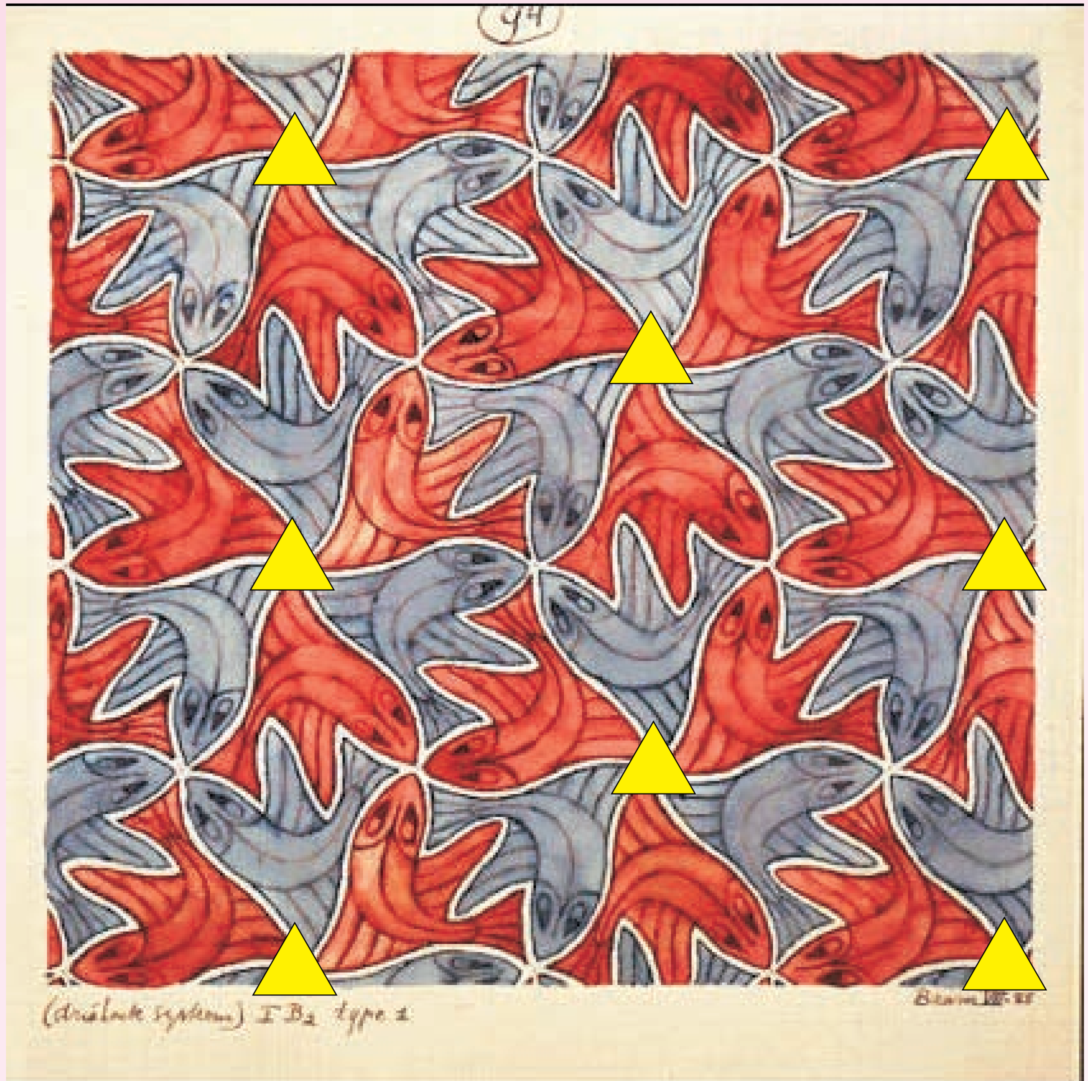


The symmetry group W_6 automatically contains threefold rotations, with centers in the middle of the equilateral triangles formed by the 6-fold centers.

The group also has half turns, at midpoints between 6-fold centers.



Some of the symmetries map fish of different color to each other. If we consider also the colors then the 6-fold symmetries are only 3-fold.



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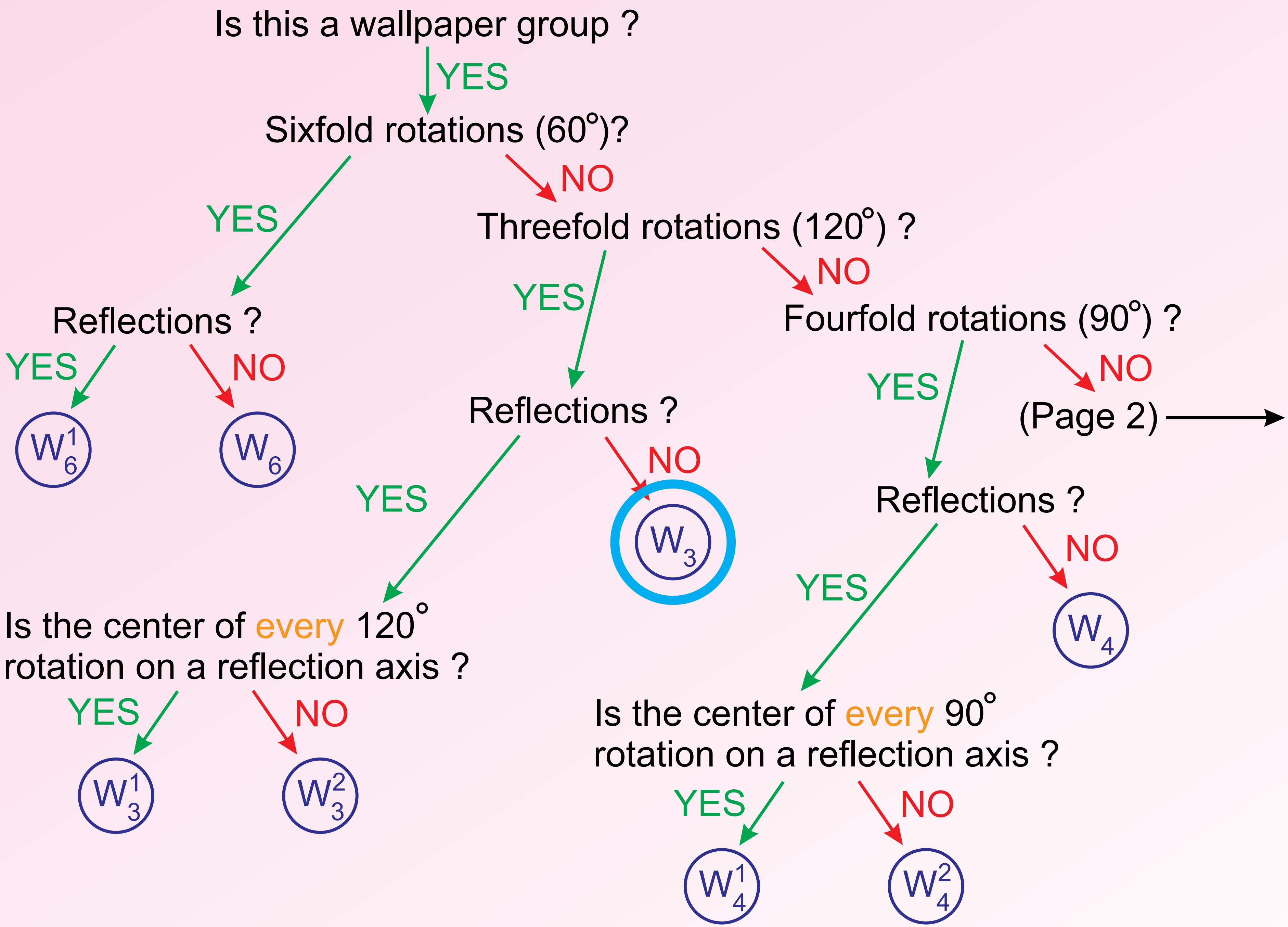
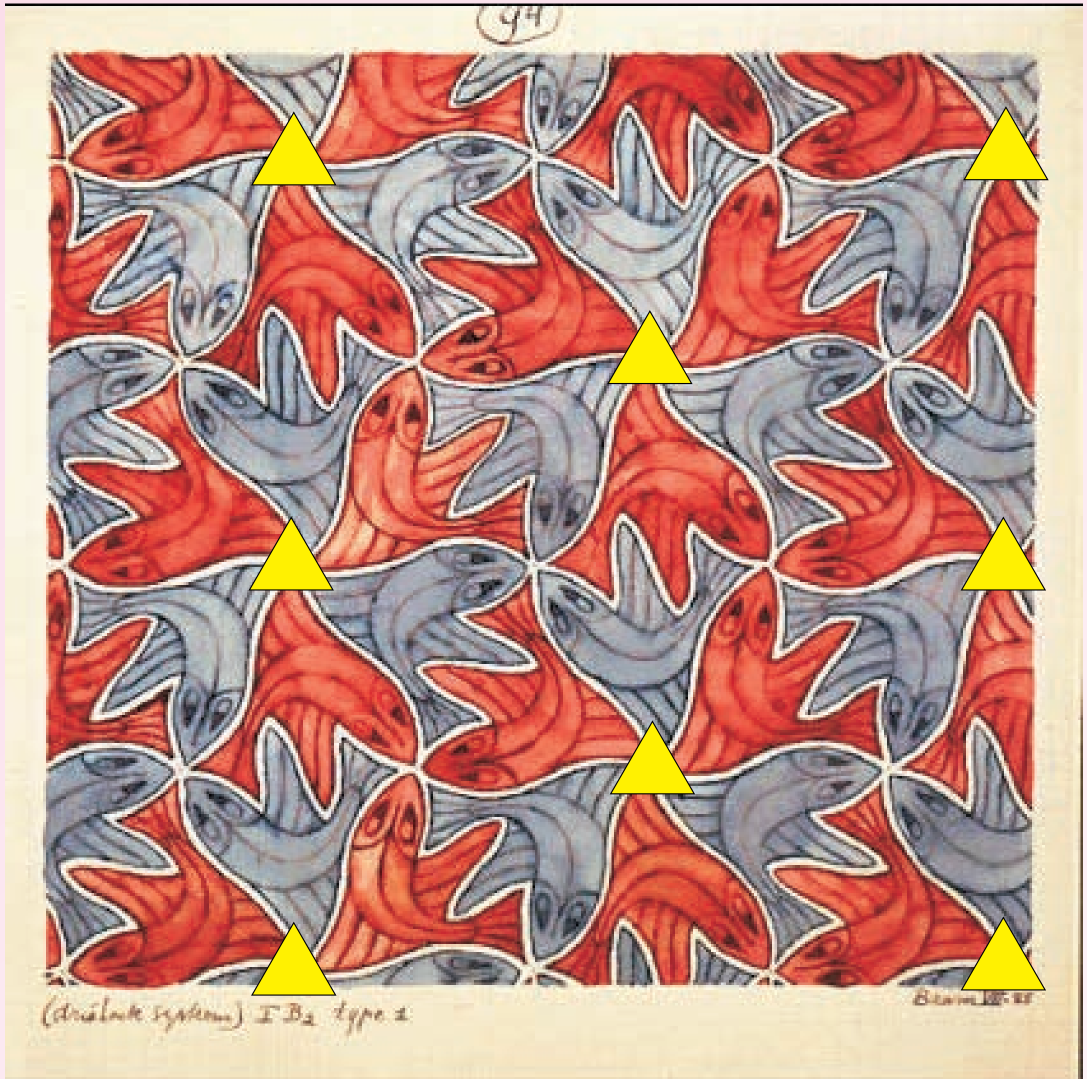
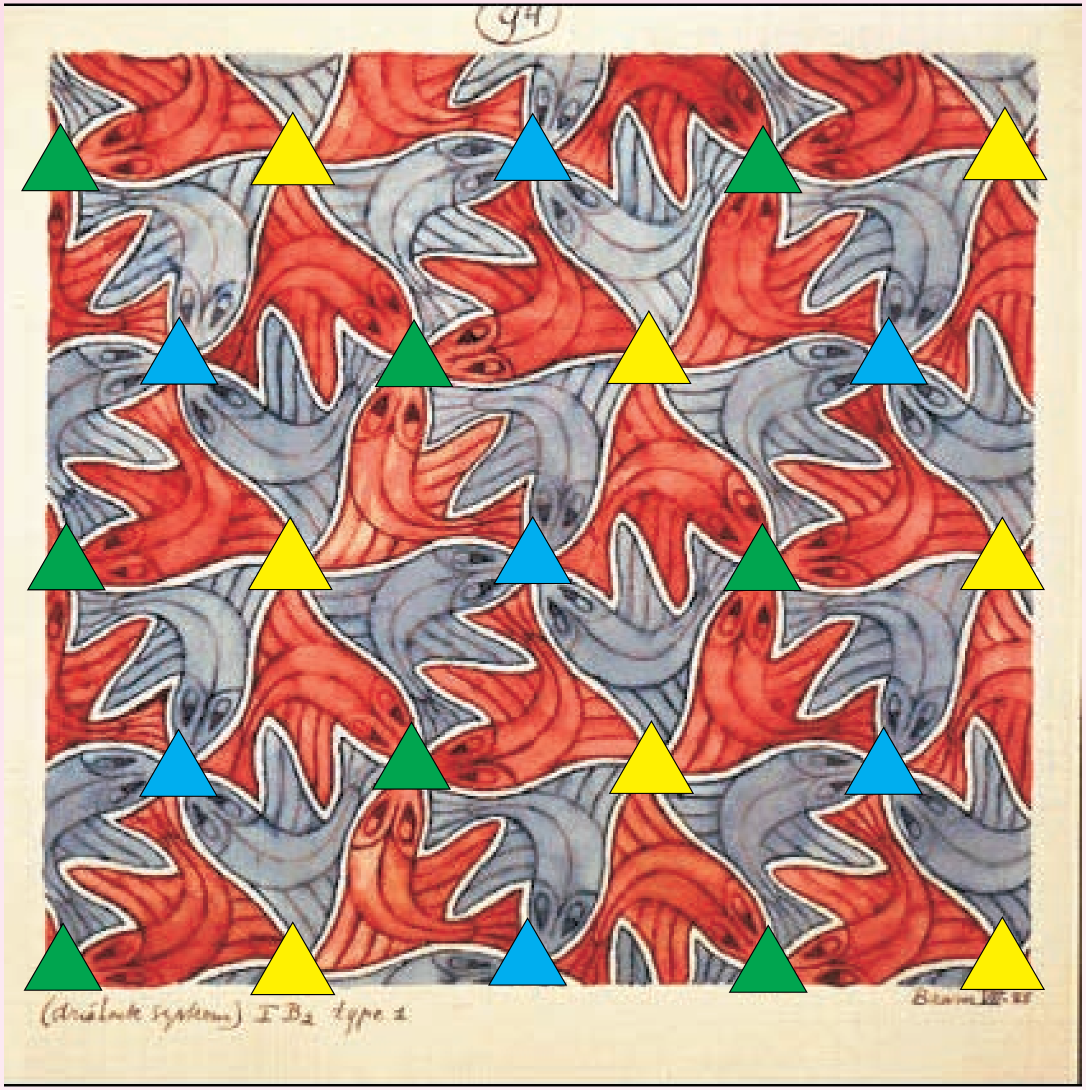


Is this a wallpaper group ?
YES
Sixfold rotations (60°)?
YES
NO
Reflections ?
YES
NO
W_6^1
W_6
Threefold rotations (120°) ?
YES
NO
Reflections ?
YES
NO
W_3
Is the center of every 120° rotation on a reflection axis ?
YES
NO
W_3^1
W_3^2
Fourfold rotations (90°) ?
YES
NO
(Page 2)
Reflections ?
YES
NO
W_4
Is the center of every 90° rotation on a reflection axis ?
YES
NO
W_4^1
W_4^2



The symmetry group is smaller if also colors must be preserved: it is the subgroup W_3 of W_6.

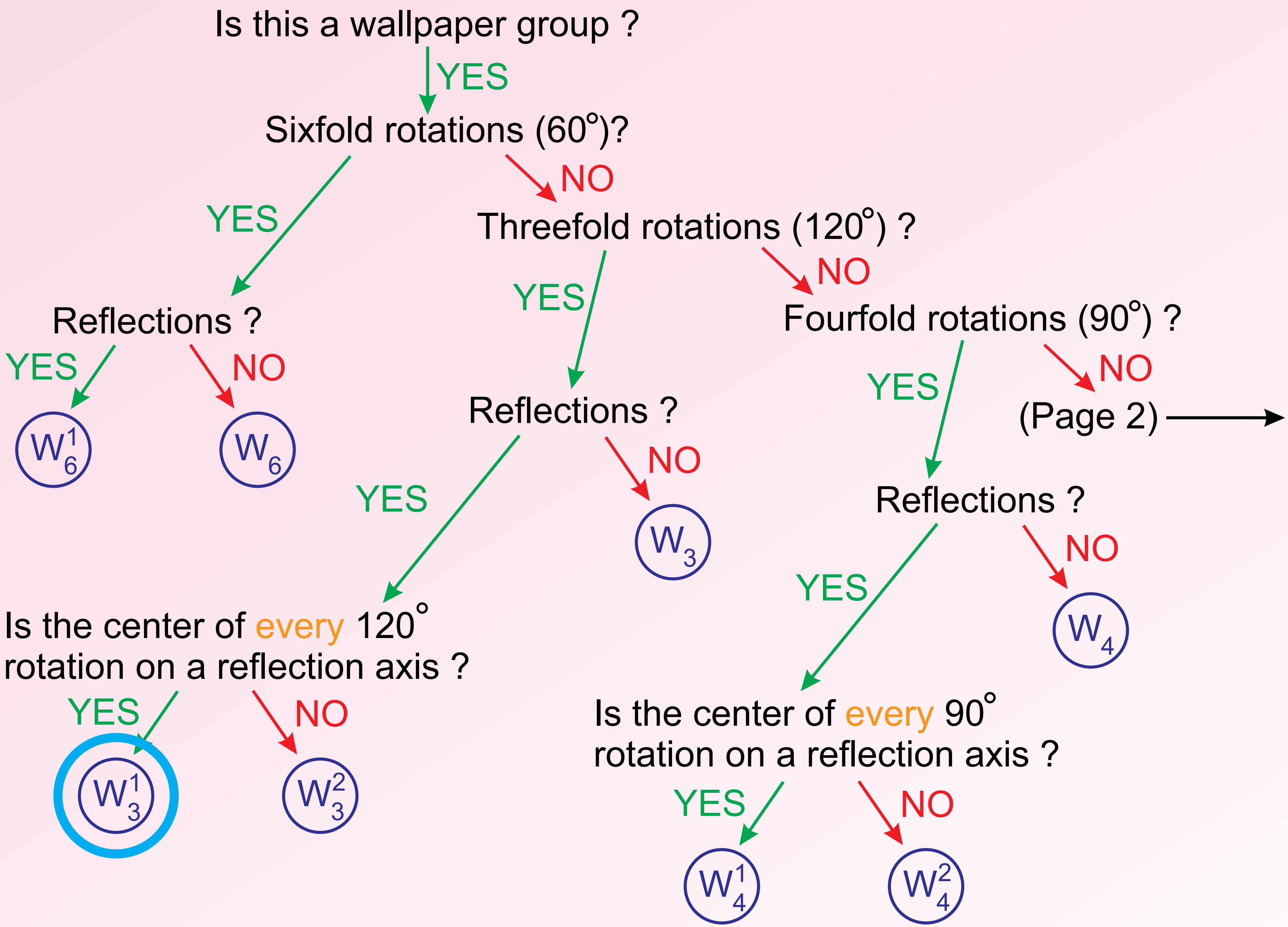


The symmetry group is smaller if also colors must be preserved: it is the subgroup W_3 of W_6.

Another example.

The largest order of rotations is three.

There is a symmetry line through every center of rotation. The symmetry group is W_3^1.


Is this a wallpaper group ?
YES
Sixfold rotations (60°)?
YES
NO
Reflections ?
YES
NO
W_6^1
W_6
Threefold rotations (120°) ?
YES
NO
Reflections ?
YES
NO
W_3
Is the center of every 120° rotation on a reflection axis ?
YES
NO
W_3^1
W_3^2
Fourfold rotations (90°) ?
YES
NO
(Page 2)
Reflections ?
YES
NO
W_4
Is the center of every 90° rotation on a reflection axis ?
YES
NO
W_4^1
W_4^2

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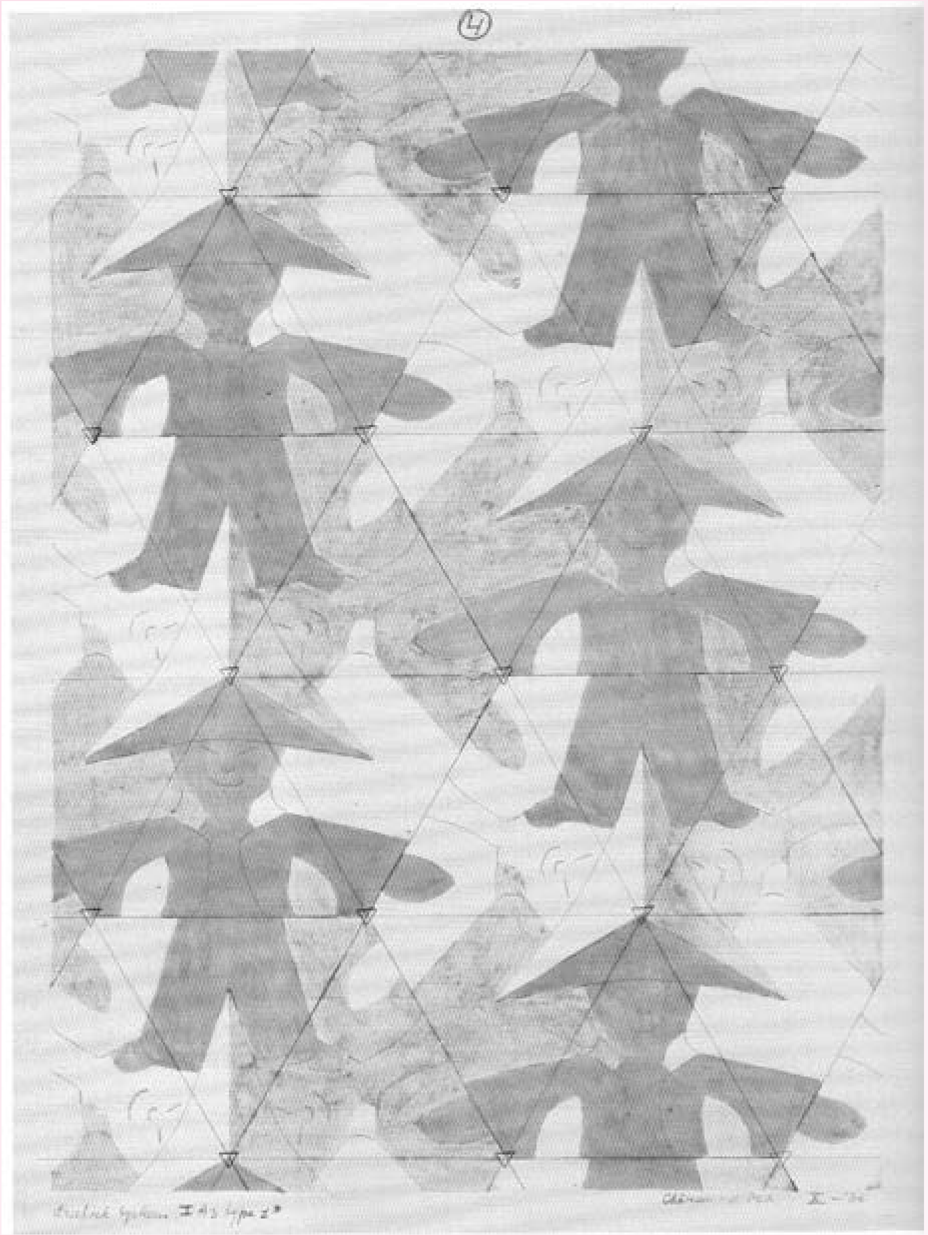
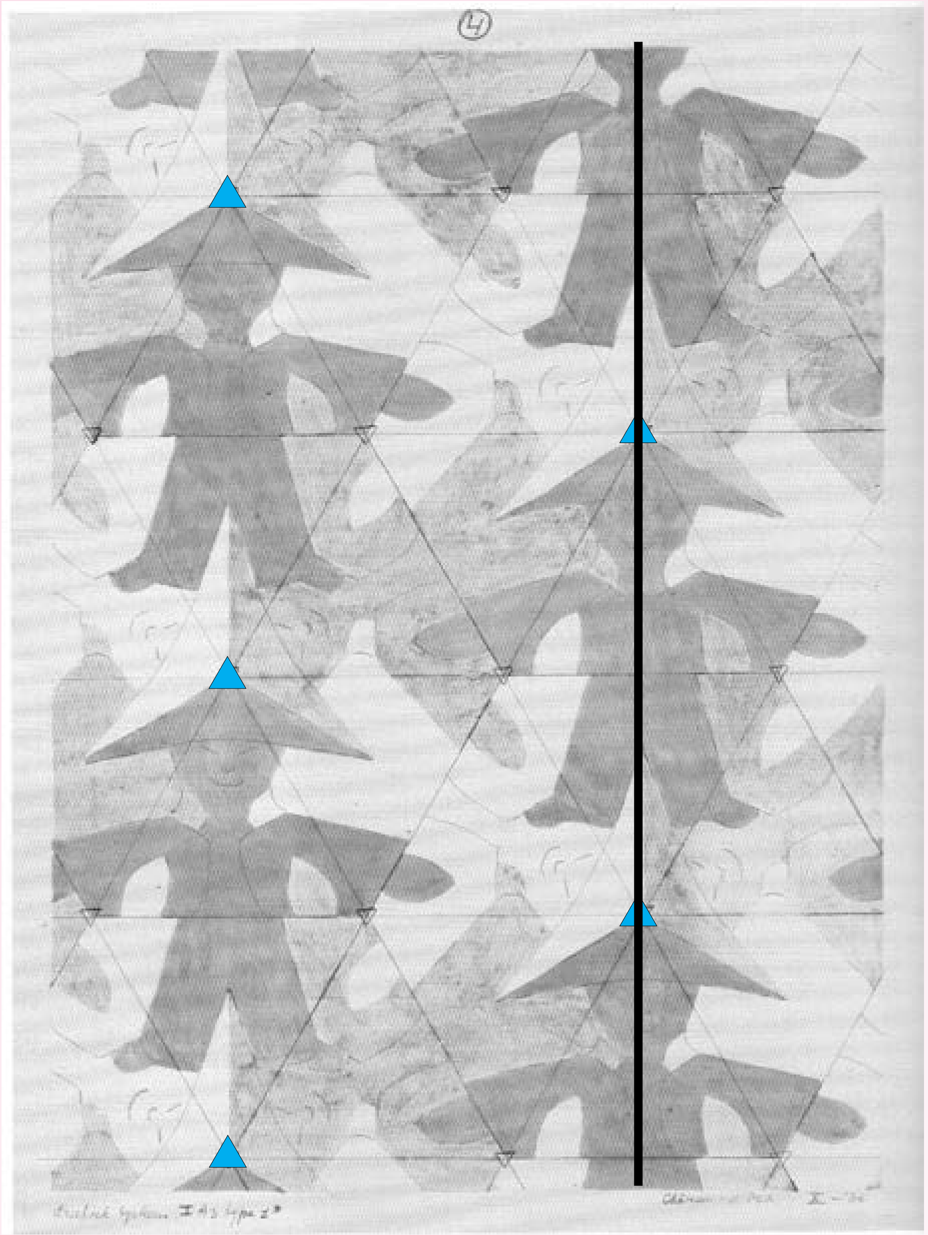
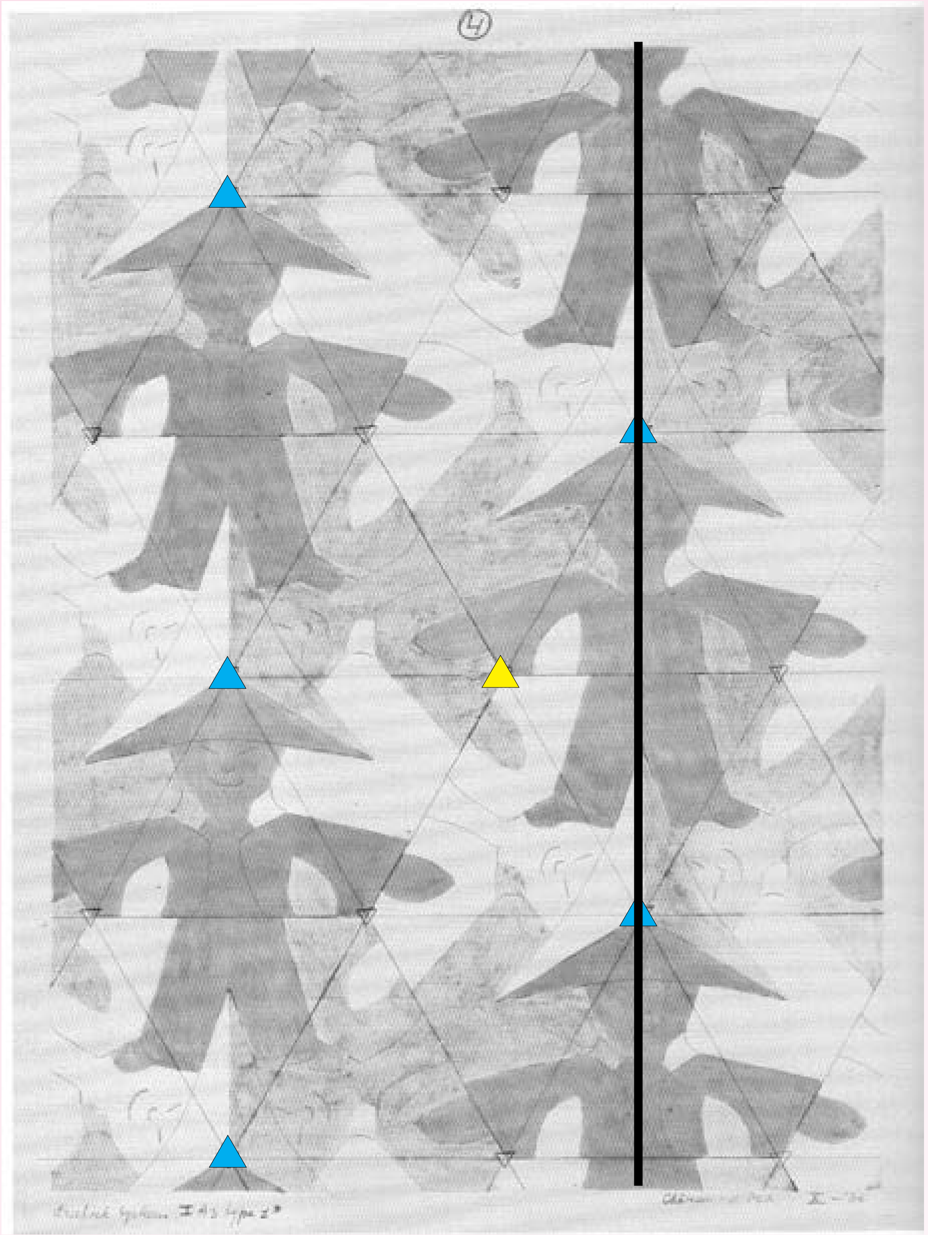


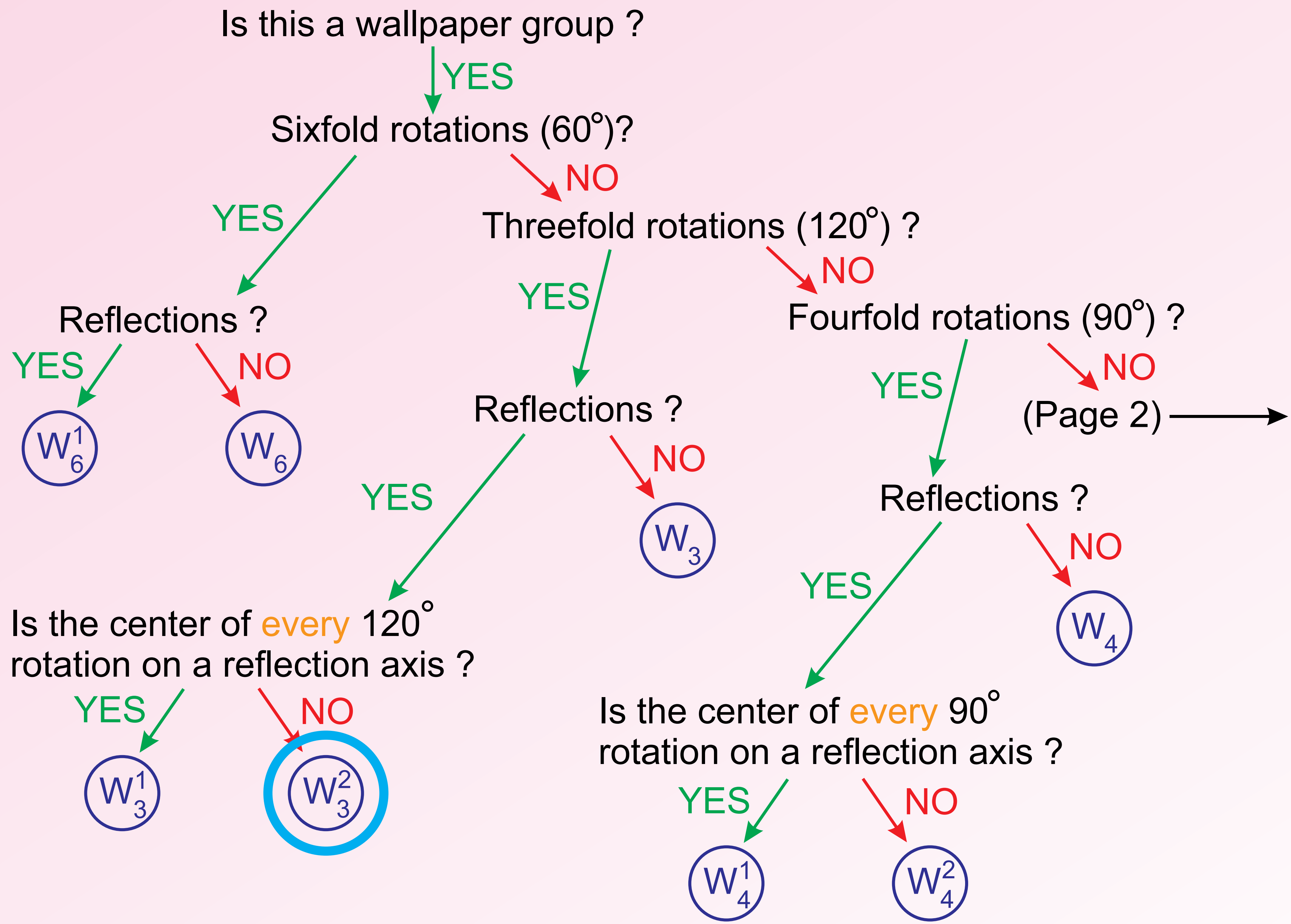
Yet another example.

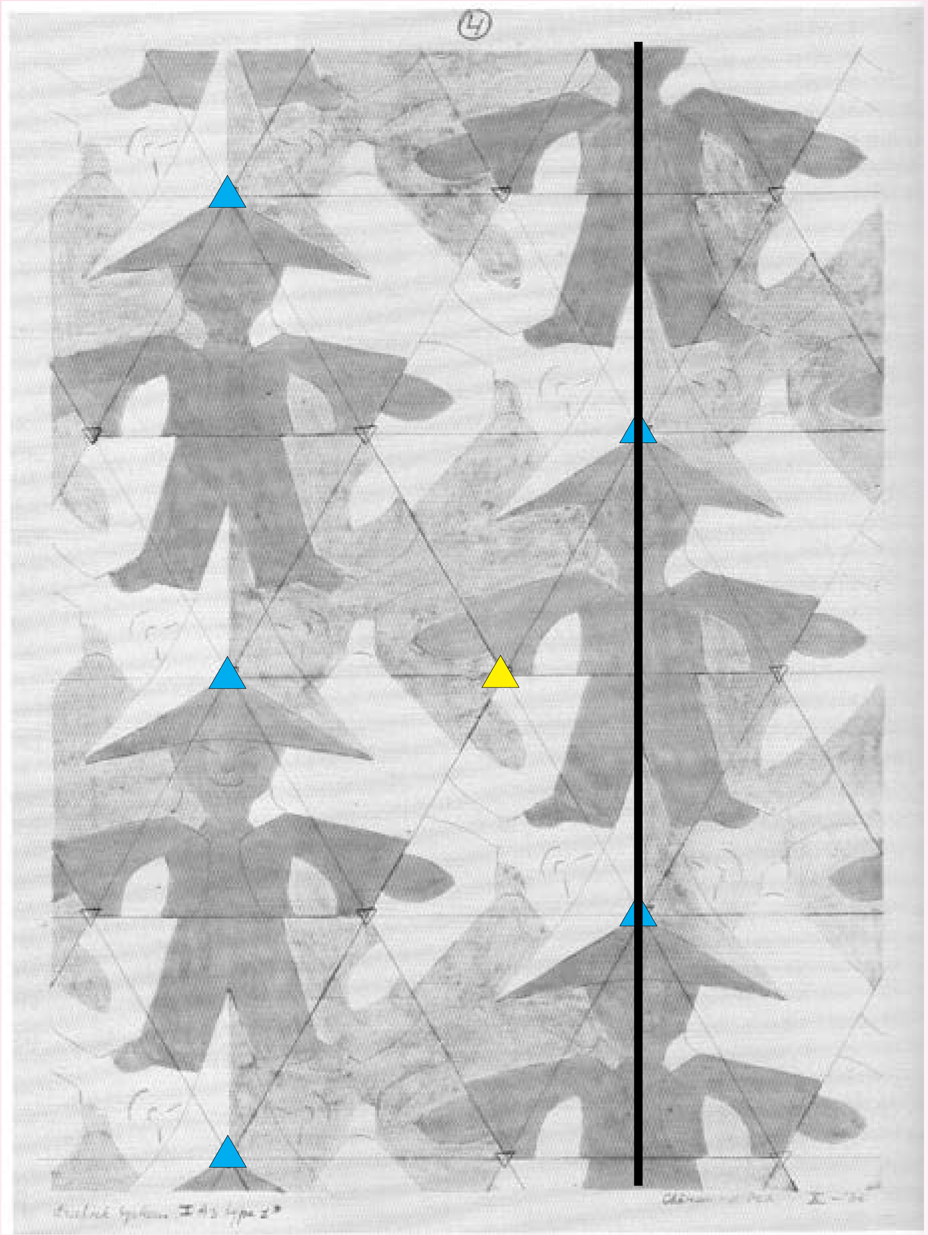


Again, the largest order of rotations is three. There is a reflection line through some rotation centers.



But there are also rotation centers that are not on a reflection line. The symmetry group is W_3^2.


Is this a wallpaper group ?
YES
Sixfold rotations (60°)?
YES
NO
Reflections ?
YES
NO
W_6^1
W_6
Threefold rotations (120°) ?
YES
NO
Reflections ?
YES
NO
W_3
Is the center of every 120°
rotation on a reflection axis ?
YES
NO
W_3^1
W_3^2
Fourfold rotations (90°) ?
YES
NO
(Page 2)
Reflections ?
YES
NO
W_4
Is the center of every 90°
rotation on a reflection axis ?
YES
NO
W_4^1
W_4^2



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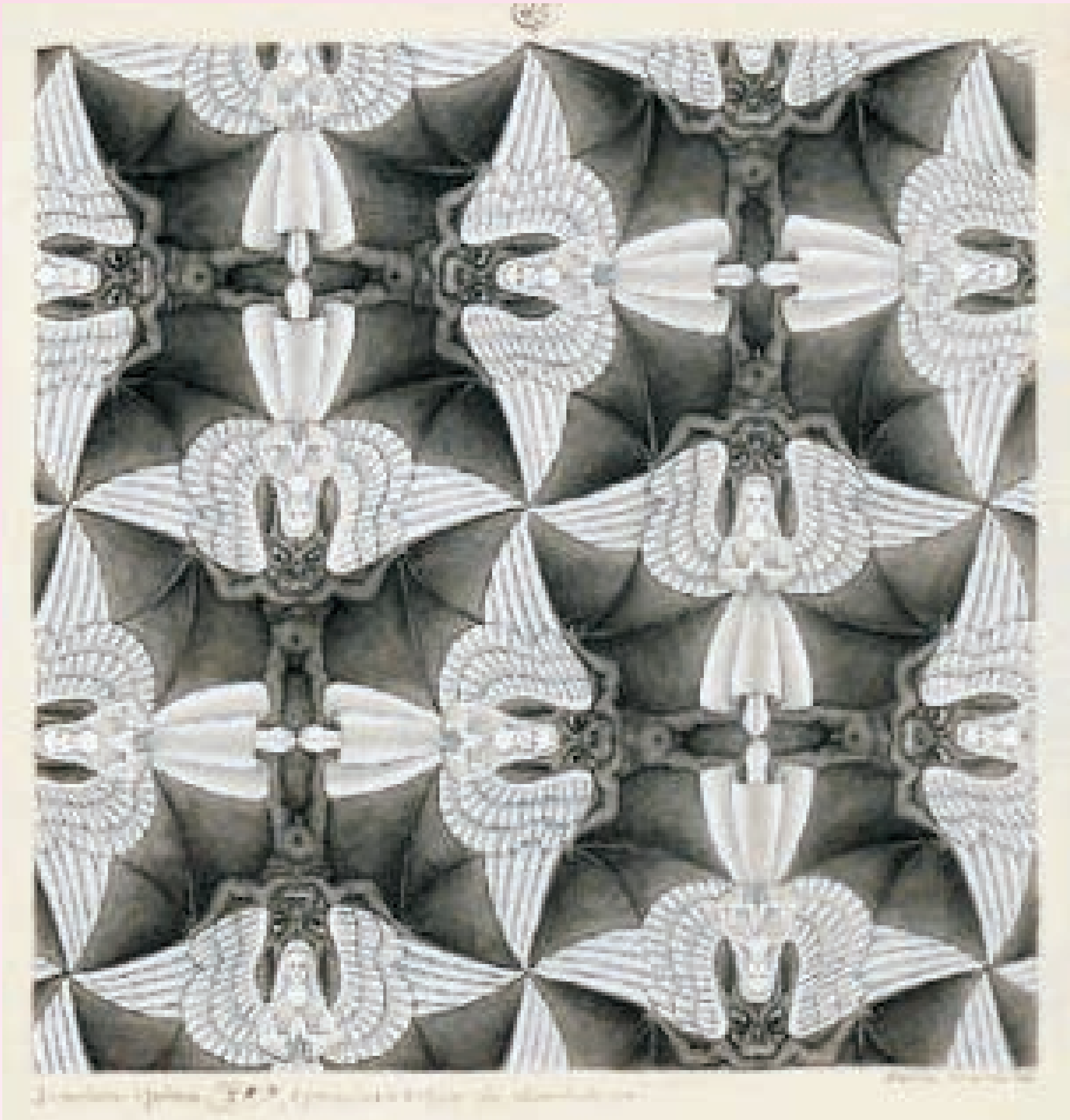
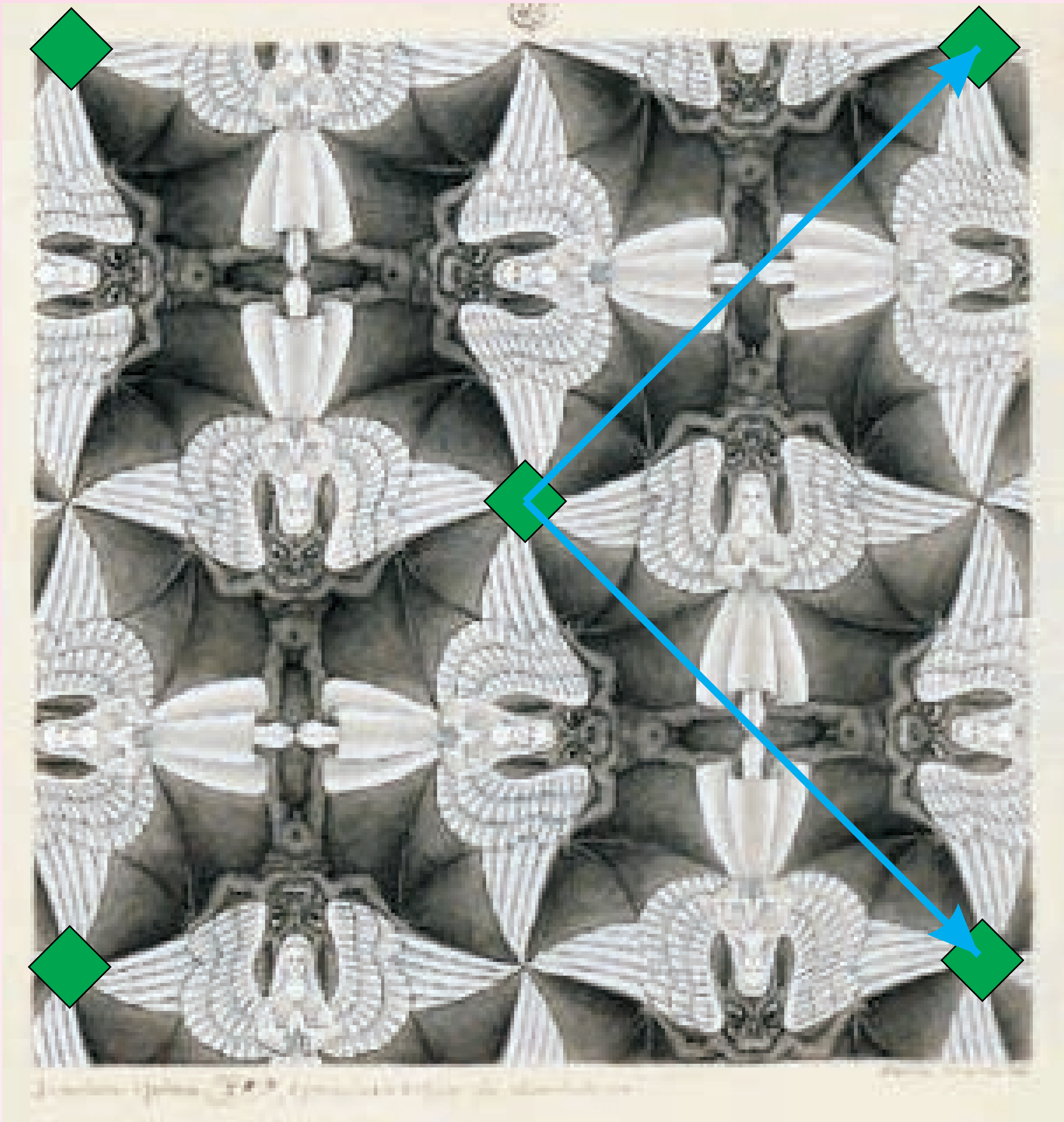
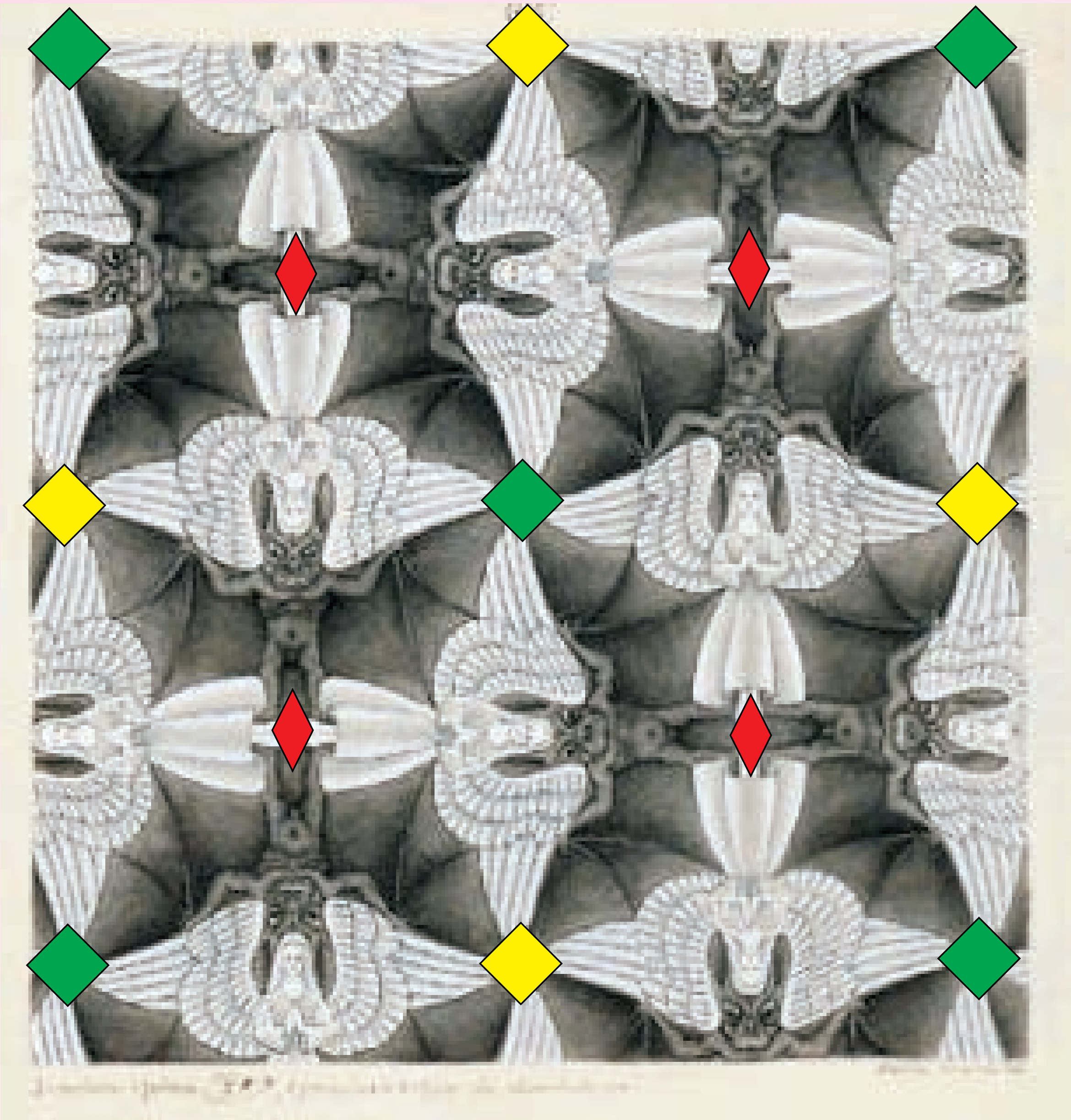
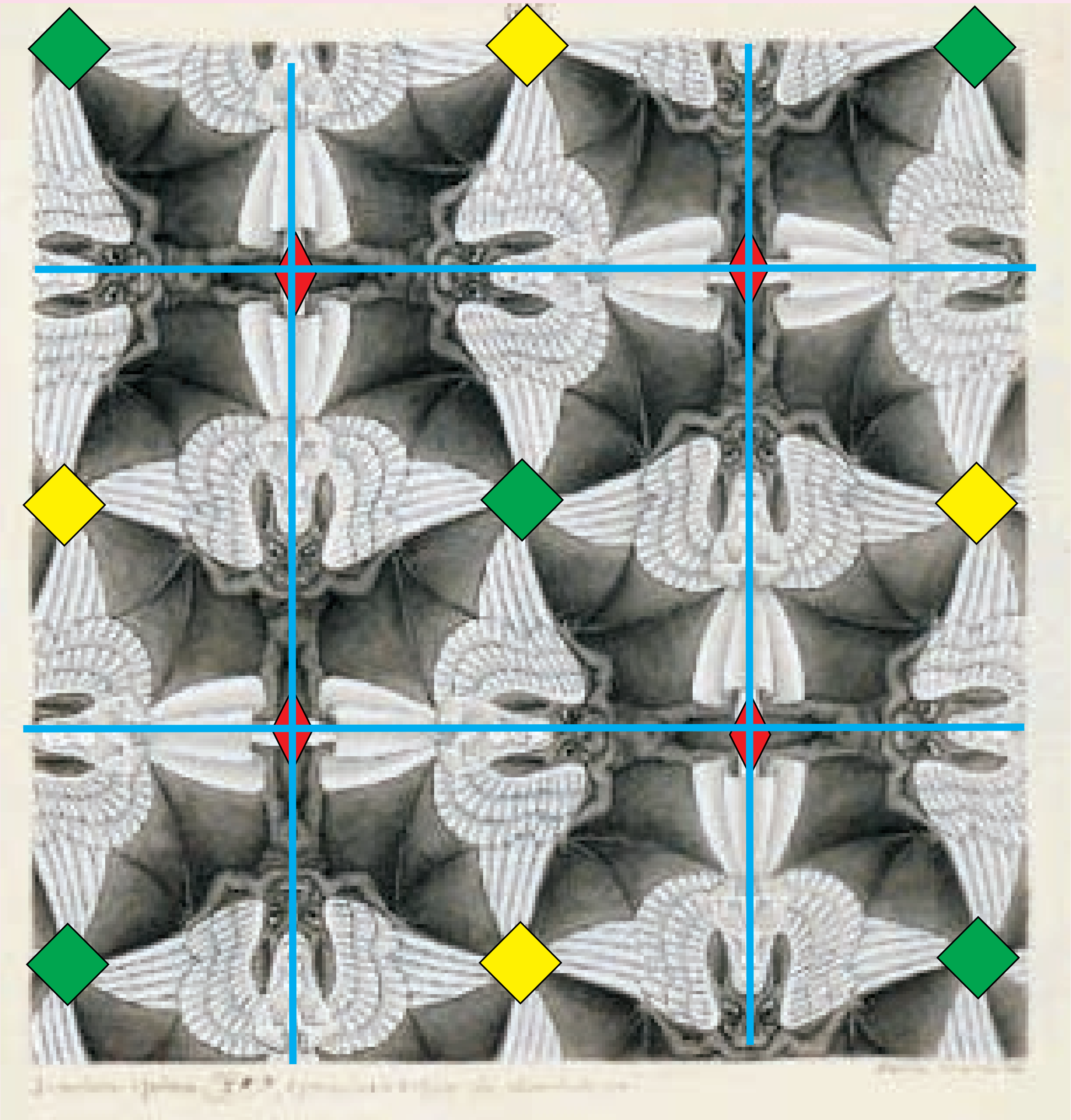




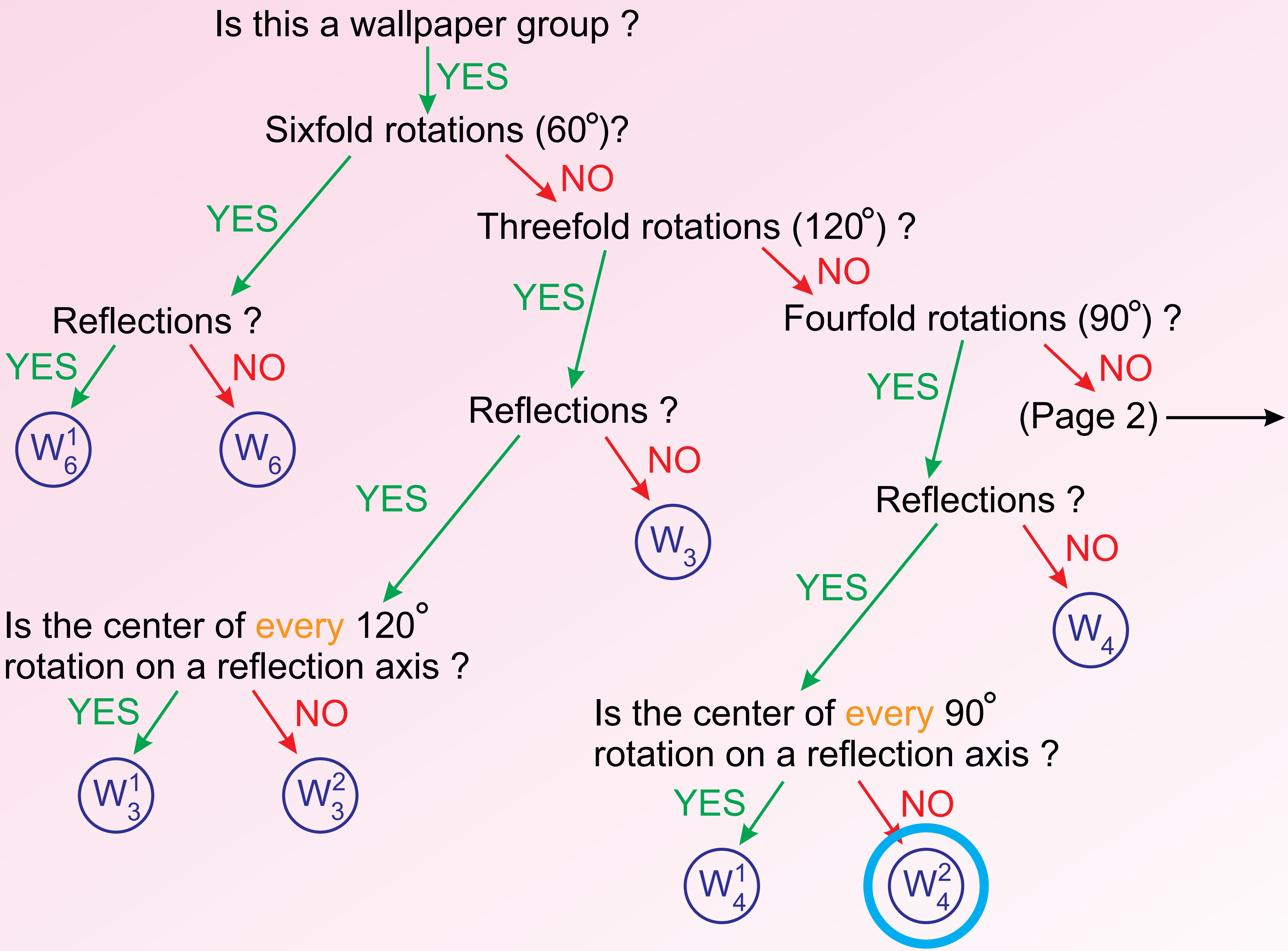
Fourfold rotations, and two generating translations.



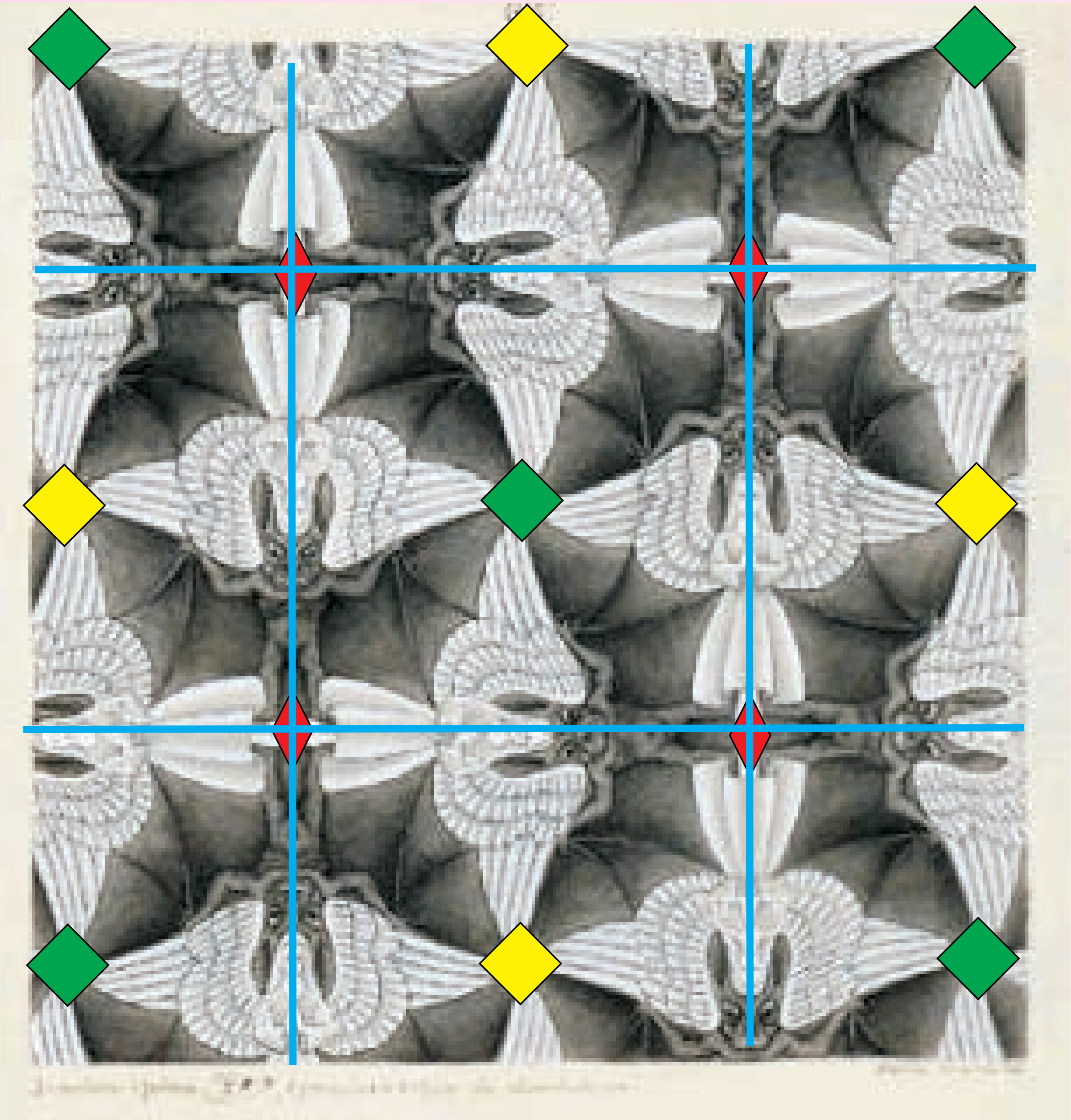
All rotation centers.



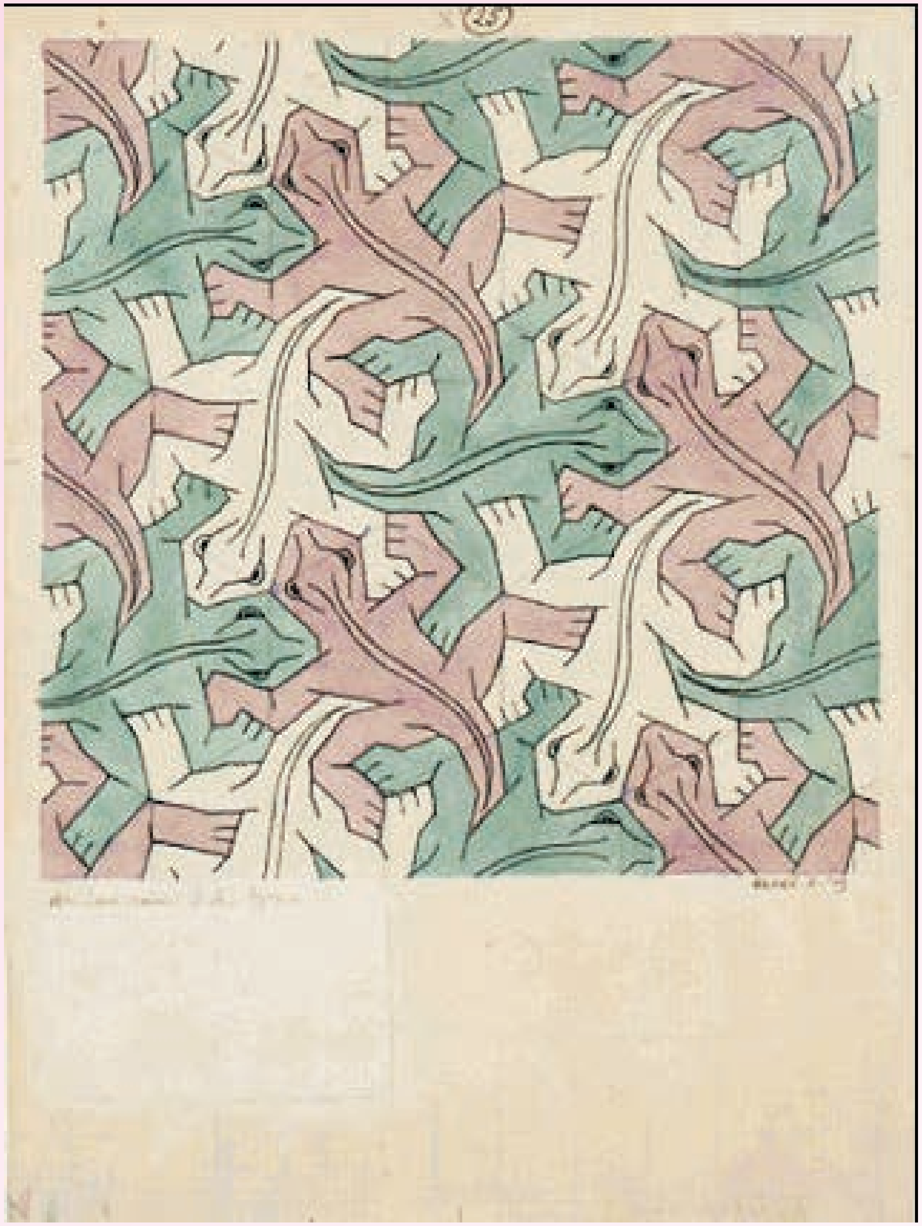
Reflection lines. There are no reflection lines through some fourfold rotation centers. The symmetry group is W_4^2.



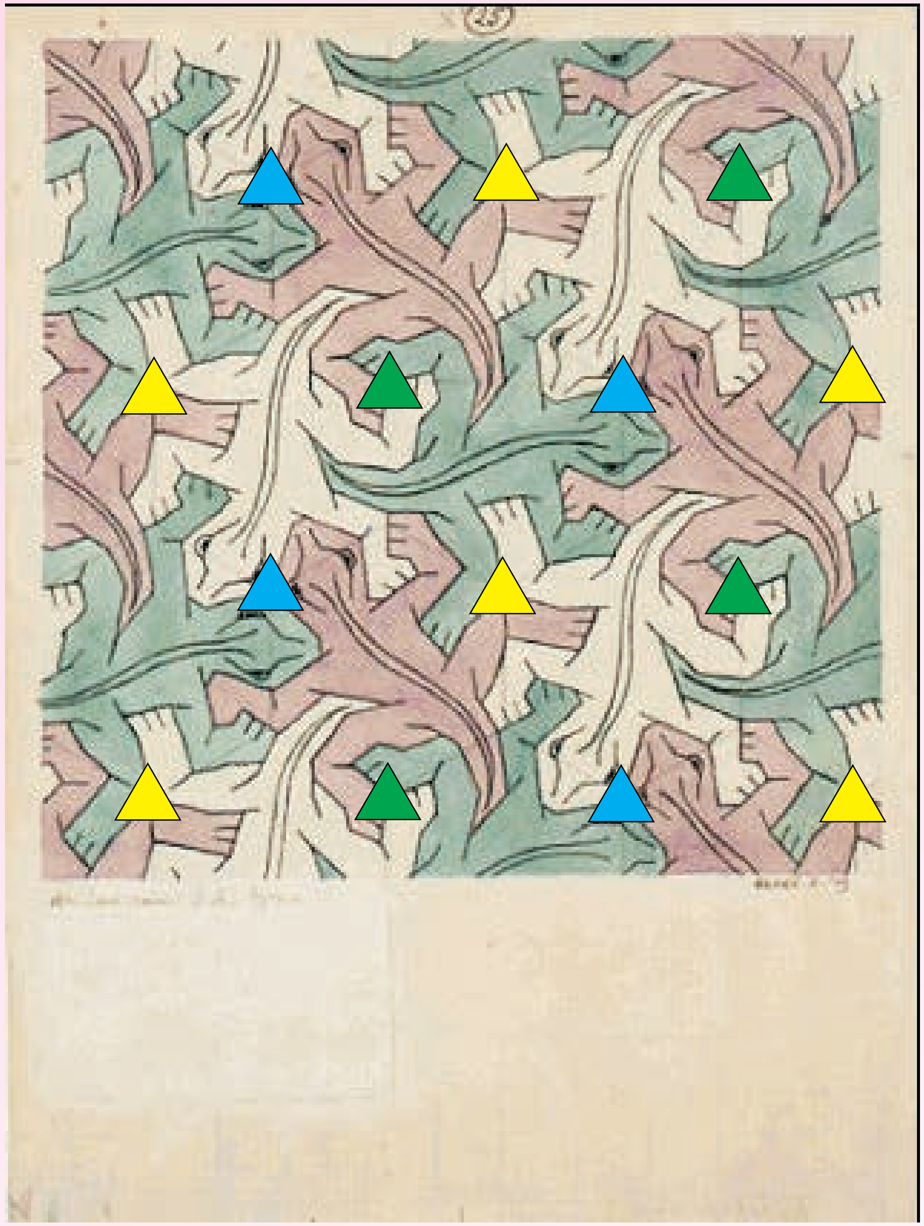
Is this a wallpaper group ?
YES
Sixfold rotations (60°)?
YES
NO
Reflections ?
YES
NO
W_6^1
W_6
Threefold rotations (120°) ?
YES
NO
Reflections ?
YES
NO
W_3
Is the center of every 120° rotation on a reflection axis ?
YES
NO
W_3^1
W_3^2
Fourfold rotations (90°) ?
YES
NO
(Page 2)
Reflections ?
YES
NO
W_4
Is the center of every 90° rotation on a reflection axis ?
YES
NO
W_4^1
W_4^2



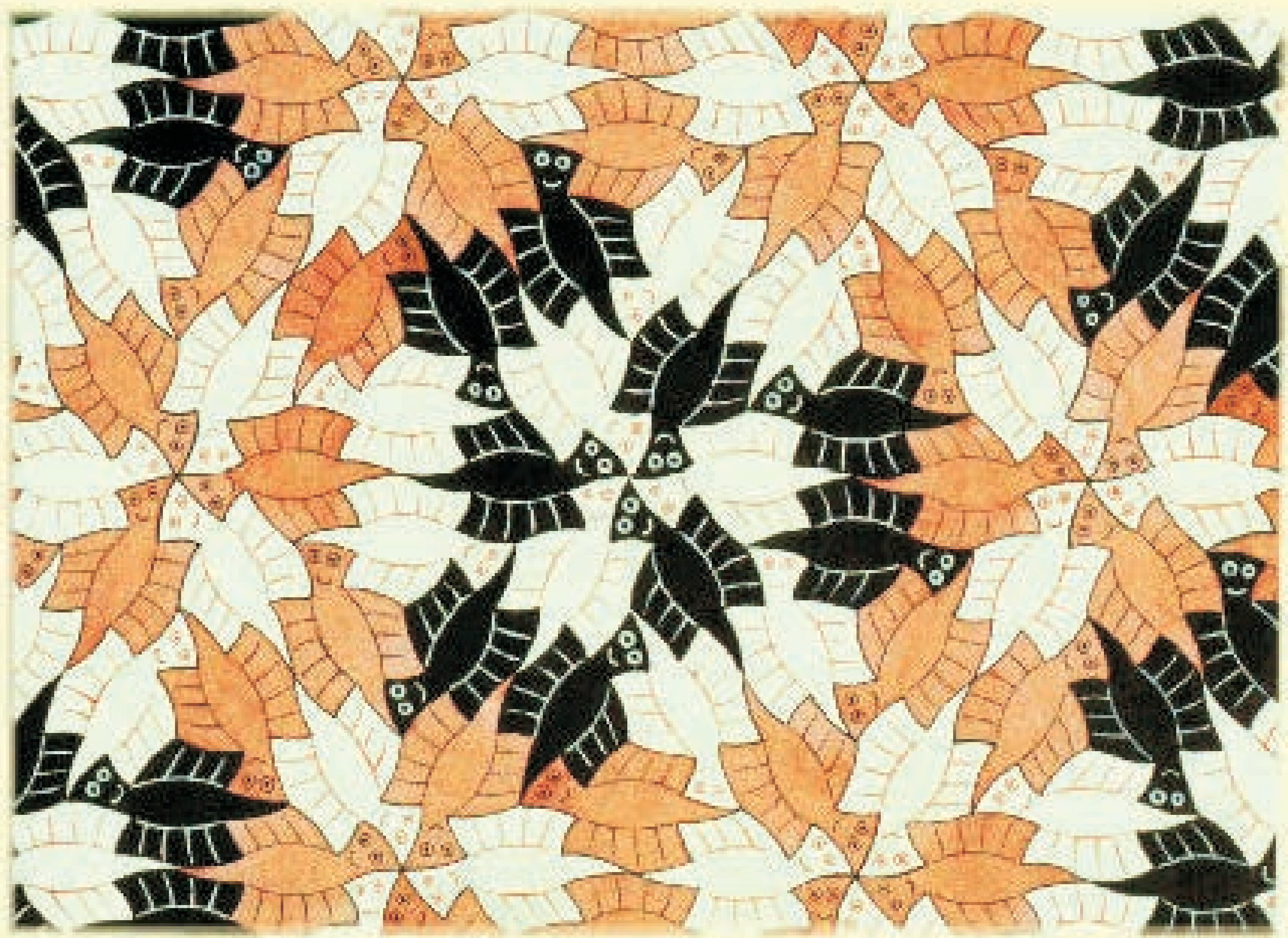
Reflection lines. There are no reflection lines through some fourfold rotation centers. The symmetry group is W_4^2.



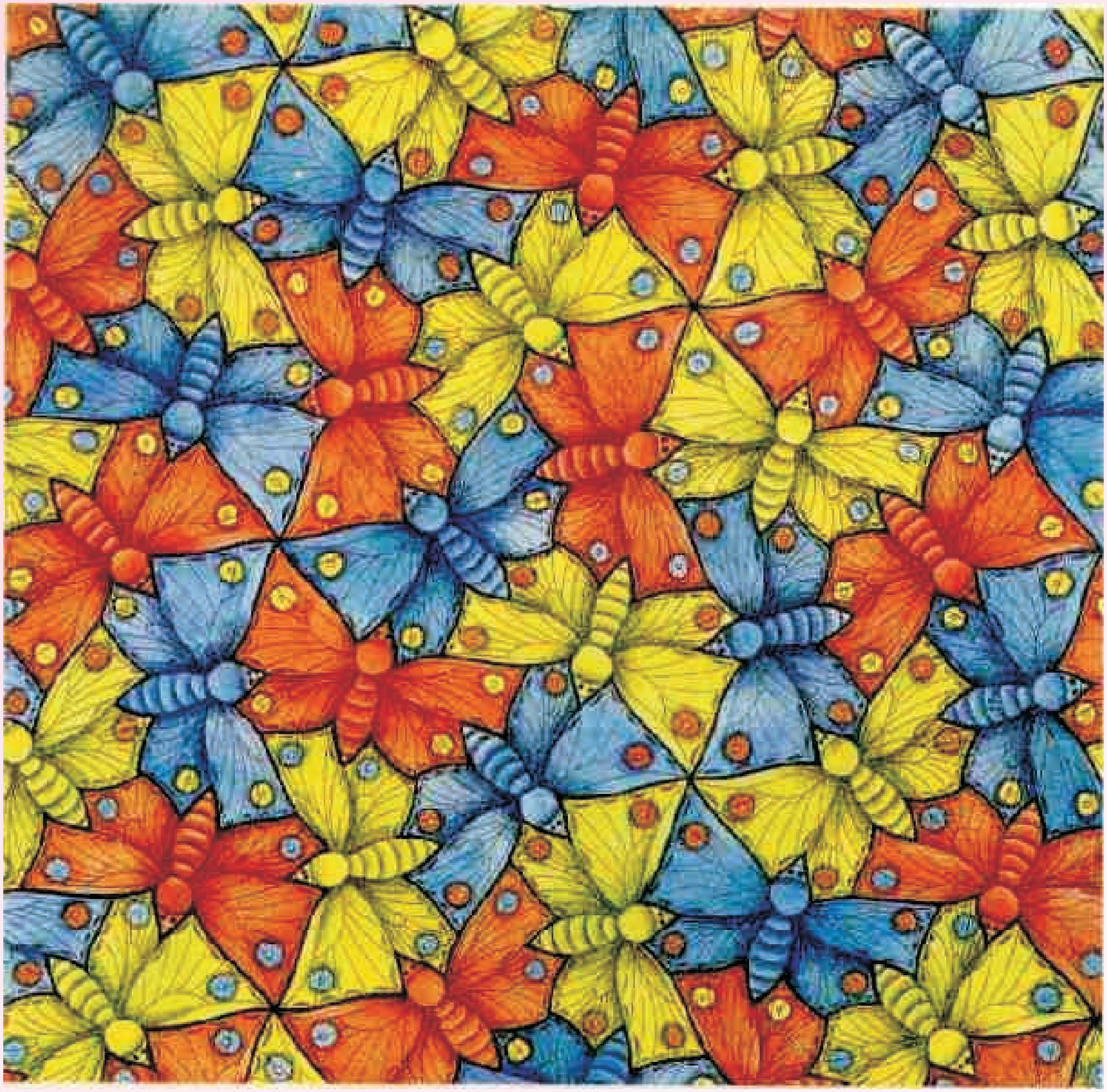
Ignoring colors. . .

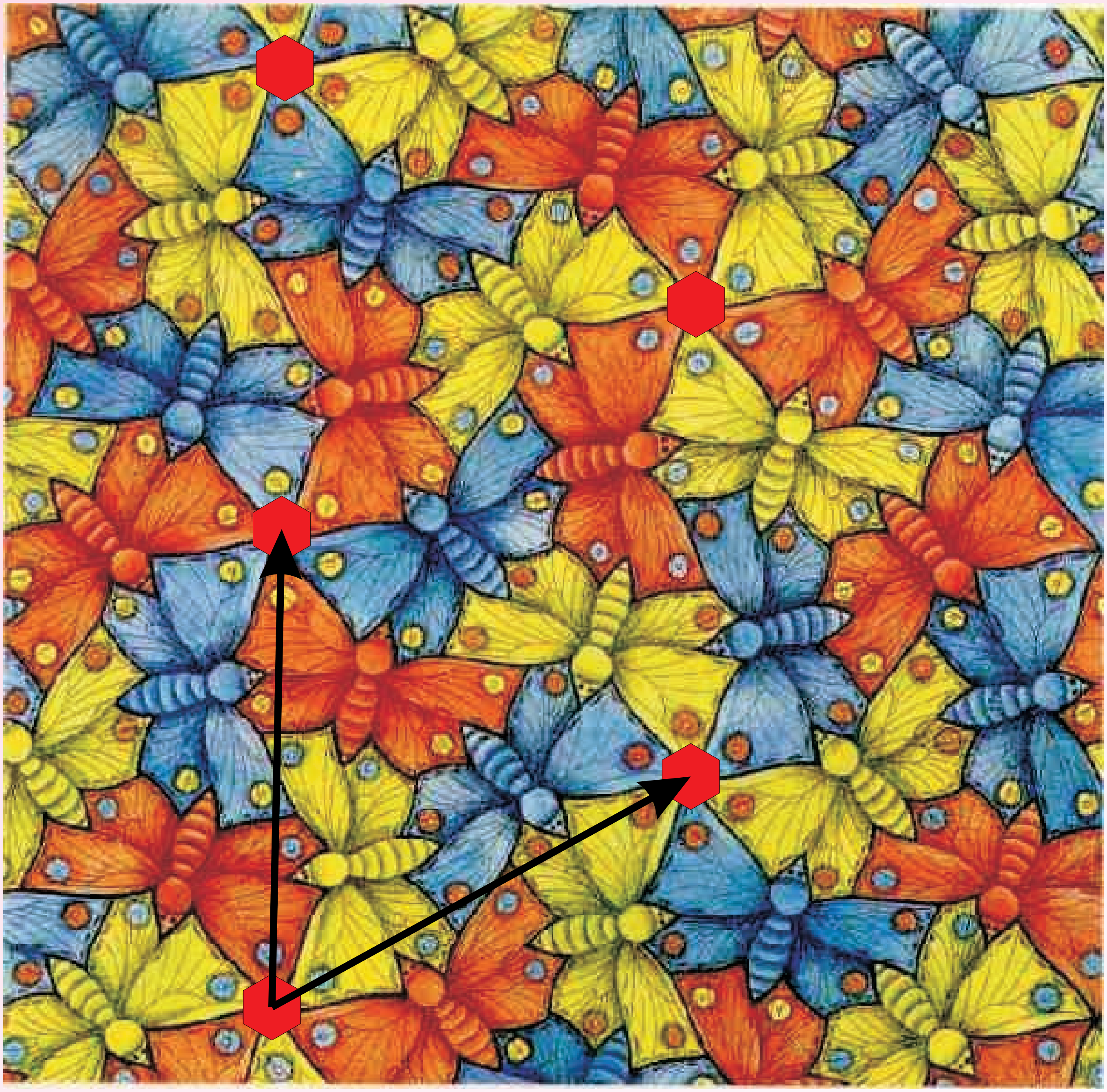


The painting only has threefold rotations. There are no reflections so the symmetry group is W_3.

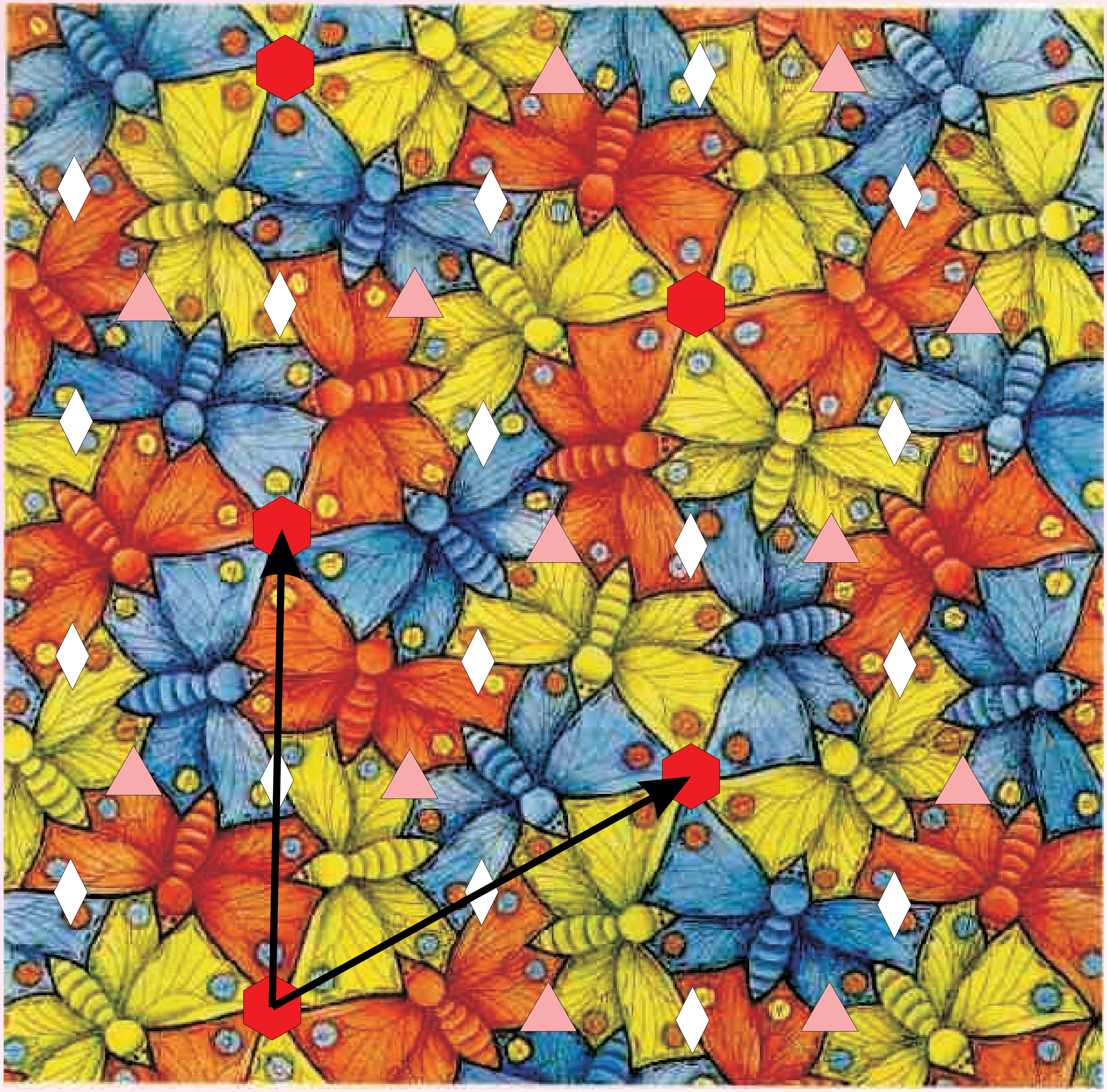


W_6 (ignoring colors) or W_3 (considering colors).

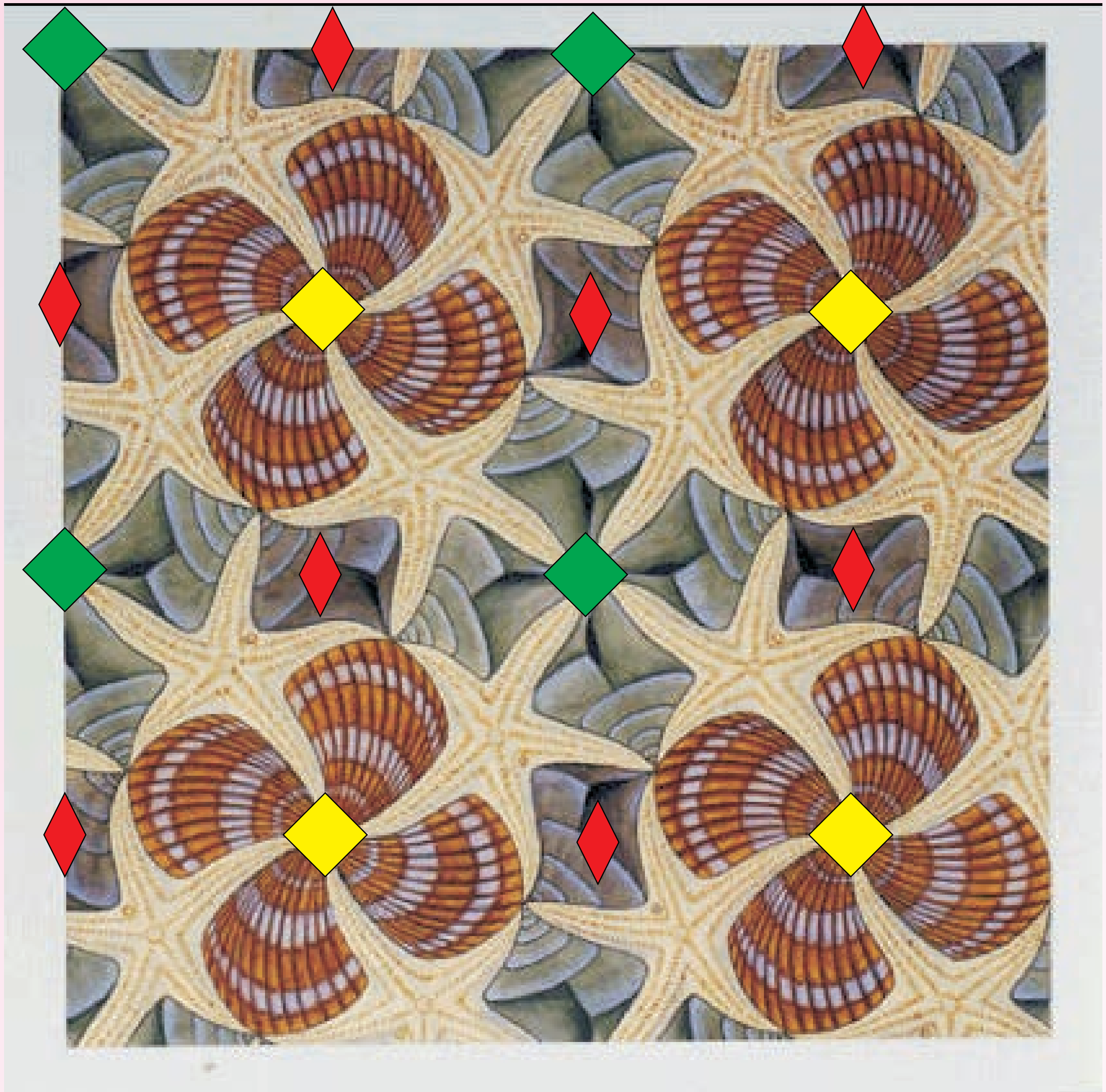




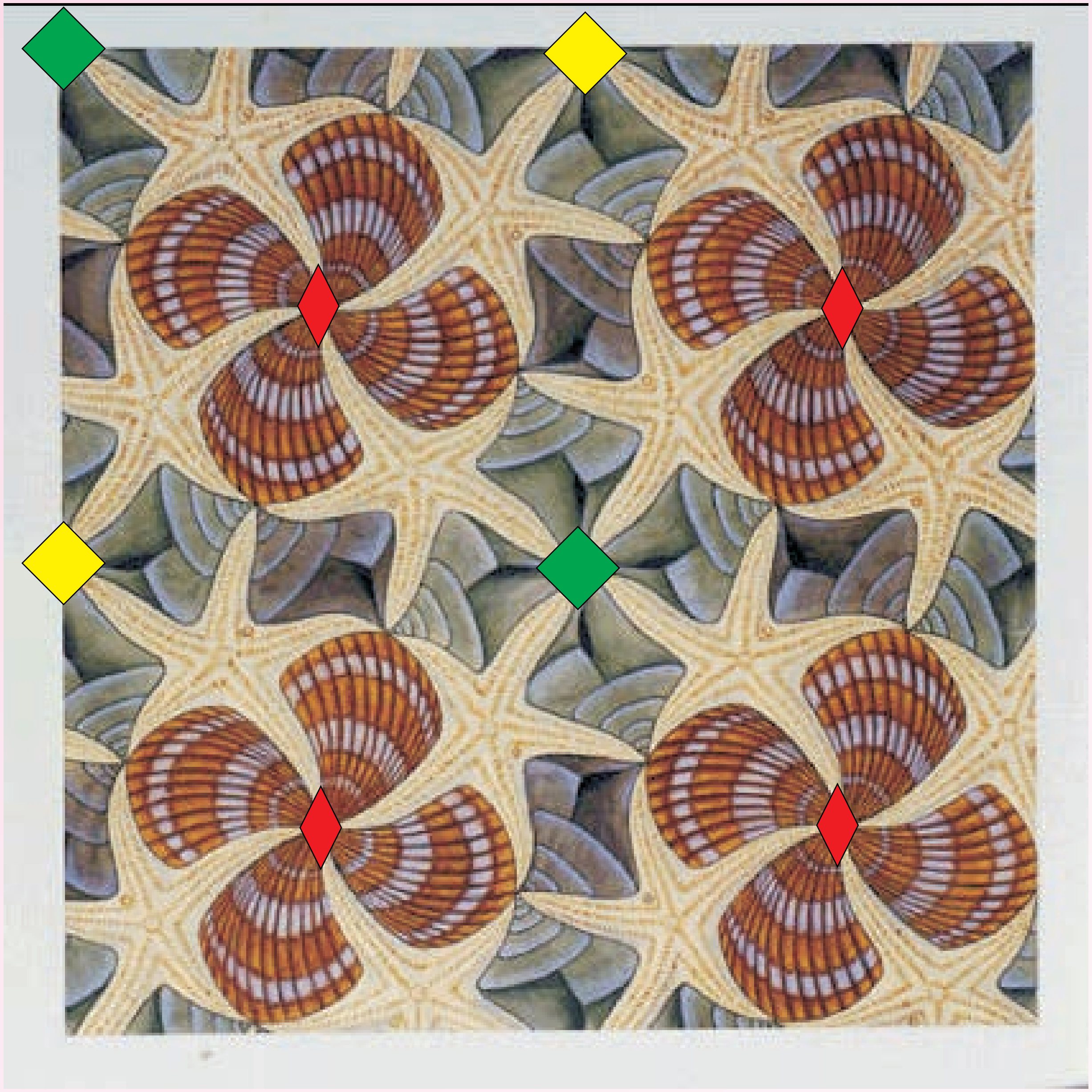
W_6



W_6



Considering the shells and the starfish the symmetry group is W_4, and here are the rotation centers.

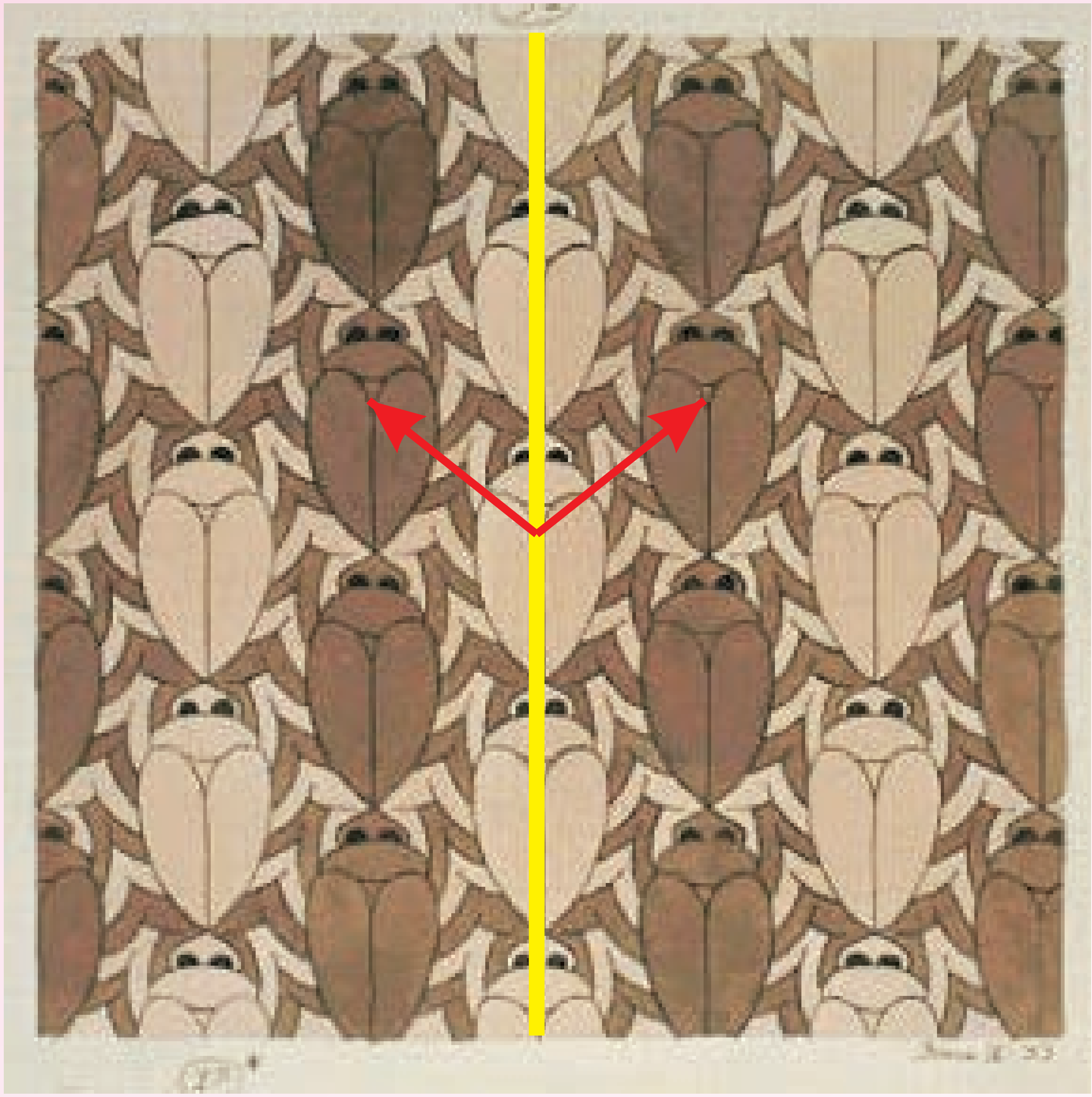


There is less rotational symmetry if also the rock underneath are considered, but the group is still W_4.

Half turns and the lattice of translations. The painting has no odd symmetries, so the group is W_2.

There are other centers of half rotation at the midpoints between the first ones.

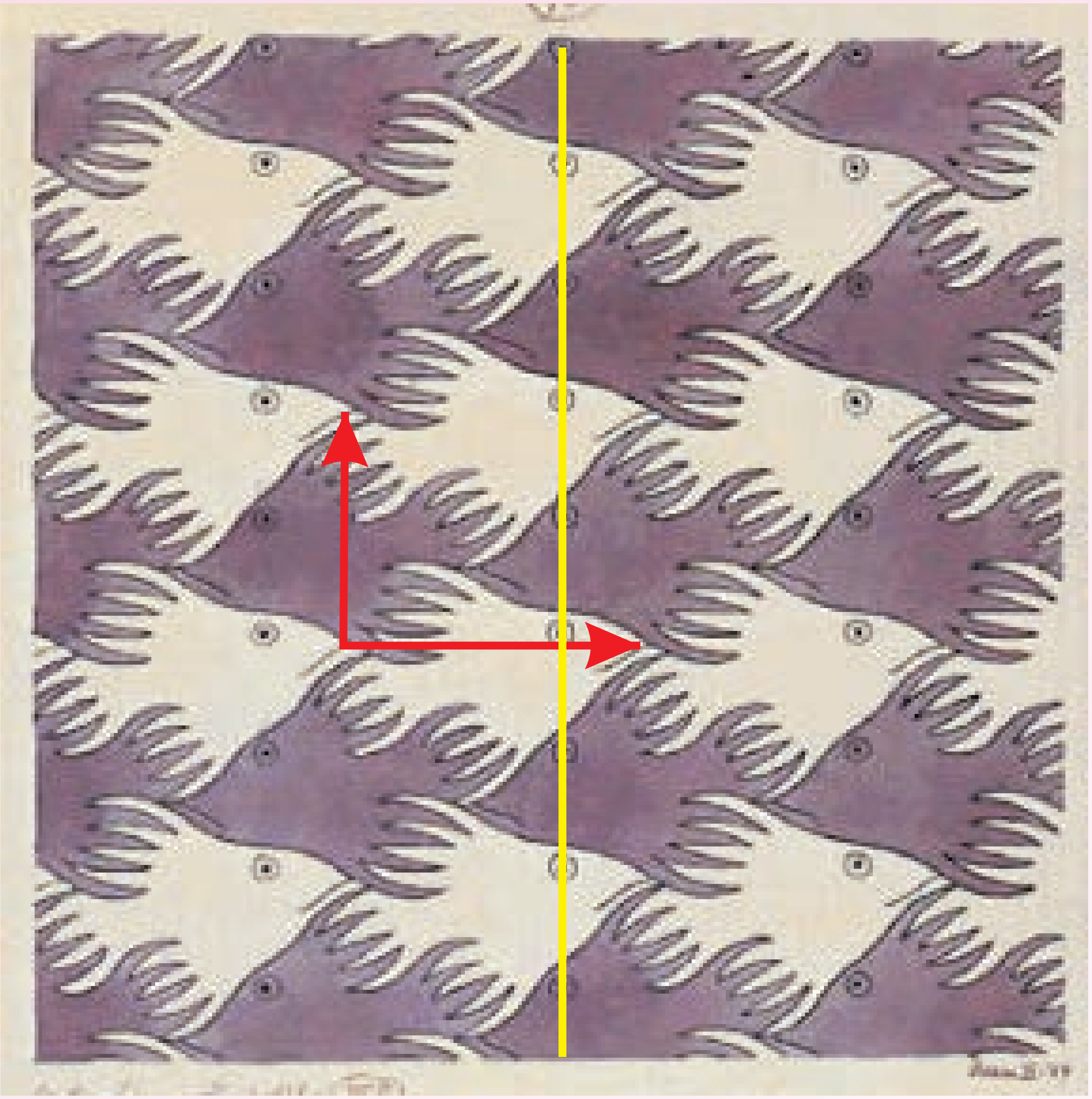
No rotational symmetries.



A symmetry axis and generating translations (ignoring colors).

There is a glide reflection whose axis not an axis of reflection.
So the symmetry group is W_1^1.

No rotational symmetries, no reflections.



However, there are glide reflections so the group is W_1^3.

M.C.Escher (1898 — 1972) was a master in creating plane tilings with various wallpaper symmetries. Skillfully he managed to use tiles of shapes from real world objects, often various animals.

It is a hard task: symmetries force in the objects identically shaped edges.

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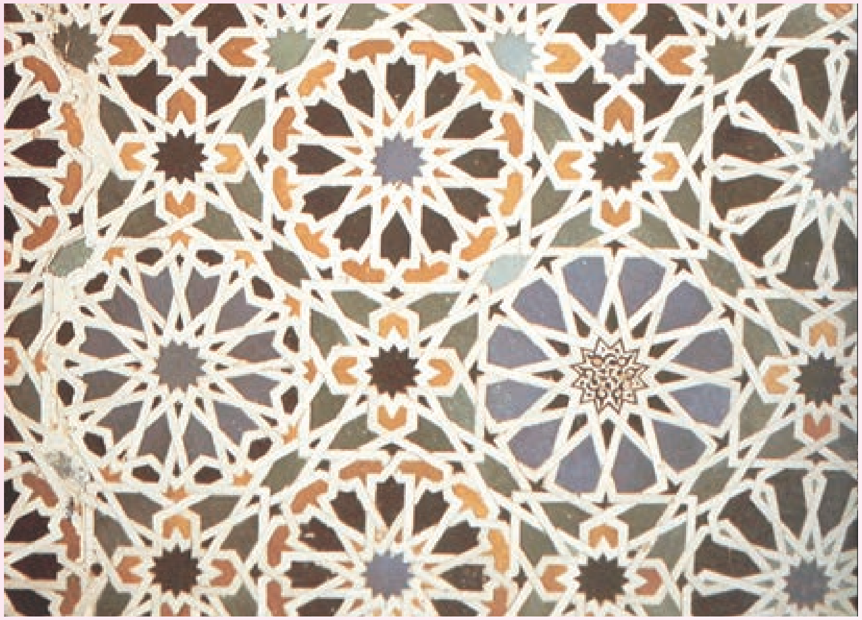
It is a hard task: symmetries force in the objects identically shaped edges.

Escherization can be tried with various “kaleidoscopic” computer software. The user first chooses a symmetry group and starts drawing. The program replicates each drawn line using all isometries in the chosen group.

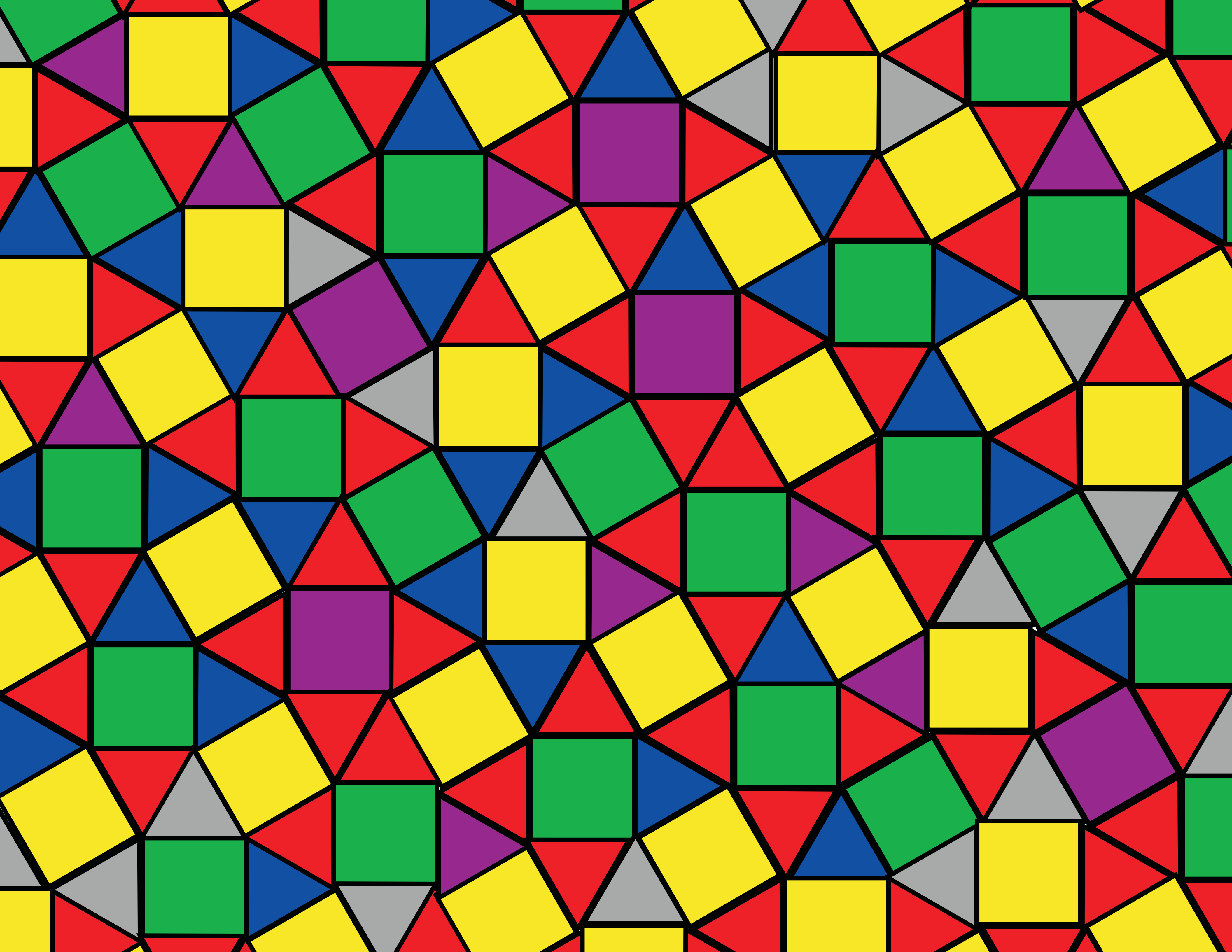
Tilings of the plane

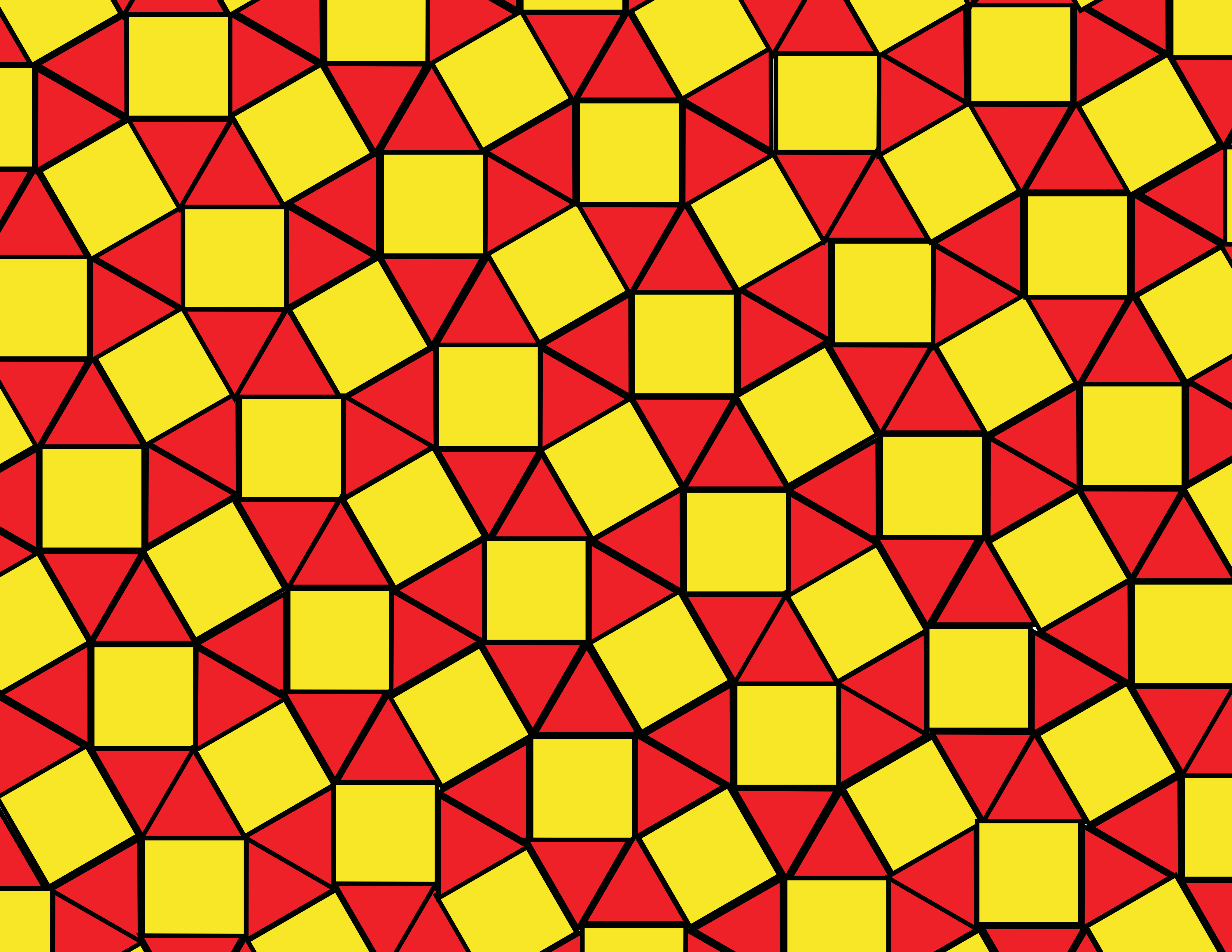
A tiling (or a tessellation) covers the infinite plane without gaps or overlaps.



Single tiles have simple shapes – typically polygons – and one only uses a small, finite number of differently shaped tiles. There are infinitely many copies of each tile available.

Colors are used to emphasize various aspects of the tiling.



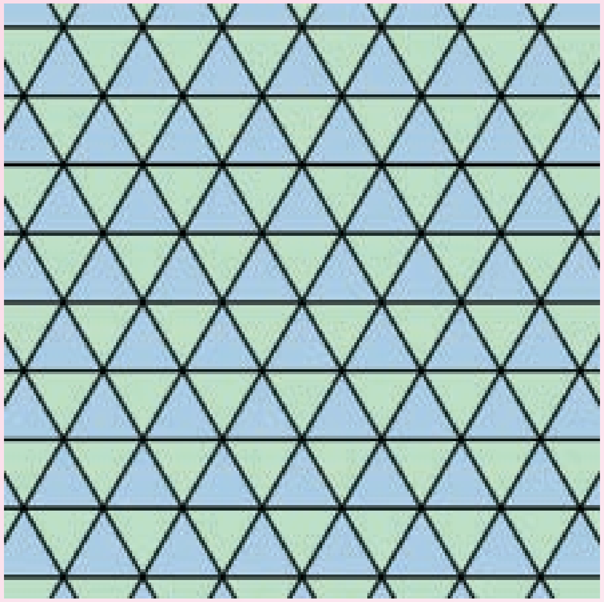


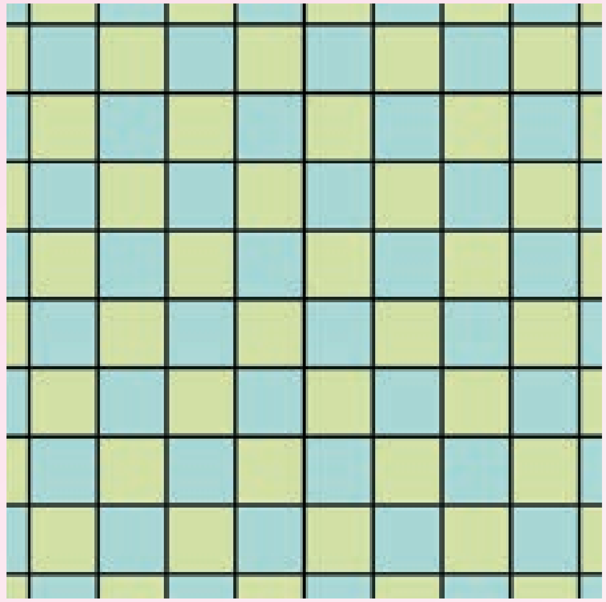
The symmetry group of a tiling is **discrete**: The only possibilities are

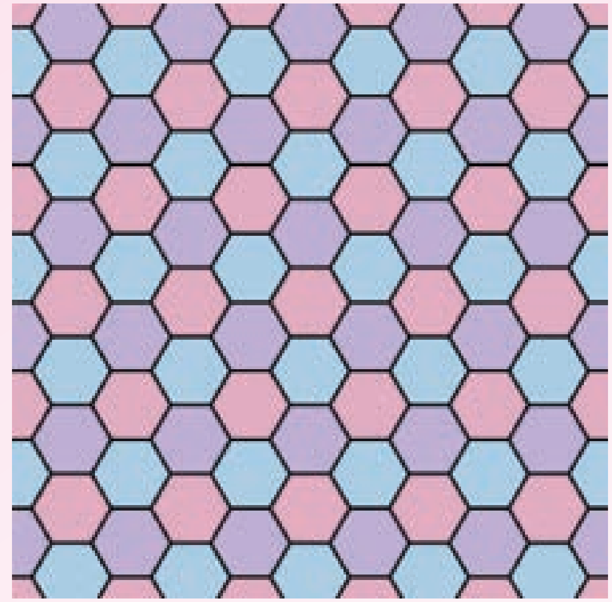
- rosette groups (i.e., dihedral and cyclic groups),
- frieze groups, and
- wallpaper groups.

A tiling is called **periodic**, if its symmetry group is a wallpaper group. In this case the tiling "repeats the same" throughout the infinite plane.

Everyone knows the three **regular** tilings:

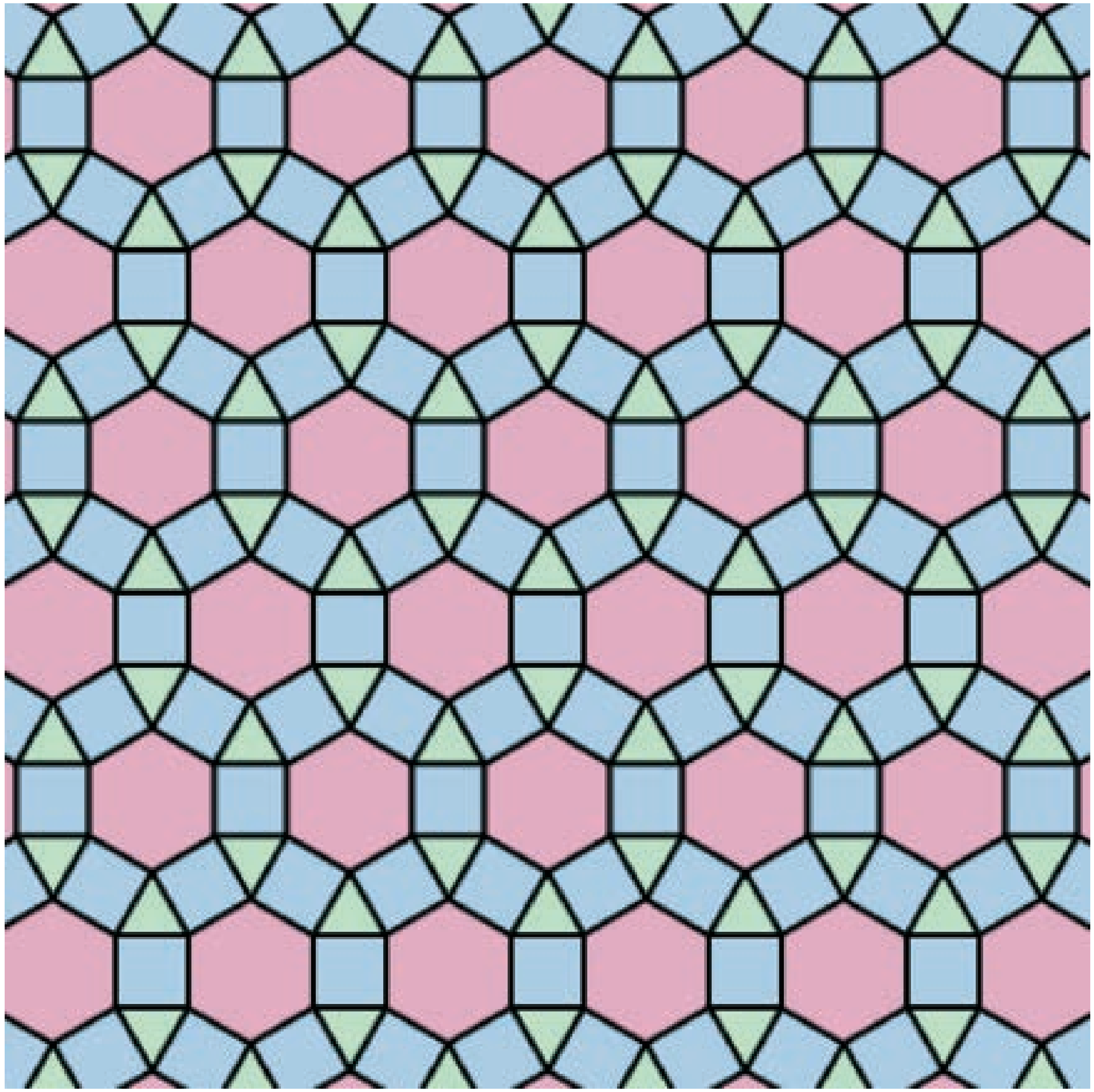




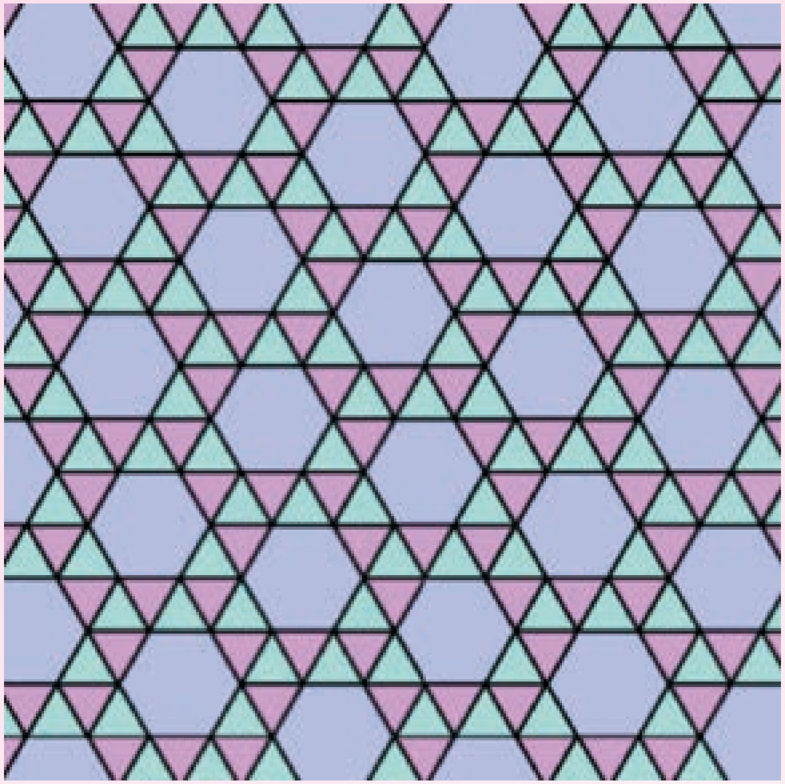


These are all periodic.

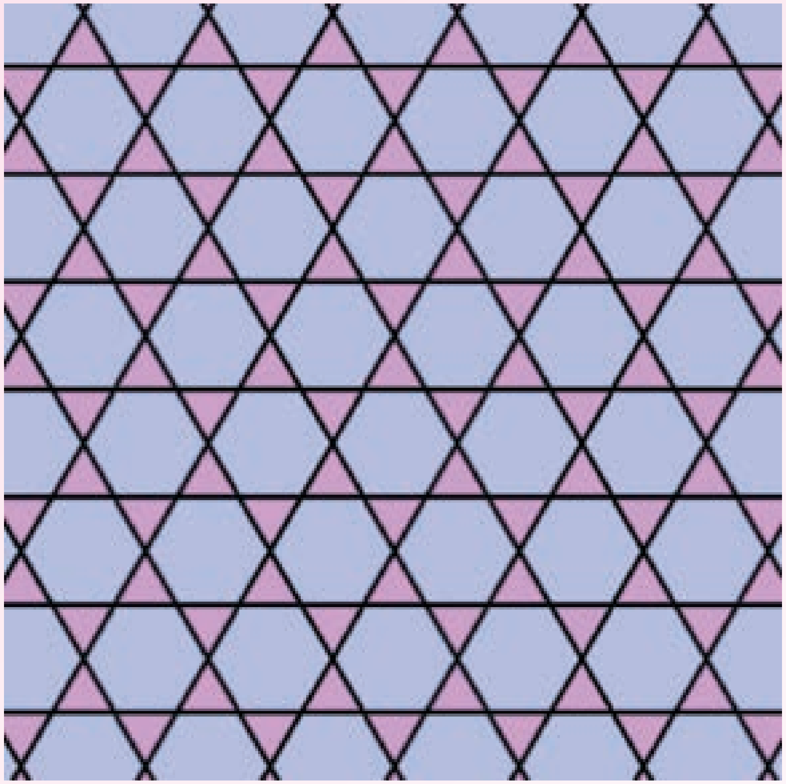
An archimedean tiling uses regular polygons with unit sides. The additional constraint is that all vertices are identical: in each vertex the same polygons meet in the same order.



Kepler 1619: There are 11 archimedean tilings (incl. 3 regular tilings)

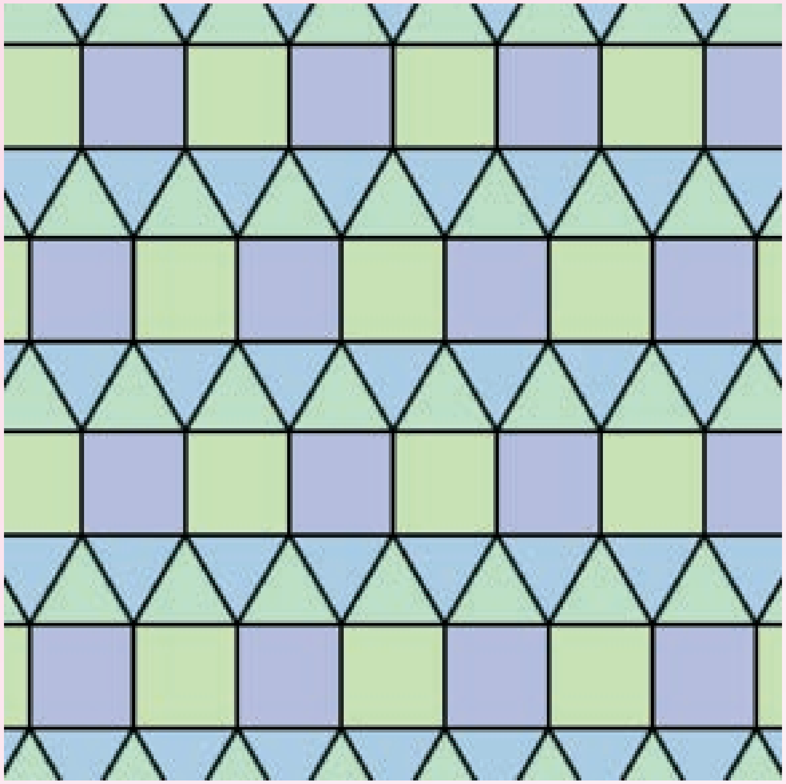


3.3.3.3.6

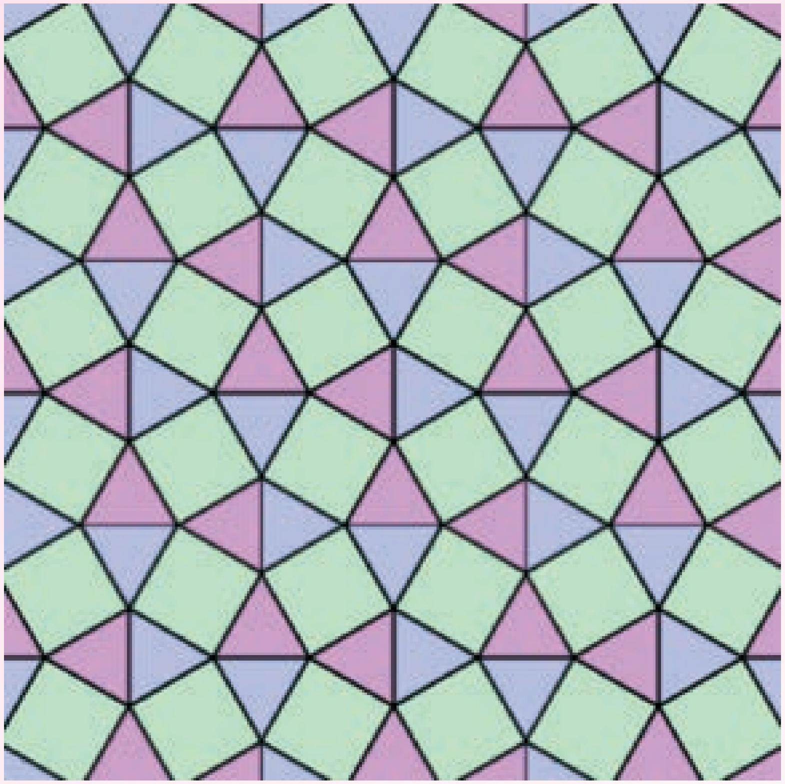


3.6.3.6

Kepler 1619: There are 11 archimedean tilings (incl. 3 regular tilings)

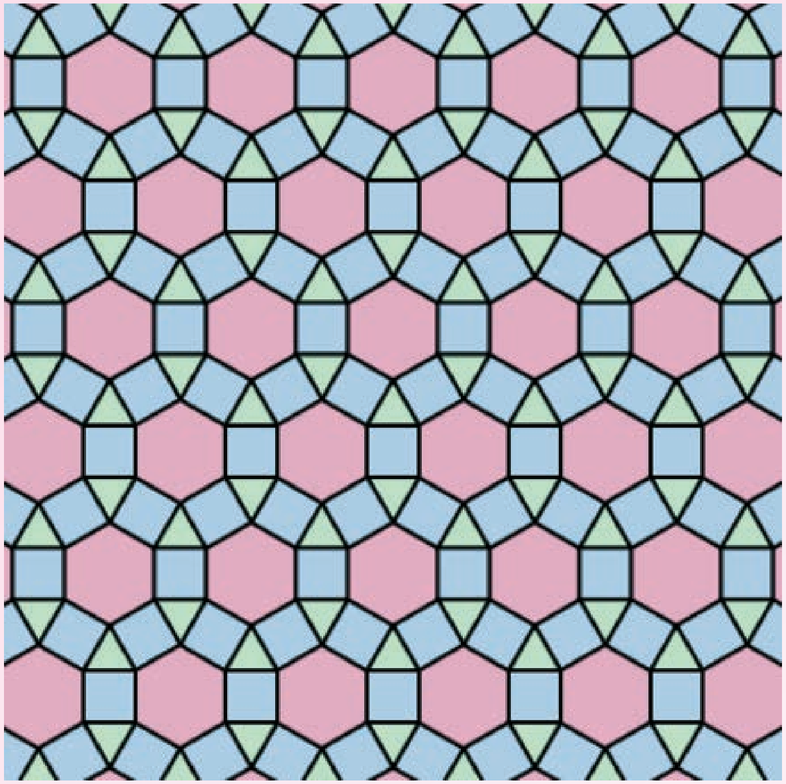


3.3.3.4.4

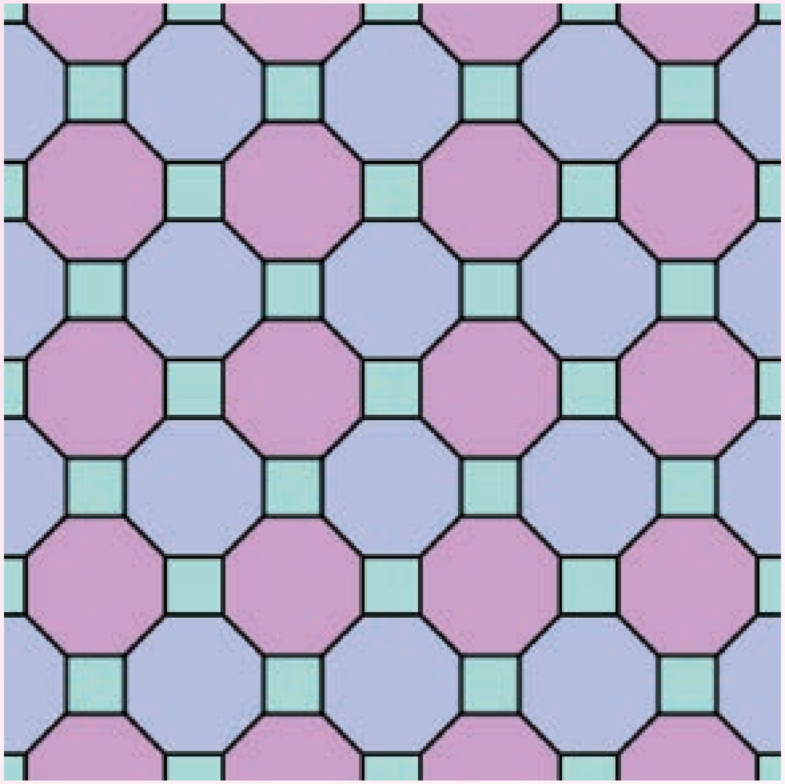


3.3.4.3.4

Kepler 1619: There are 11 archimedean tilings (incl. 3 regular tilings)

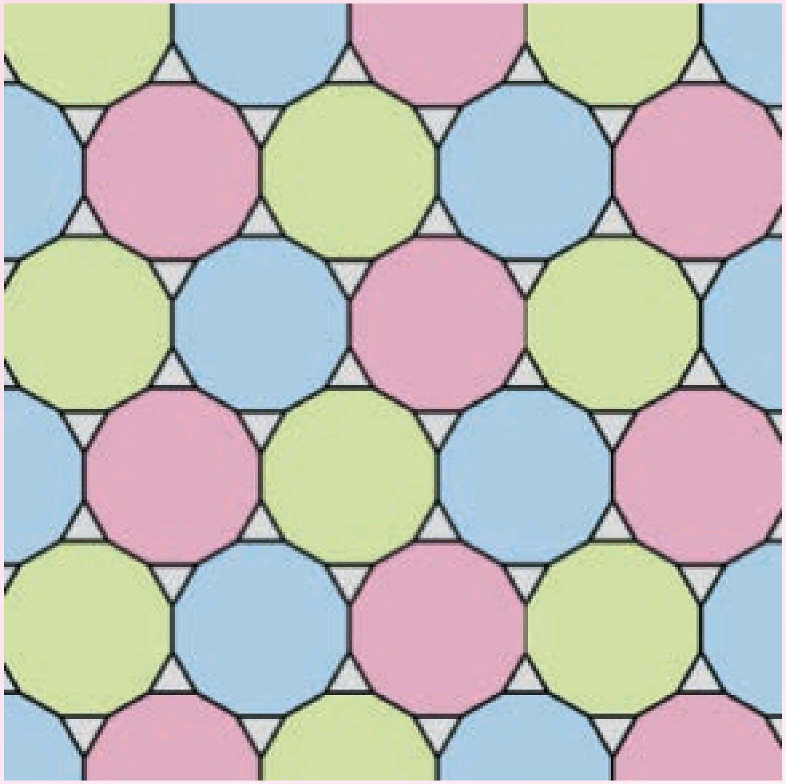


3.4.6.4

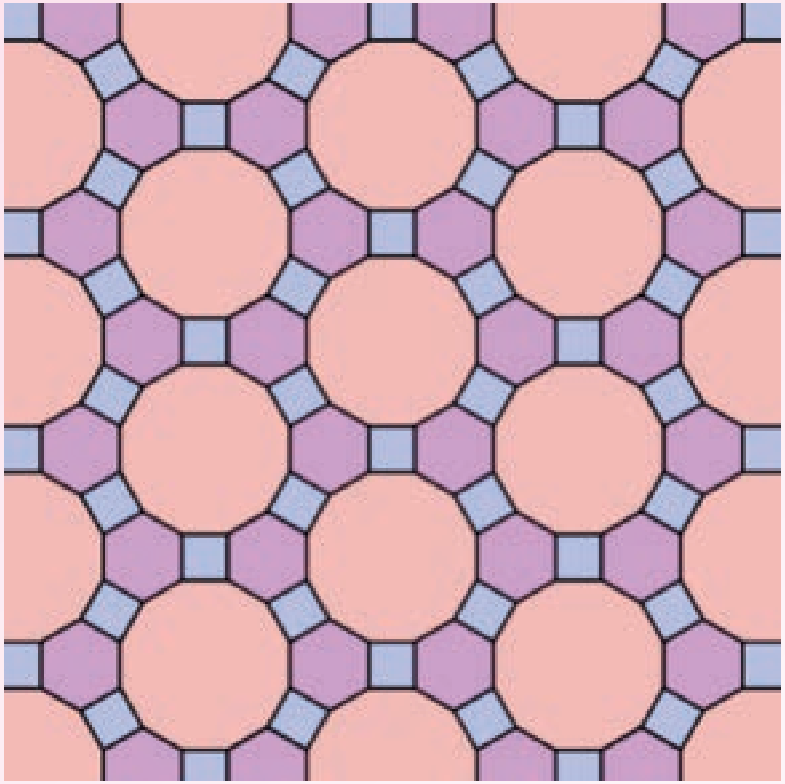


4.8.8

Kepler 1619: There are 11 archimedean tilings (incl. 3 regular tilings)



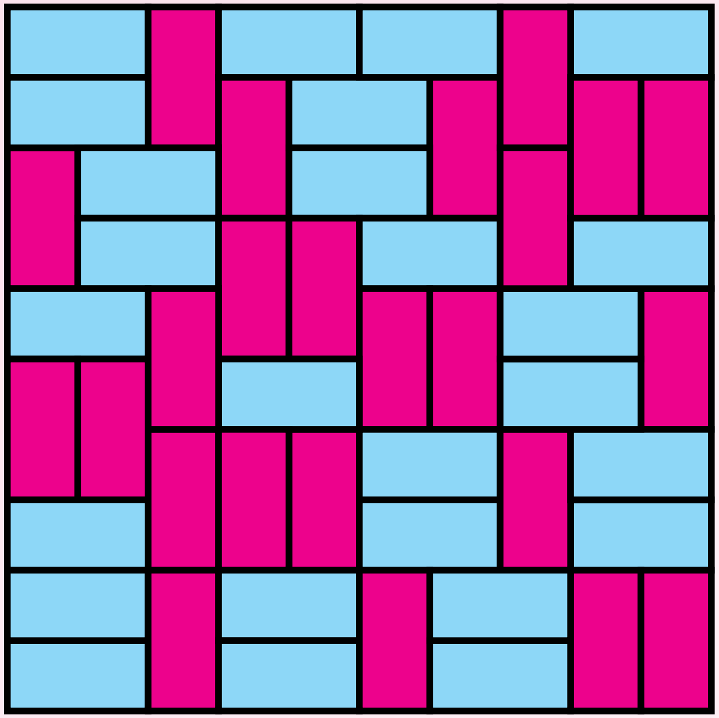
3.12.12



4.6.12

All archimedean tilings are periodic.

It is also very easy to create non-periodic tilings:



The tiles in this example (dominoes) can be rearranged so that they tile periodically.

Can one do this with every non-periodic tiling ?

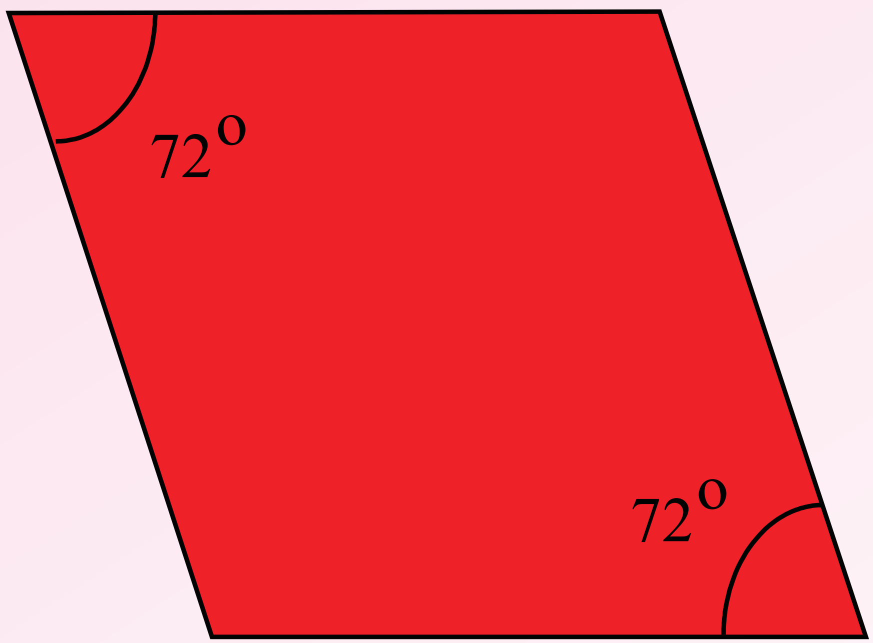
It was conjectured for a long time that any finite tile set that admits a tiling of the plane also admit a periodic tiling of the plane. This conjecture turned out to be false:

Berger 1966: There are finite tile sets that can tile the plane, but only non-periodically.

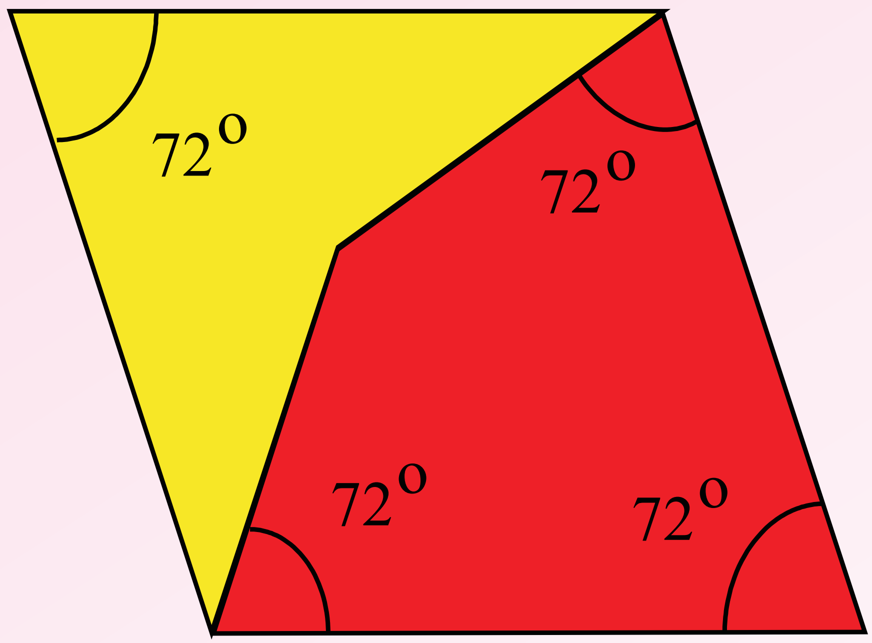
Such tile sets are called **aperiodic**.

Note that aperiodicity is a property of the tiles, not the tiling: one cannot make periodic tiling with these tiles – they force non-periodicity.

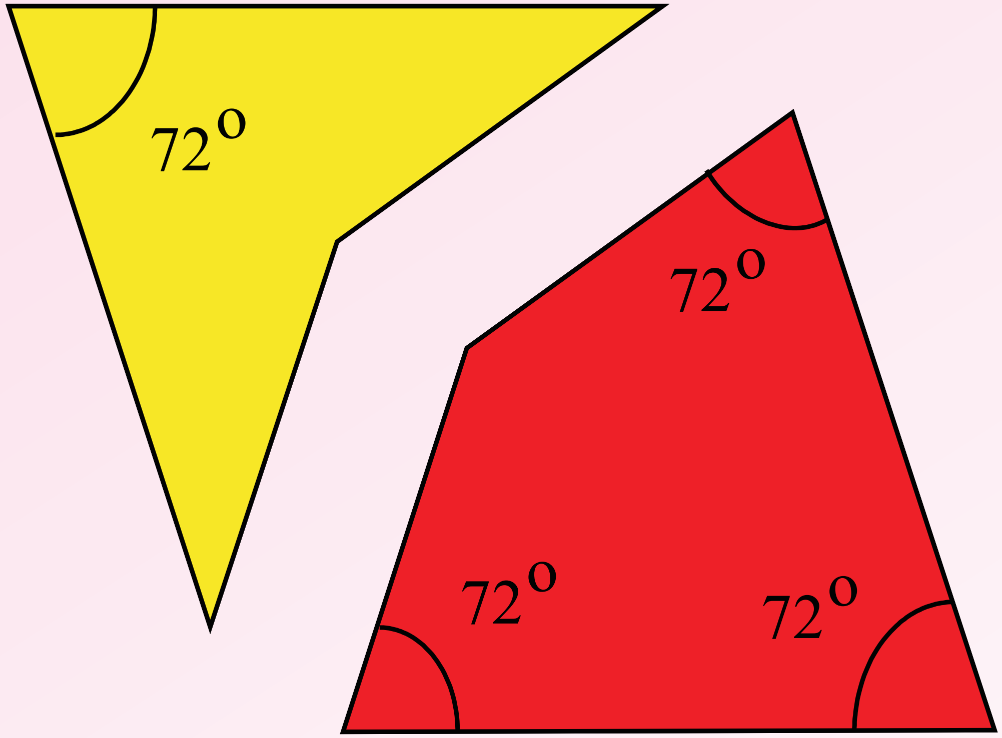
Kite and dart tiles (R.Penrose)



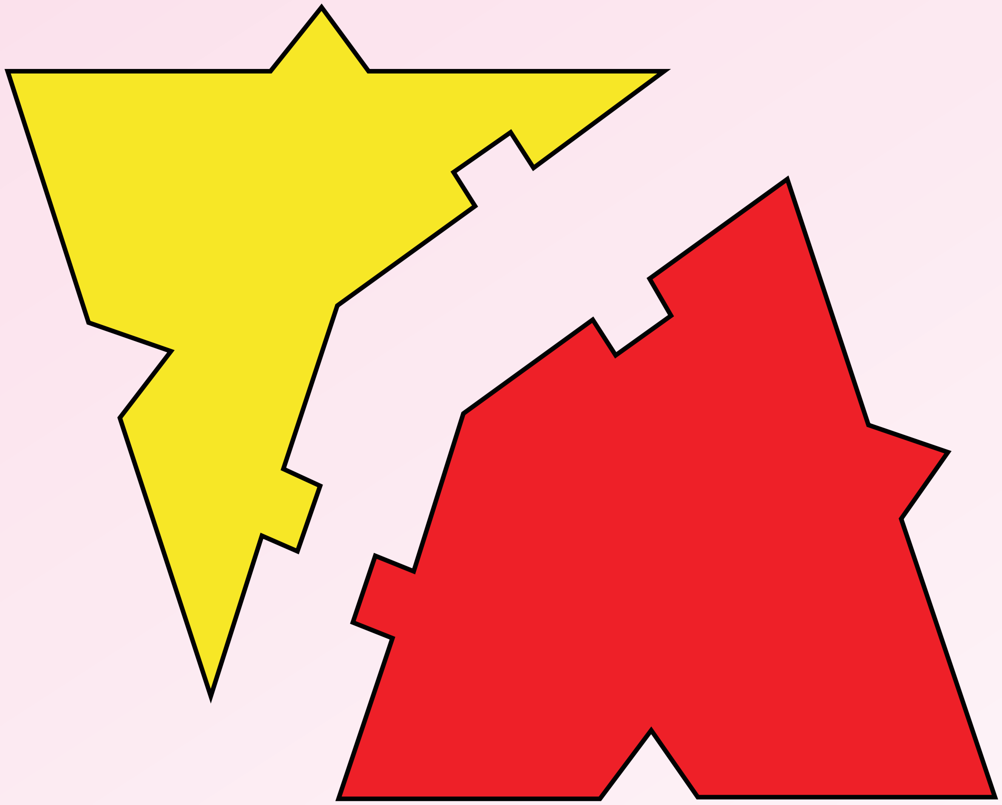
Kite and dart tiles (R.Penrose)

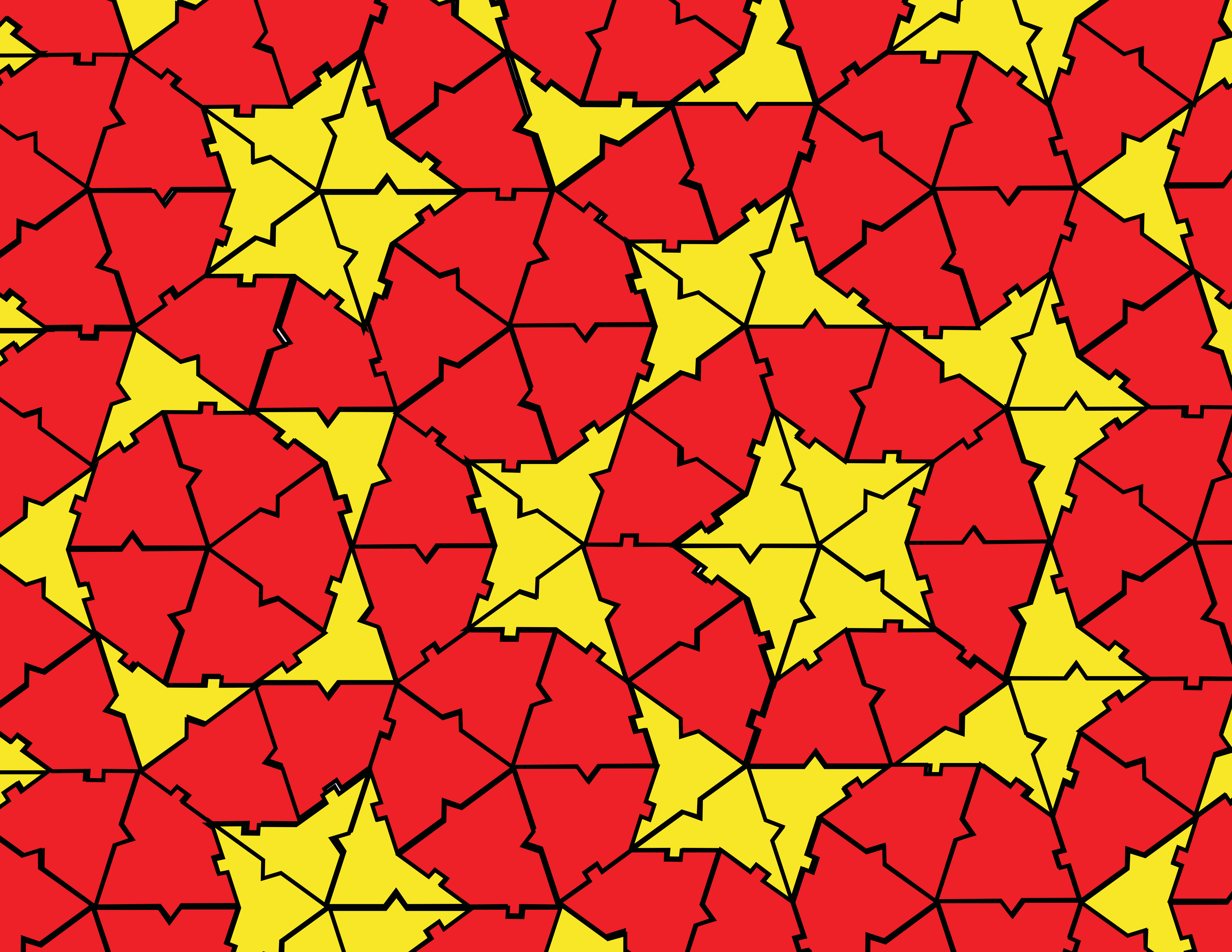


Kite and dart tiles (R.Penrose)



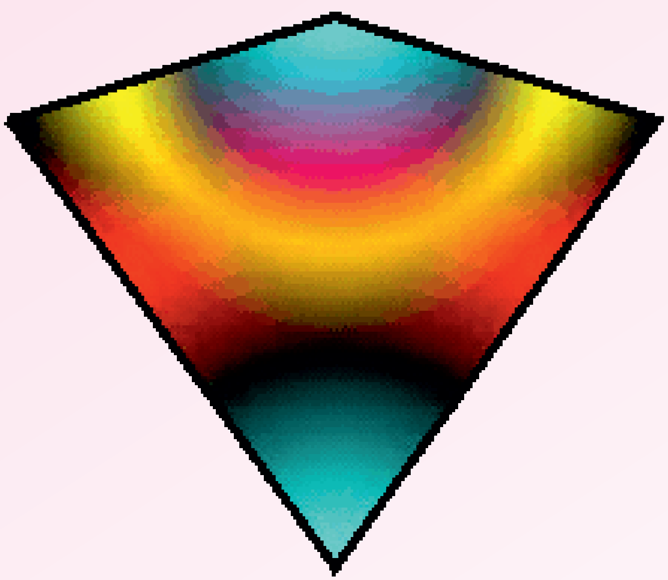
Kite and dart tiles (R.Penrose)



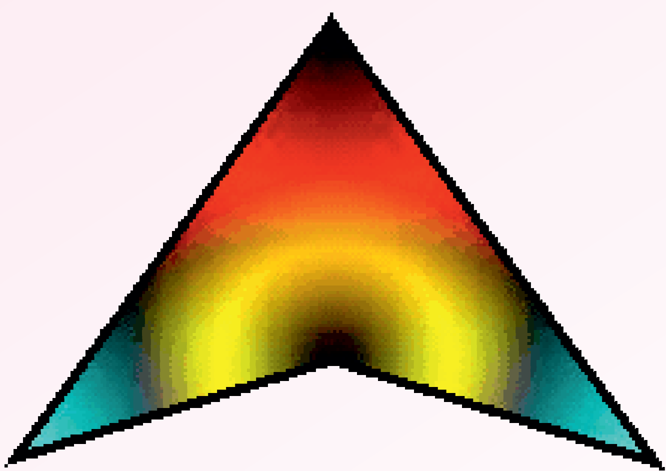


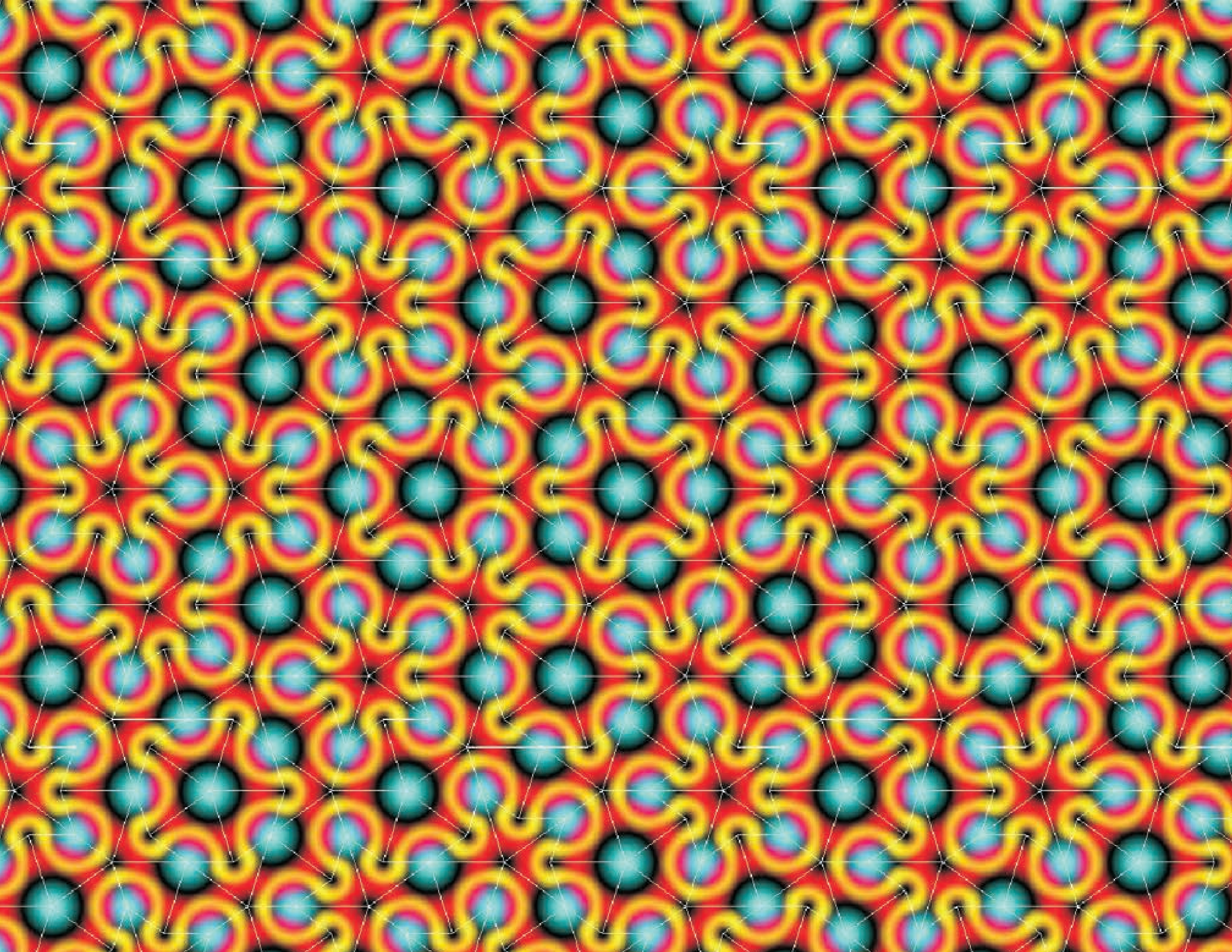
The Kite and the dart are a well-known aperiodic tile pair that admit a tiling with a fivefold rotational symmetry. One can visualize the fivefold symmetry by a suitable coloring of the tiles.

Kite



Dart





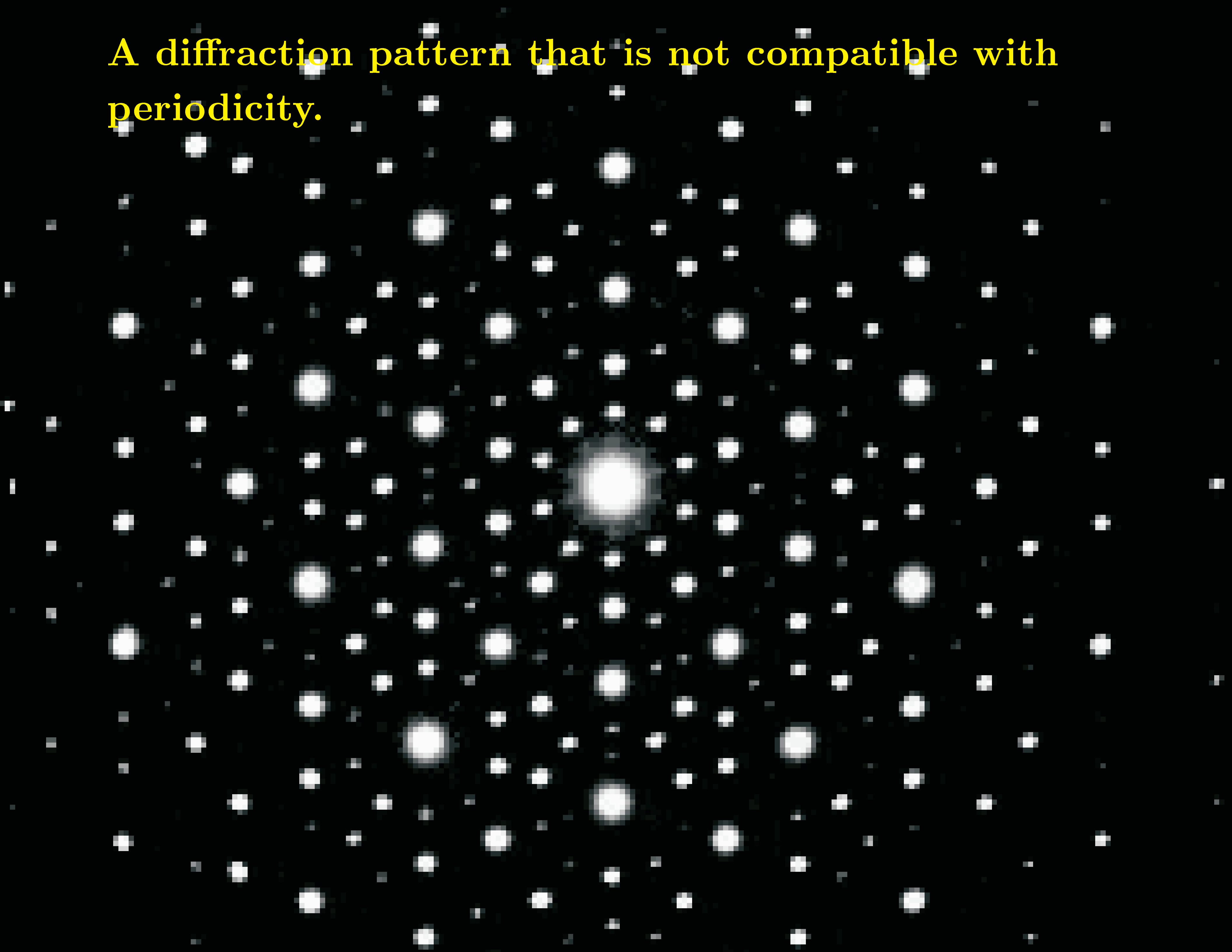
Keskuskatu, Helsinki

Until the early 1980's crystallographers believed that all crystals must be periodic.

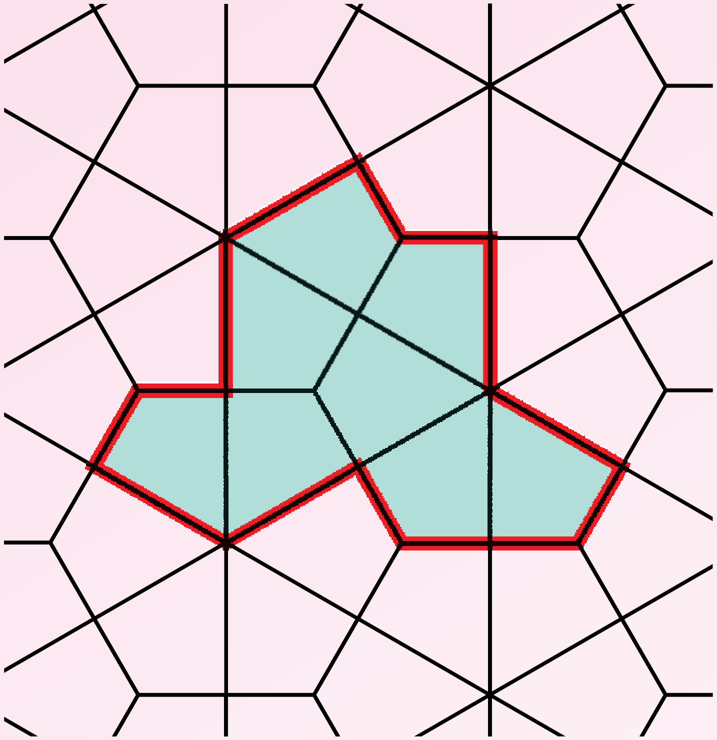
The invention of aperiodic tile sets opened up the theoretical possibility for non-periodic order in crystallography.

First such **quasicrystals** were discovered by Dan Shechtman in 1982. This discovery gave him the 2011 Nobel prize in chemistry.

A diffraction pattern that is not compatible with periodicity.



Very recently (in 2023) a single tile, the **hat**, was discovered that alone is an aperiodic tile:



Very recently (in 2023) a single tile, the **hat**, was discovered that alone is an aperiodic tile: