Let

$$\mathcal{T} = \{t_1, t_2, t_3, \dots\}$$

be a tiling. If $h : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ is a homeomorphism then also

$$h(\mathcal{T}) = \{h(t_1), h(t_2), h(t_3), \dots\}$$

is a tiling. We say that tilings $\mathcal{T}$ and $h(\mathcal{T})$ are **topologically equivalent**. This is an equivalence relation among tilings.

Every isometry is a homeomorphism, so if α is an isometry then $\alpha(\mathcal{T}) = \{\alpha(t_1), \alpha(t_2), \alpha(t_3), \ldots\}$ is a tiling. We say that that $\alpha(\mathcal{T})$ is **congruent** to tiling $\mathcal{T}$. Also congruence is an equivalence relation among tilings.

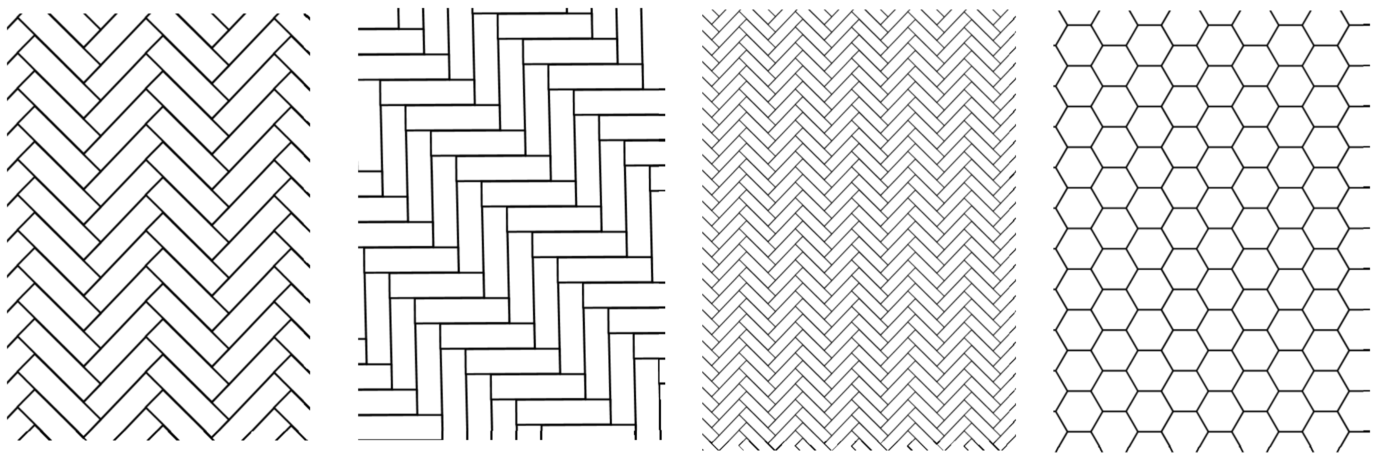
A **similarity** $s : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ is a composition of an isometry and a **stretch** map

$$(x, y) \mapsto (kx, ky)$$

for some $k > 0$. We say that tilings $\mathcal{T}$ and $s(\mathcal{T})$ are **similar**.

Similarity of two tiling means that they look the same when one of them is watched under a suitable magnifying class. Usually we consider similar tilings to be the same tiling.

Example. Four topologically equivalent monohedral tilings. First two are congruent with each other, and they are similar to the third one:



Tilings by regular polygons

A tiling whose tiles are polygons is **edge-to-edge** if the intersection of two tiles is either empty, a vertex of the polygons, or the entire edges of the two neighboring polygons.

Two tiles are

- **edge neighbors** if their intersection is an edge of both polygons,
- **vertex neighbors** if their intersection is a vertex of both polygons.

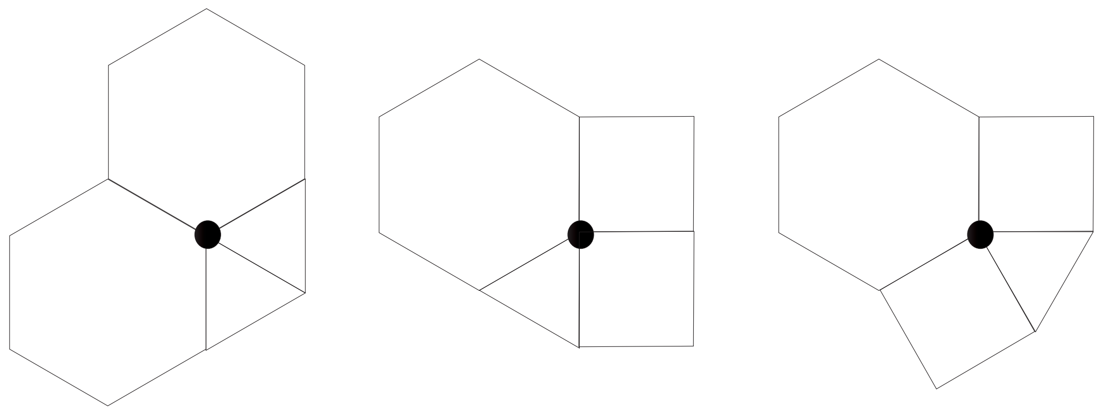
Consider a vertex P where r **regular polygons** of orders

$$n_1, n_2, \ldots, n_r$$

meet, in this order (counted clockwise or counterclockwise). We say that the vertex is of **type**

$$n_1 \cdot n_2 \cdot \cdots \cdot n_r.$$

Example. Vertices of types $3 \cdot 3 \cdot 6 \cdot 6$, $3 \cdot 4 \cdot 4 \cdot 6$ and $3 \cdot 4 \cdot 6 \cdot 4$:



Remark: Types $3 \cdot 4 \cdot 4 \cdot 6$ and $4 \cdot 6 \cdot 3 \cdot 4$ and $4 \cdot 3 \cdot 6 \cdot 4$ are all identical, as they are obtained by changing the starting point and/or the direction of reading the polygons.

We also adapt a shorthand notations for repetitions:
$3 \cdot 3 \cdot 6 \cdot 6$ may be abbreviated as $3^2 \cdot 6^2$.

The interior angle of a regular n-gon is

Consequently, if P is a vertex of type $n_1 \cdot n_2 \cdot \dots \cdot n_r$ then

$$\sum_{i=1}^{r} \left(1 - \frac{2}{n_i}\right) =$$

(The interior angles of the polygons that meet at P must sum up to 360°.)

This limits the possible vertex types: only finitely many possibilities remain.

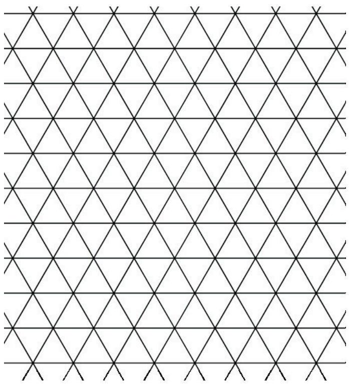
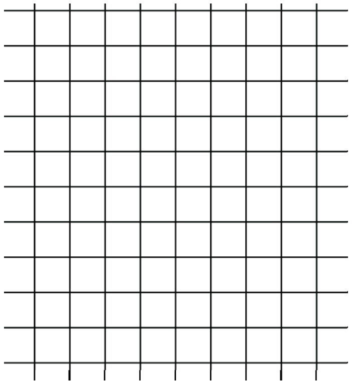
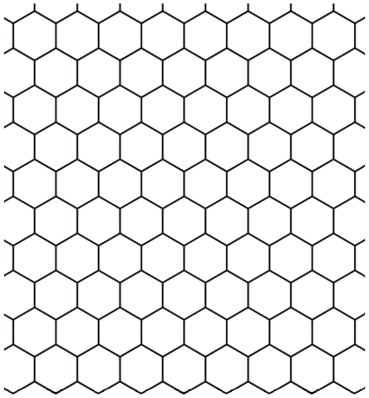
$$\sum_{i=1}^{r}\left(1-\frac{2}{n_i}\right)=2$$

Suppose first a vertex type n^r, where r copies of regular n-gons meet. We then get the condition

$$n = \frac{2r}{r-2}$$

Because n is positive, we have $r \geq 3$, and because $n \geq 3$ we have $r \leq 6$.

With $r = 3, 4, 5$ and 6 we get $n = 6, 4, \frac{10}{3}$ and 3. Number n is an integer so we only have three solutions. These vertex types appear in the familiar **regular** tilings

Theorem. The only edge-to-edge monohedral tilings by regular polygons are the three regular tilings above.

$$\sum_{i=1}^{r}\left(1-\frac{2}{n_i}\right)=2$$

Theorem. There are only finitely many vertex types.

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Theorem. There are only finitely many vertex types.

Proof. It is enough to show that there are finitely many solutions that satisfy

$$n_1 \le n_2 \le \cdots \le n_r.$$

We have $r \ge 3$ and $n_i \ge 3$, which implies that $r \le 6$.

Also $n_{r-1} \le 12$: If $n_{r-1} > 12$ then also $n_r > 12$ so

$$\sum_{i=1}^{r}\left(1-\frac{2}{n_i}\right) \ge \left(1-\tfrac{2}{n_1}\right)+\left(1-\tfrac{2}{n_{r-1}}\right)+\left(1-\tfrac{2}{n_r}\right)$$

$$> \left(1-\tfrac{2}{3}\right)+\left(1-\tfrac{2}{12}\right)+\left(1-\tfrac{2}{12}\right)$$

a contradiction.

There are only finitely many tuples $(n_1, n_2, \ldots, n_{r-1})$ with $r \le 6$ and $3 \le n_1 \le \cdots \le n_{r-1} \le 12$. The last number n_r is uniquely determined by $n_1, n_2, \ldots, n_{r-1}$ so there are finitely many vertex types.

The possible vertex types:

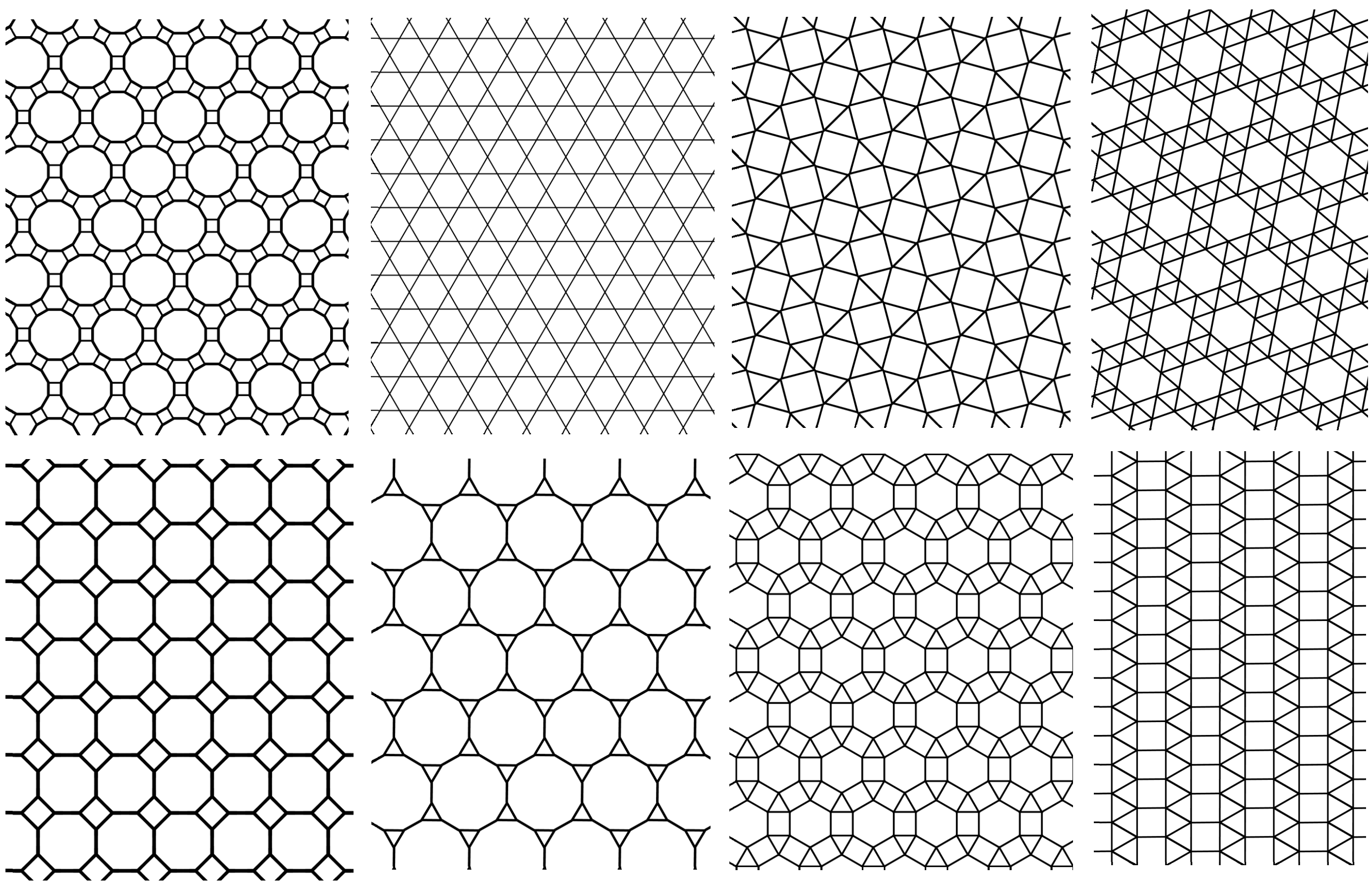
type	archimedean
$3 \cdot 3 \cdot 3 \cdot 3 \cdot 3 \cdot 3$	A
$3 \cdot 3 \cdot 3 \cdot 3 \cdot 6$	A
$3 \cdot 3 \cdot 3 \cdot 4 \cdot 4$	A
$3 \cdot 3 \cdot 4 \cdot 3 \cdot 4$	A
$3 \cdot 3 \cdot 4 \cdot 12$	
$3 \cdot 3 \cdot 6 \cdot 6$	
$3 \cdot 4 \cdot 3 \cdot 12$	
$3 \cdot 4 \cdot 4 \cdot 6$	
$3 \cdot 4 \cdot 6 \cdot 4$	A
$3 \cdot 6 \cdot 3 \cdot 6$	A
$3 \cdot 7 \cdot 42$	
$3 \cdot 8 \cdot 24$	
$3 \cdot 9 \cdot 18$	
$3 \cdot 10 \cdot 15$	
$3 \cdot 12 \cdot 12$	A
$4 \cdot 4 \cdot 4 \cdot 4$	A
$4 \cdot 5 \cdot 20$	
$4 \cdot 6 \cdot 12$	A
$4 \cdot 8 \cdot 8$	A
$5 \cdot 5 \cdot 10$	
$6 \cdot 6 \cdot 6$	A

An edge-to-edge tiling by regular polygons is **archimedean** if all vertices of the tiling are of the same type.

The three regular tilings are all archimedean, corresponding to vertex types 6^3, 4^4 and 3^6.

There are only eight non-regular archimedean tilings, corresponding to the vertex types marked by "A" in the table.

Theorem [Kepler 1619]. There are exactly eleven different archimedean tilings, one of each type indicated by "A" in the table.



Proof. We show that

(i) the vertex types without "A" in the table are not possible in archimedean tilings, and

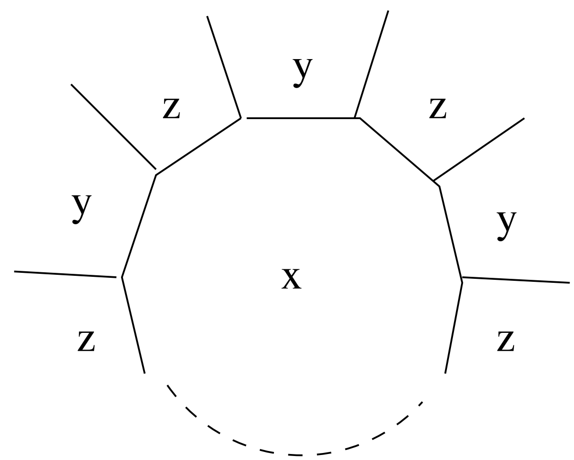
(ii) each type with "A" leads to a unique tiling (up to similarity).

Some terminology:

A polygon is **incident** to its vertices and edges, and an edge is **incident** to its endpoints.

Two vertices are **neighbors** if they are the two endpoints of an edge.

(i) Vertex type $x \cdot y \cdot z$ where x is odd and $y \neq z$ is not possible in any archimedean tiling.



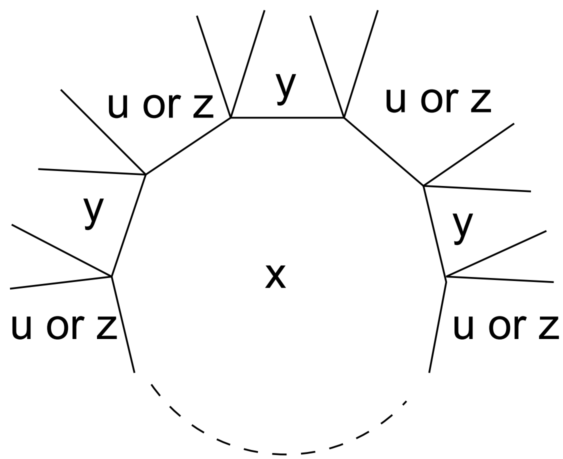
This rules out six vertex types $3 \cdot 7 \cdot 42$, $3 \cdot 8 \cdot 24$, $3 \cdot 9 \cdot 18$, $3 \cdot 10 \cdot 15$, $4 \cdot 5 \cdot 20$ and $5 \cdot 5 \cdot 10$.

Similarly: vertex type $x \cdot y \cdot u \cdot z$ is not possible when x is odd, $y \neq z$, and no three of the numbers are equal.

Reason: $x = y = z$ is not possible; let $x \neq z$.

- If $x \neq y$ consecutive edge-neighbors of x are y **and** z.
- If $x = y$ then $u \neq x, y$, and consecutive edge-neighbors of x are y **and** z, or $x = y$ **and** u.

In any case, edge neighbors $= y$ and $\neq y$ alternate.



This rules out four vertex types $3 \cdot 3 \cdot 4 \cdot 12$, $3 \cdot 3 \cdot 6 \cdot 6$, $3 \cdot 4 \cdot 3 \cdot 12$ and $3 \cdot 4 \cdot 4 \cdot 6$.

The only remaining types are the ones marked with “A”.

(ii) Uniqueness of the tiling for each vertex type.

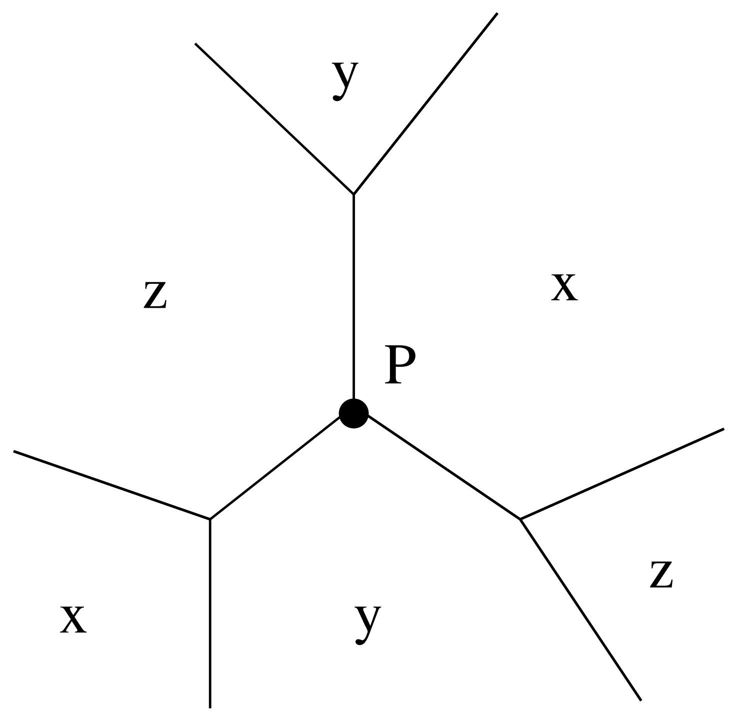
For each remaining vertex type there exists an archimedean tiling.

To prove uniqueness we show that, starting from known tiles around some vertex P of an archimedean tiling, the tiles around the neighboring vertices are uniquely forced.

Then (by induction on the distance of vertices from P) it follows that the tiles around all vertices are uniquely forced, so that the whole tiling is forced to be the same as the known archimedean tiling.

(The vertex type $3 \cdot 3 \cdot 3 \cdot 3 \cdot 6$ is an exception: there are two ways to complete the initial patch around P into an archimedean tiling. But these tilings are also isometric via an odd isometry.)

1) Vertex types $x \cdot y \cdot z$, that is, $3 \cdot 12 \cdot 12$, $4 \cdot 6 \cdot 12$, $4 \cdot 8 \cdot 8$, and $6 \cdot 6 \cdot 6$:

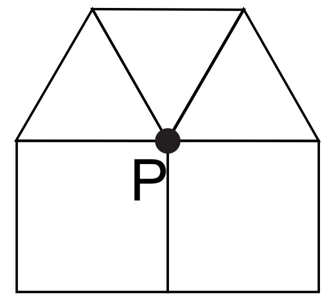


2) Vertex types $4 \cdot 4 \cdot 4 \cdot 4$ and $3 \cdot 3 \cdot 3 \cdot 3 \cdot 3 \cdot 3$: Trivial because all tiles are congruent to each other.

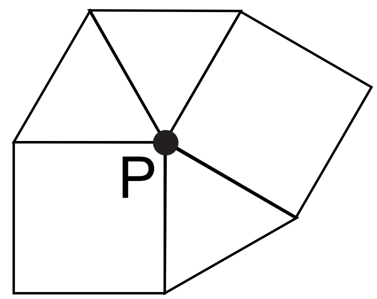
3) vertex types $x \cdot y \cdot x \cdot z$ where $y, z \neq x$, that is, types $3 \cdot 4 \cdot 6 \cdot 4$ and $3 \cdot 6 \cdot 3 \cdot 6$:

Opposite to x must be x.

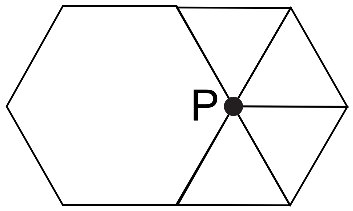
4) Vertex type $3 \cdot 3 \cdot 3 \cdot 4 \cdot 4$:

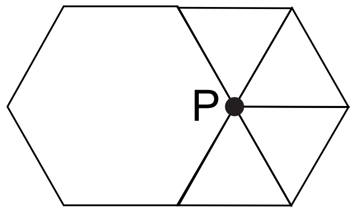


5) Vertex type $3 \cdot 3 \cdot 4 \cdot 3 \cdot 4$:



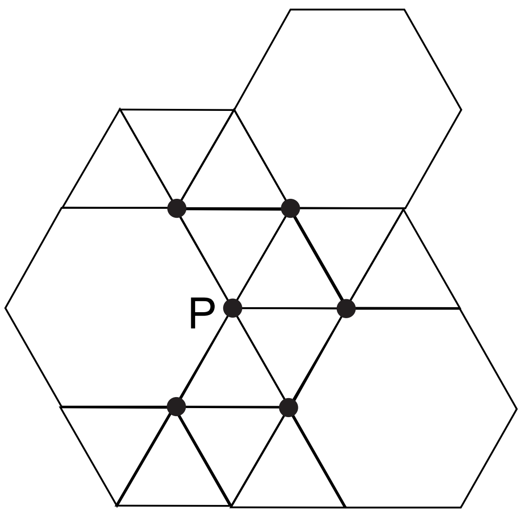
6) Vertex type $3 \cdot 3 \cdot 3 \cdot 3 \cdot 6$:





There are two ways of completing the polygons around the neighboring vertices. These ways are equivalent under an odd isometry.

Once the polygons around vertices of distance one from P are fixed, the rest of the tiling is uniquely forced:



Theorem. All archimedean tilings are **vertex-transitive**: For any two vertices P and Q there is a symmetry of the tiling that maps $P \mapsto Q$.

So we started with the weaker assumption that all vertices have the same type, and concluded the stronger property of vertex transitivity.

In fact: **(Except for** $3 \cdot 3 \cdot 3 \cdot 3 \cdot 6$**)** any isometry that maps the incident vertices around P onto the incident vertices around Q is a symmetry of the tiling. (In the case of $3 \cdot 3 \cdot 3 \cdot 3 \cdot 6$: there are two such isometries and one of them is a symmetry of the tiling.)