Wang tiles

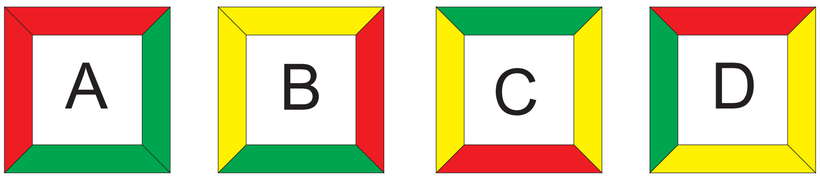
We are interested not only single tilings but the **sets of tilings** that a fixed protoset of tiles admits.

Questions like: do given prototiles admit any tiling of the plane; are there any periodic tilings, etc.

We study these questions on a setup where the role of geometry has been minimized and the spatial relation of tiles is simple.

A **Wang tile** is a unit square tile with colored (i.e. labeled) edges.

A **Wang tile set** is a **finite** set of Wang prototiles.



We tile the plane in the regular **grid** fashion **without rotating** the tiles. All tiles are thus congruent to the given prototiles by translations only.

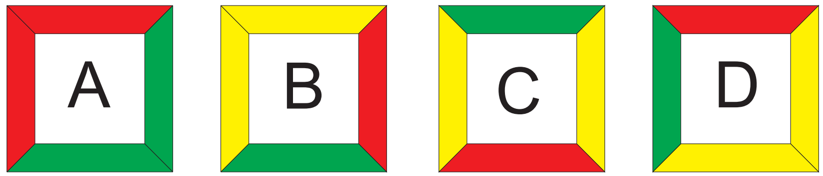
A tiling can then be represented as a function

$$f : \mathbb{Z}^2 \to A$$

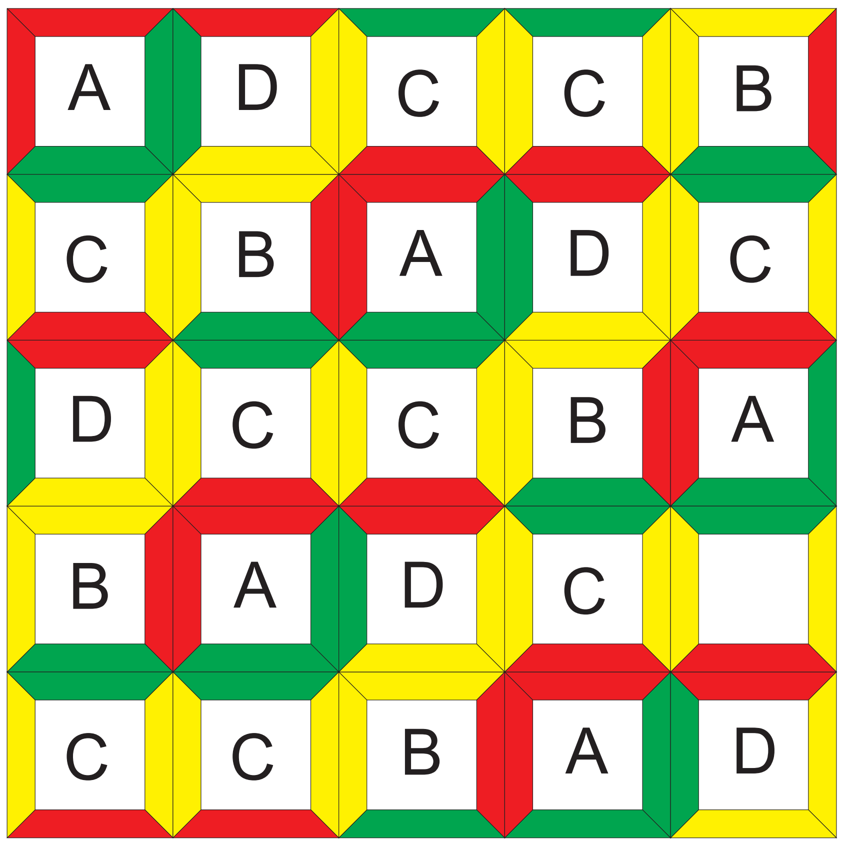
where A is the set of prototiles and $f(i, j)$ gives the tile at position $(i, j) \in \mathbb{Z}^2$. Such functions are called **configurations**.

The **matching rule** is that the shared edge between any two neighboring tiles must have the same color in both tiles. If the matching rule is satisfied everywhere in a configuration then the configuration is a valid **tiling**.

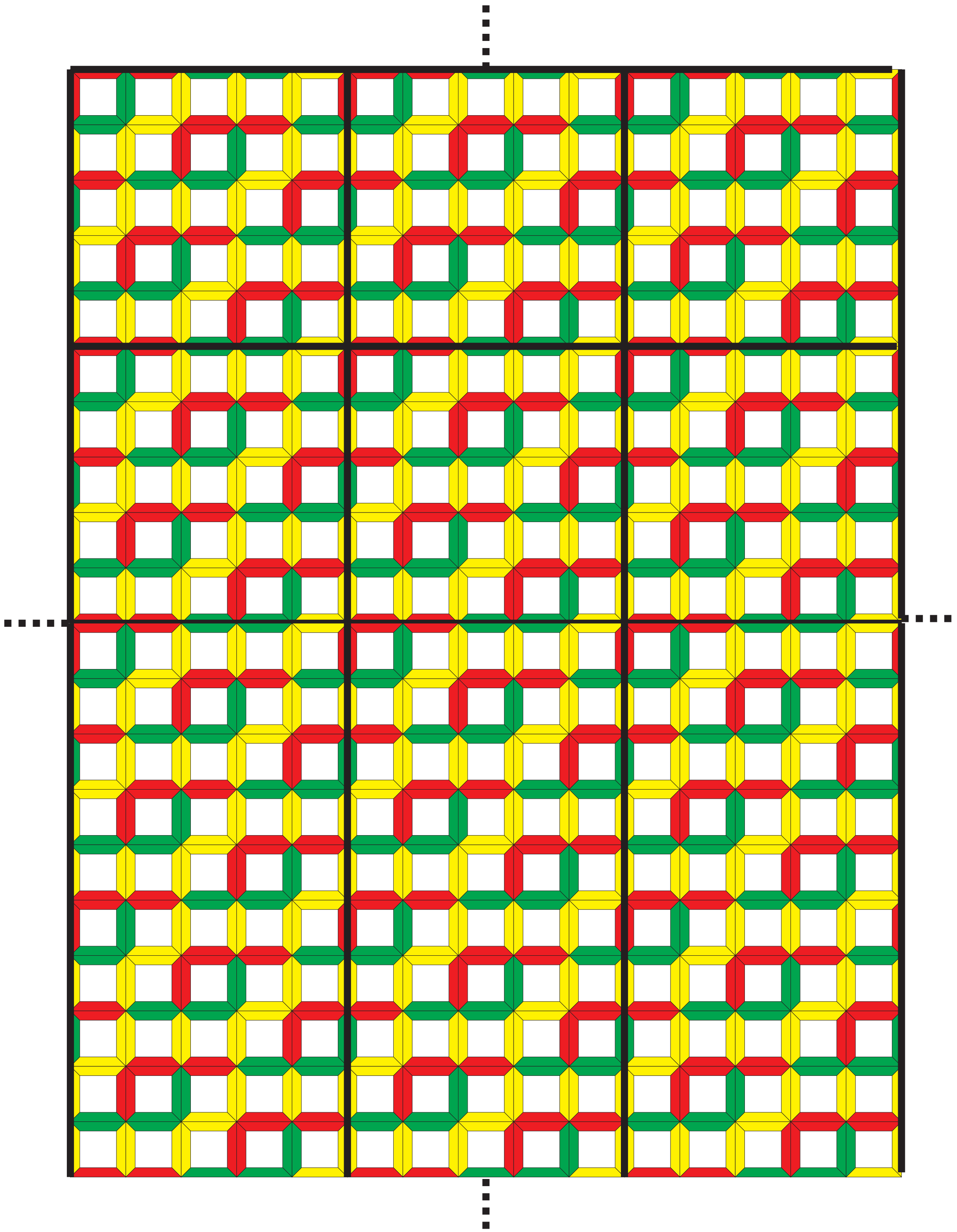
Example. With



we can tile:



. . . and since the colors on the borders match this square can be repeated to form a valid periodic tiling of the whole plane.



The **set of all configurations** $\mathbb{Z}^2 \longrightarrow A$ is denoted by

$$A^{\mathbb{Z}^2}.$$

(In general, Y^X denotes the set of functions $X \longrightarrow Y$.)

Set $A^{\mathbb{Z}^2}$ contains all assignments of tiles to $\mathbb{Z}^2$, including the ones where colors do not match. The set of valid tilings is a subset of $A^{\mathbb{Z}^2}$.

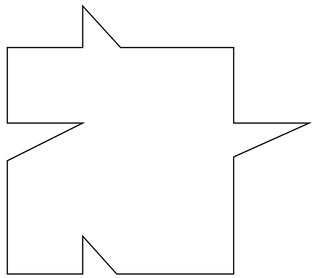
Wang tiles provide a **discrete** abstraction of tilings that allows us to study tilings using tools of discrete mathematics rather than geometry.

This is especially useful when investigating **computational properties**.

Although Wang tilings are on a square lattice $\mathbb{Z}^2$ only, the computational problems on Wang tiles are as hard as on more general types of tiles.

Remark. Wang tiles fit our original definition of tiles as **topological disks**: We can represent Wang tiles as "equivalent" polygons whose basic shape is a unit square.

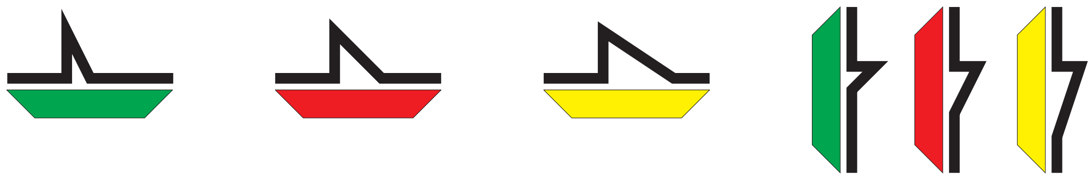
The middle of the north and east sides of each tile contain triangular **"bumps"** and the south and west sides have **"dents"** that exactly fit the bumps:

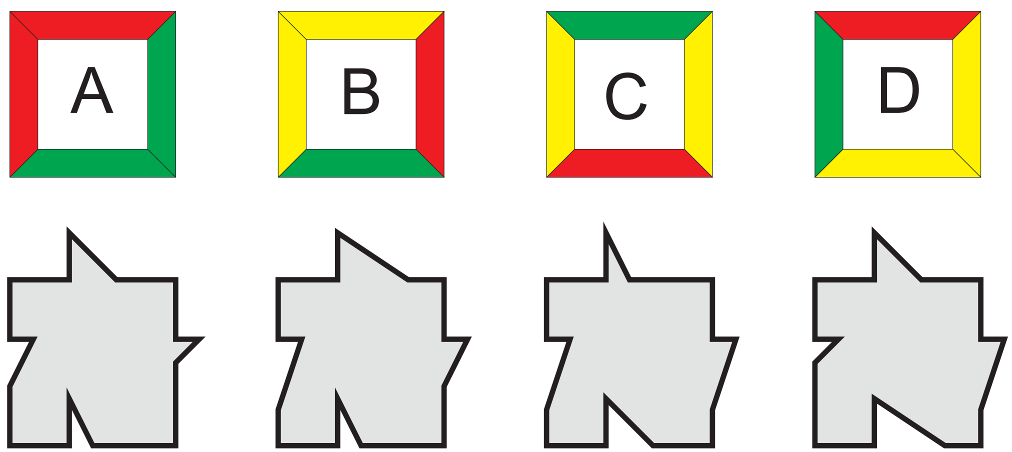


The bump/dent pairs are different in the horizontal and the vertical directions (to disallow rotating the tiles), and they are **asymmetric** so that flipped and non-flipped tiles do not match.

Moreover, **each color** has its own bump/dent shape that does not fit with any other color.

Example. Using six different shapes of bump/dents we represent the three horizontal and three vertical colors of our sample Wang tile set:





Obviously tilings by the polygons and tilings by the Wang tiles are "the same".

We call a configuration $f : \mathbb{Z}^2 \longrightarrow A$

- **(one-way) periodic** if there exists $(a, b) \neq (0, 0)$ such that

  $$\forall (x, y) \in \mathbb{Z}^2 \ : \ f((x, y) + (a, b)) = f(x, y)$$

  (in other words, the tiling has non-trivial translational symmetry), and we call any such vector (a, b) a **period** of the configuration.

- **strongly periodic** or **two-way periodic** if it has periods in non-parallel directions. (So the symmetry group is a wallpaper group.)

Remark: if $f : \mathbb{Z}^2 \longrightarrow A$ is strongly periodic with non-parallel periods (a, b) and (c, d) then it is also periodic with the **horizontal period**

$$d(a, b) - b(c, d) = (ad - bc, 0)$$

and the **vertical period**

$$a(c, d) - c(a, b) = (0, ad - bc).$$

(Note that $ad - bc \neq 0$ as vectors (a, b) and (c, d) are not parallel.)

In other words, a two-way periodic configuration has horizontal and vertical periods: It consists of a periodic repetition of a square pattern, as in earlier our example.

In the following we prove two preliminary lemmas:

(1) If a Wang tile set admits a one-way periodic tiling then it also admits a two-way periodic tiling.

(2) If a Wang tile set admits valid tilings of arbitrarily large finite squares then it also admits a tiling of the infinite plane.

One-periodic $\Longrightarrow$ two-periodic

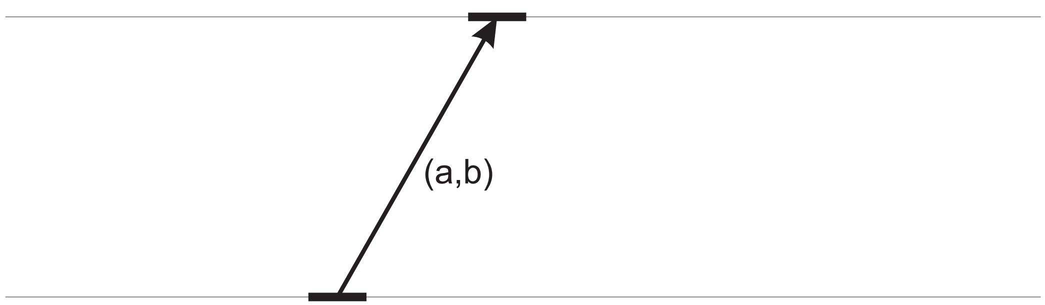
Theorem. If a Wang tile set admits a tiling with a period, then it also admits a tiling with two periods in non-parallel directions.

Proof. Let $f : \mathbb{Z}^2 \longrightarrow A$ be a periodic tiling using a prototile set A, and let

$$(a, b) \neq (0, 0)$$

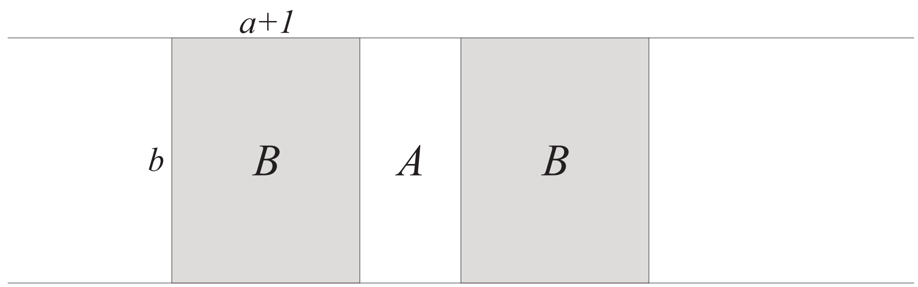
be a period. By symmetry we may assume $b > 0$ and $a \geq 0$.

Extract a **horizontal strip** of height b from the tiling:

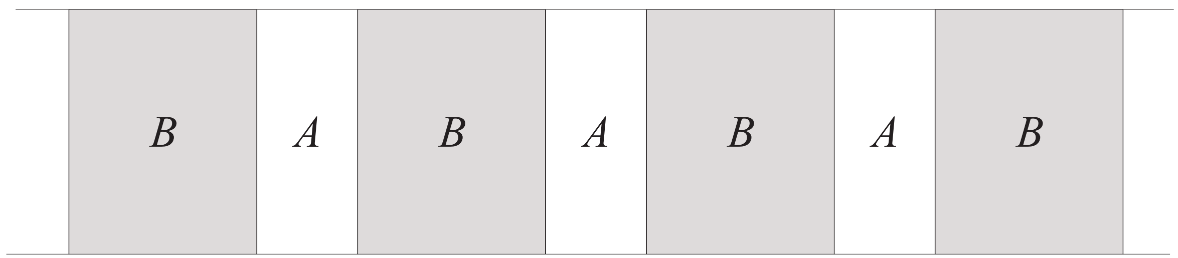


The **infinite sequences of colors** on top and bottom of the strip are the same, with a horizontal offset a.

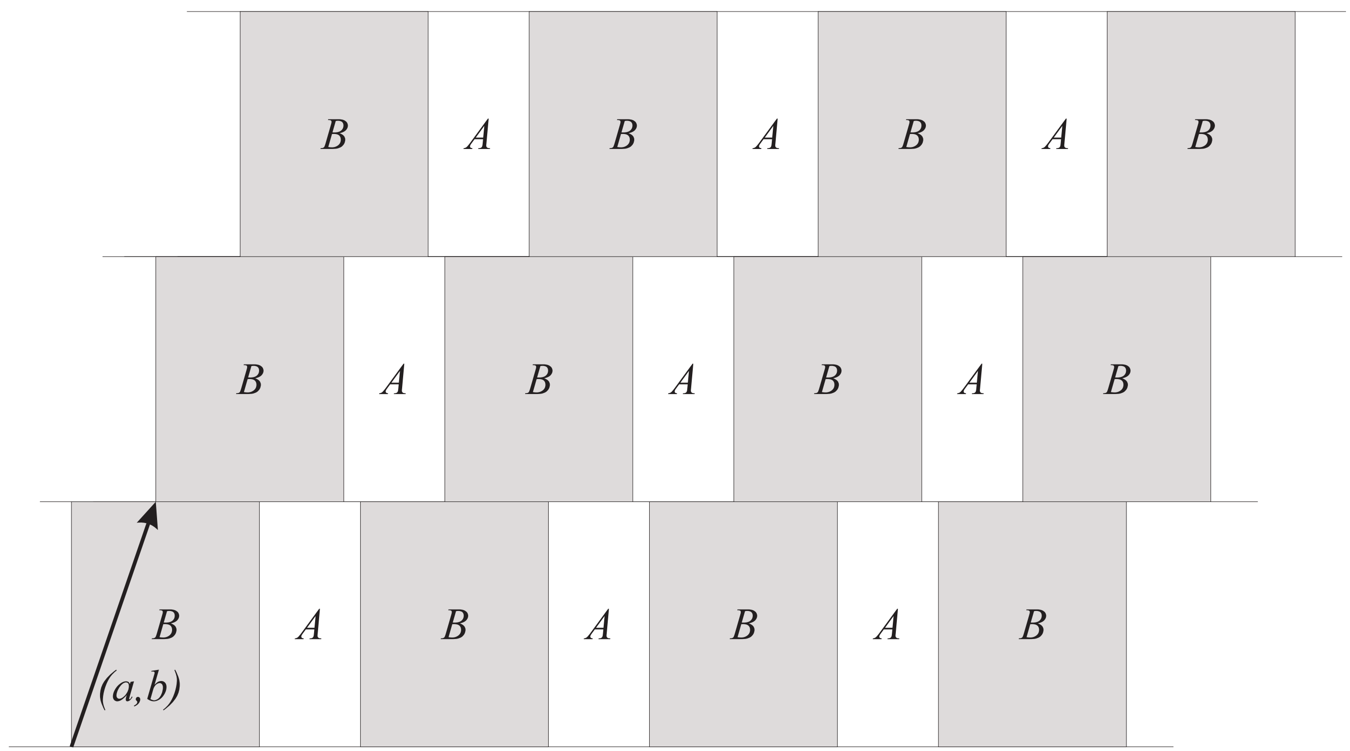
Consider **blocks of tiles** of size $(a+1) \times b$ in the strip. There are only finitely many different such blocks so some block B repeats with some block A in between:



Then the strip where blocks A and B repeat periodically is correctly tiled. It has identical infinite sequences of colors on the top and on the bottom, with offset a:

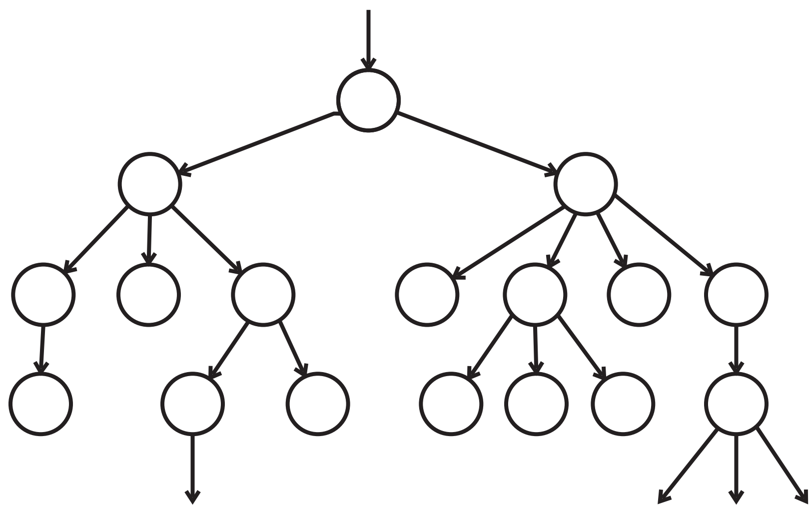


Thus we can stack copies of the periodic strip in top of each other with offsets a, and obtain a valid two-way periodic tiling, with a horizontal period and the non-horizontal period (a, b):



All finite squares $\Longrightarrow$ whole plane

Consider an infinite directed rooted tree where each node has a finite number of children:



Claim: The tree contains an infinite path down from the root.

Proof: If a node is the root of an infinite subtree, then it has a child that is also the root of an infinite subtree (because the node has just finite number of children).

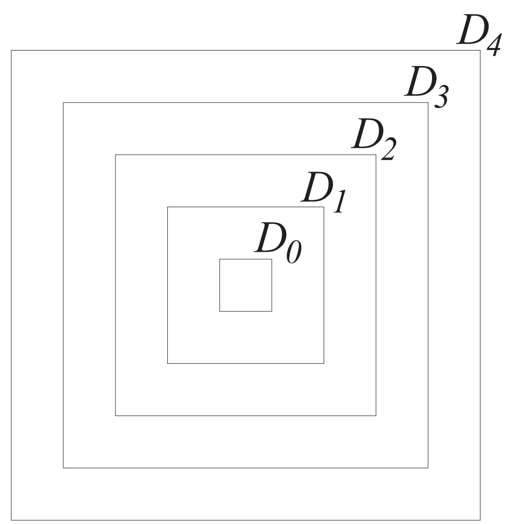
So starting from the root one can move down the tree by always moving to the child that is the root of an infinite subtree. This path is never blocked so the path follows an infinite branch of the tree.

Theorem. If a Wang tile set admits valid tilings of arbitrarily large finite squares then it also admits a tiling of the infinite plane.

Proof. For $n = 0, 1, \ldots,$ let

$$D_n = \{-n, \ldots, n\} \times \{-n, \ldots, n\},$$

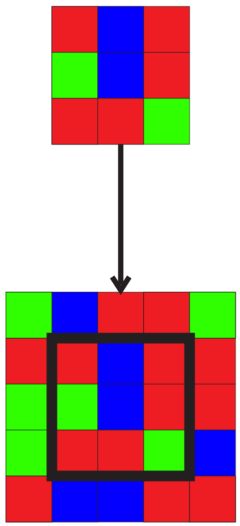
that is, the $(2n+1) \times (2n+1)$ size square centered at cell $(0, 0)$.



The assumption is that the set A of prototiles is such that every D_n can be tiled correctly.

For each tile $t \in A$ consider the rooted tree such that:

- Nodes at level n are correct tilings of D_n with t at the center cell $(0,0)$.
- The parent of a tiling x of D_n is $x|D_{n-1}$. Thus children extend the parent pattern to a larger domain:



- The root is the unique node of level 0: it is the tiling of the single cell $(0,0)$ with tile t.

One of the $|A|$ many trees is infinite, so it has an infinite branch b. The infinite branch defines a valid tiling of $\mathbb{Z}^2$ where the tile in any position (x, y) is the unique tile put in that position by patterns of the branch b.

Compactness

The previous reasoning is in fact a compactness argument.

Let $c_1, c_2, \ldots$ be a **sequence** of configurations in $A^{\mathbb{Z}^2}$. The sequence **converges** to a **limit** configuration c if

$$(\forall (x,y) \in \mathbb{Z}^2)\ (\exists k)\ (\forall i \geq k)\ :\ c_i(x,y) = c(x,y).$$

Such a limit (if it exists) is unique and we denote

$$c = \lim_{i \to \infty} c_i.$$

In other words: if we look at any cell (x,y) and scan $c_1, c_2, \ldots$ then from some moment on we always see the same tile $c(x,y)$ in position (x,y).

Later we give the set $A^{\mathbb{Z}^2}$ of configurations a **metric**. The convergence of sequences under this metric is exactly this convergence concept.

A **subsequence** of $c_1, c_2, \ldots$ is a sequence

$$c_{i_1}, c_{i_2}, \ldots$$

where $i_1 < i_2 < \ldots$.

(So a subsequence is obtained by picking infinitely many elements of the sequence, preserving their relative order.)

Obviously every subsequence of a converging sequence also converges and has the same limit.

The following theorem states the compactness of the configuration space:

Theorem. Every sequence of configurations has a converging subsequence.

Proof. Let $c_1, c_2, \ldots$ be an arbitrary sequence in $A^{\mathbb{Z}^2}$.

Fix one $t \in A$ such that $c_i(0,0) = t$ for infinitely many i. (Such t exists since A is finite.)

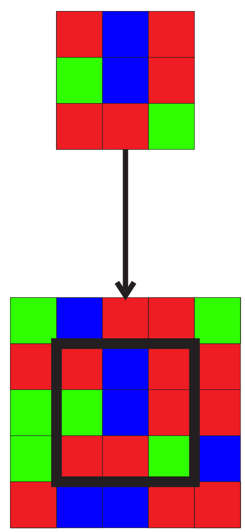
Define the following infinite tree:

- Nodes at level n are patterns $p : D_n \longrightarrow A$ with t at the center cell $(0,0)$, such that for infinitely many i we have

$$c_i|_{D_n} = p.$$

(Patterns that appear centered around origin in infinitely many elements of the sequence.)

- The parent of a pattern $x : D_n \longrightarrow A$ is the pattern $x|_{D_{n-1}}$. Thus children extend the parent pattern to a larger domain:



- The root is the unique node of level 0: it is the tiling of the single cell $(0,0)$ with tile t.

The tree is infinite so it has an infinite branch b. The infinite branch defines a configuration c where the tile in any position (x, y) is the unique tile put in that position by patterns of the branch b.

There is a subsequence that converges to c: we pick indices $i_1, i_2, \ldots$ such that for every n

- $i_n < i_{n+1}$, and
- $c_{i_n}|_{D_n}$ is the level n node in branch b.

Robinson's aperiodic tile set

For a long time it was thought that any finite set of prototiles that admits a non-periodic tiling must also admit a periodic one. This conjecture was refuted by **R.Berger** in 1966 when he constructed a set of Wang prototiles that only admit non-periodic tilings.

A finite set of prototiles is called **aperiodic** if

(i) it admits valid tilings, and

(ii) it does not admit any periodic valid tilings.

The first aperiodic Wang tile set by Berger contains 20426 tiles. (The main point of his work was the undecidability of the domino problem, discussed later in the course, and the number of tiles is irrelevant in that context.)

As our first example of an aperiodic tile set we take 56 Wang tiles due to **R.M.Robinson**. This set will be also useful later in our undecidability proofs.

Instead of colors we use **arrows** to describe the matching rules between tiles. In valid tilings arrow heads and tails in neighboring tiles must match. This formalism can be easily converted into a color-based matching simply by assigning a different color for each orientation and positioning of arrows.