Frieze groups

The set $\mathcal{T}$ of translations is a subgroup of $\mathcal{I}$. Thus for any group G of isometries the **translations in G** form a subgroup

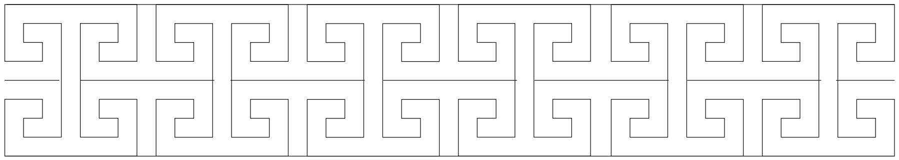
$$G \cap \mathcal{T}.$$

Group G is a **frieze group** if

$$G \cap \mathcal{T} = \langle \tau \rangle$$

for some non-trivial translation τ. So for some non-zero vector A the translations in G are precisely by multiples of A.

Example. The symmetry group of the bi-infinite horizontally repeating pattern



is a frieze group.

Example. The symmetry group of a horizontal line is not a frieze group although all translations are in the horizontal direction. However, there is no smallest positive translation.

There are **seven** different frieze groups (when we ignore the position, orientation and the size of the frieze).

Convention: The direction of the translations in the frieze group is fixed to be **horizontal**.

Lemma. Let G be a subgroup of $\mathcal{I}$ such that all translations in G are horizontal, and assume that there is at least one non-trivial translation. Then there exists a horizontal line m such that all elements of G are products of reflections in vertical lines, possibly followed by the reflection σ_m in line m.

These products are:

- horizontal translations,
- reflections in vertical lines,
- reflection σ_m in line m,
- halfturns about points of line m, and
- glide reflections with axis m.

Proof.

Classifying possible frieze groups

Let G be frieze group. Notations:

- Translations are generated by the shortest translation τ_A by vector A.
- Let m be the horizontal line from the previous lemma, called the **axis** of the frieze group.
- Let $2d$ be the length of vector A, so that τ_A is a product of two reflections in vertical lines at distance d. The translations in G are then exactly the products of two reflections in any two vertical lines whose distance is a multiple of d.

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$$\textbf{glide reflection } \sigma_l \sigma_k \sigma_m \in G \Longleftrightarrow \textbf{translation } \sigma_l \sigma_k \in G.$$

Translations are known (by multiples of A) so glide reflections exist in G and they are known (glides are by multiples of A.)

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Thus half turns and vertical reflections uniquely determine each other. We have two subcases:

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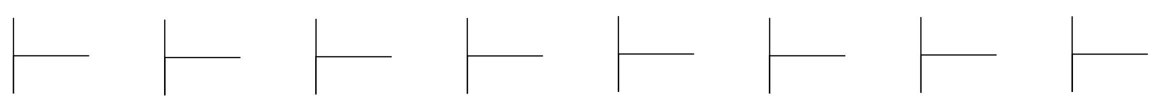
Subcase 1(a) No vertical reflections in G, so no half turns either: $G = \langle \tau_A, \sigma_m \rangle$. It is the symmetry group of



We call this the frieze group F_{1001}.

Our naming convention (not standard!):

Frieze group F_{abcd} where $a, b, c, d \in \{0, 1\}$ indicate the following isometries in G:

- $a = 1$ iff G contains σ_m,
- $b = 1$ iff G contains a reflection σ_ℓ on some vertical line ℓ,
- $c = 1$ iff G contains a half turn,
- $d = 1$ iff G contains a glide reflection.

It turns out that these four values uniquely identify the frieze group. Only seven of the a, b, c, d combinations are possible.

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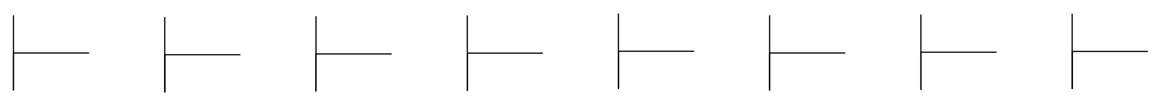

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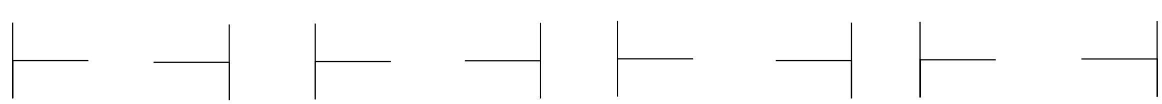
Thus half turns and vertical reflections uniquely determine each other. We have two subcases:

Subcase 1(b) There is a vertical reflection $\sigma_\ell \in G$. Then other vertical reflections $\sigma_k \in G$ are uniquely determined since

$$\sigma_\ell \sigma_k = \tau_{2B}$$

is a translation where B is the vector from ℓ to k. Thus the vertical symmetry lines are uniquely determined by ℓ and the translations: they are at distances that are multiples of d.

Half turns in G are also uniquely determined. The group is $G = \langle \tau_A, \sigma_m, \sigma_\ell \rangle$ and has name F_{1111}. It is the symmetry group of



Case 2 Assume that $\sigma_m \notin G$ (m is the horizontal axis)

Subcase 2(a) If there are no other symmetries except the translations we get the group $G = \langle \tau_A \rangle$ that has name F_{0000}. It is the symmetry group of

Case 2 Assume that $\sigma_m \notin G$ (m is the horizontal axis)

Subcase 2(b) Assume there are also other symmetries. Possibilities are **vertical reflections** σ_ℓ, **half turns** σ_P and **glide reflections** γ. Note that

- $\sigma_\ell \sigma_P$ is a glide reflection,
- $\gamma \sigma_P$ is a vertical reflection,
- $\gamma \sigma_\ell$ is a half turn.

Thus you cannot have two of these types without the third. Thus only possibilities F_{0100}, F_{0010}, F_{0001} and F_{0111} remain.

We need to show that each of these four cases is a possibility and a unique group.

Case 2 Assume that $\sigma_m \notin G$ (m is the horizontal axis)

Subcase F_{0100}. Suppose G contains a vertical reflection σ_ℓ. Then other vertical reflections are products of translations and σ_ℓ, so they are uniquely identified. The distances between vertical symmetry lines are multiples of d (=half of the translation). We have the symmetry group $G = \langle \tau_A, \sigma_\ell \rangle$ of

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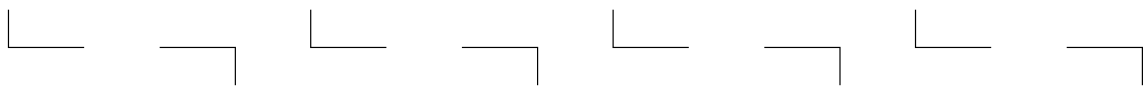
Subcase F_{0010}. Suppose G contains a half turn $\sigma_P = \sigma_m \sigma_\ell$ for a vertical line ℓ. Then other half turns are products of translations and σ_P, so they are uniquely identified. The distance between rotation centers are multiples of d (=half of the translation).

We have the symmetry group $G = \langle \tau_A, \sigma_P \rangle$ of



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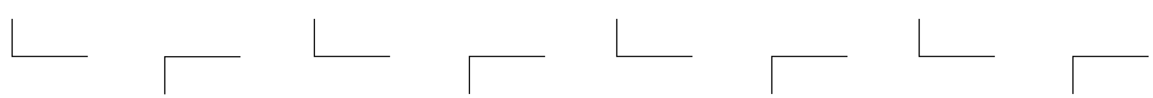
Subcase F_{0001}**.** Suppose G contains a glide reflection $\gamma = \sigma_\ell \sigma_k \sigma_m$ with axis m. Let g be the distance of lines ℓ and k, so the glide vector of γ has length $2g$.

Now $\gamma\gamma$ is a translation of length $4g$ so $4g$ is a multiple of $2d$, that is, g **is a multiple of** $d/2$**.**

But g **is not a multiple of** d because otherwise a translation in G cancels the glide leaving σ_m, which is not in G.

Conclusion: g is an odd multiple of $d/2$, i.e., the length $2g$ of the glide of γ is by an odd multiple of d. All such glide reflections are products of a single γ and the translations in G, so we may choose γ to have glide of length d, half of the shortest translations, and all glide reflections are generated by this γ and the translations.

We get the symmetry group $G = \langle \tau_A, \gamma \rangle = \langle \gamma \rangle$ of the pattern

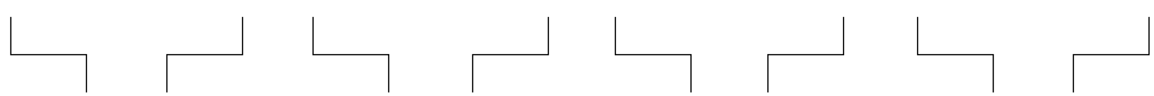


Case 2 Assume that $\sigma_m \notin G$ (m is the horizontal axis)

Final subcase F_{0111}. As seen above:

- The glide reflection γ with glide length d, half of the shortest translations, is in G and generates with translations all glide reflections in G.
- A single vertical reflection σ_ℓ determines all vertical reflections.
- Half turns are products of glide reflections and vertical reflections so they get also uniquely determined. The centers of half turns are midway between vertical reflection lines.

We thus get the symmetry group $G = \langle \gamma, \sigma_\ell \rangle$ of



All possibilities have been exhausted, and we got seven frieze groups.