Tilings and Patterns: Homework 10 (6.11.2023)

1. Let $X = \{0, 1, 2, \ldots\}$, and let $\mathcal{T} = \{\emptyset\} \cup \{U_0, U_1, U_2, \ldots\}$ where for every i

$$U_i = \{i, i + 1, i + 2, \ldots\}.$$

   (a) Show that $\mathcal{T}$ is a topology of X.

   (b) Is $\mathcal{T}$ compact ? Is $\mathcal{T}$ metric ?

   (c) Which subsets of X are compact ?

   (d) Determine which sequences of elements of X converge to 0, and determine also which sequences converge to 1.

2. Prove the following properties of the topological closure $\overline{A}$ of $A \subseteq X$.

   (a) If $A \subseteq B$ then $\overline{A} \subseteq \overline{B}$,

   (b) $\overline{A \cup B} = \overline{A} \cup \overline{B}$,

   (c) $\overline{\overline{A}} = \overline{A}$.

3. Determine if the given metric space is compact. Prove your claim.

   (a) The real interval $[0, 1]$ under the usual metric.

   (b) The rational interval $[0, 1] \cap \mathbb{Q}$ under the usual metric.

   (c) Set $\mathbb{N}$ under the discrete metric.

4. (a) Prove that in a metric space X, every compact $A \subseteq X$ is closed and bounded. (Bounded means that A is contained in some ε-ball.)

   (b) Prove that the converse is not true: there is a metric space where a closed and bounded set is not necessarily compact.

5. Let X be a compact space, and suppose the topology has a base all of whose members are clopen (closed and open). Prove that a set is clopen if and only if it is a finite union of base sets.

6. Let A be a finite set, let $f : \mathbb{Z}^2 \longrightarrow \mathbb{N}$ be any function such that $f^{-1}(n)$ is finite for all $n \in \mathbb{N}$, and let $g : \mathbb{N} \longrightarrow \mathbb{R}_+$ be any decreasing function such that $\lim_{n \longrightarrow \infty} g(n) = 0$. Define $d : A^{\mathbb{Z}^2} \times A^{\mathbb{Z}^2} \longrightarrow \mathbb{R}$ by

$$d(x, y) = g(\min\{f(i, j) \mid x(i, j) \neq y(i, j)\}) = \max\{g(f(i, j)) \mid x(i, j) \neq y(i, j)\}$$

   when $x \neq y$, and $d(x, y) = 0$ if $x = y$.

   (a) Prove that d is a metric on the configuration space $A^{\mathbb{Z}^2}$.

   (b) Prove that open balls under the metric d are cylinders.

   (c) Prove that cylinders are open under the metric d.

   (d) Conclude that the metric d defines the same topology as the metric defined in the class on the configuration space $A^{\mathbb{Z}^2}$.

   (Note that the metric defined in the lecture notes is obtained with the choices $f(i, j) = |i| + |j|$ and $g(n) = 2^{-n}$, and in the class we used $f(i, j) = \max\{|i|, |j|\}$ instead.)

7. A subset of a topological space is disconnected if it is the union of two non-empty disjoint open sets under the induced topology. A space is totally disconnected if every set with more than one element is disconnected. Prove that the space $A^{\mathbb{Z}^2}$ is totally disconnected under the topology discussed in the class.