The tiling problem
(aka the domino problem)

Now we turn to:

Tiling problem
Instance: A finite set $\mathcal{T}$ of Wang tiles
Positive instance: $\mathcal{T}$ admits a tiling of the plane

and prove its undecidability.

Again, an aperiodic tile set is needed in the proof. In our proof, this set is **Robinson's tile set**.

In contrast to **Seeded tiling problem** no specified seed tile is now required to be used. The main problem is how to force the presence of the seed (=the beginning state of the Turing machine) in every valid tiling.

By compactness, if it is possible to have arbitrarily large squares tiled without the seed, then it is also possible to make the entire tiling without the seed.

Therefore the seed must be enforced inside all $n \times n$ squares for some n. This on the other hand would seem to be contradictory to the possibility that tilings with only a single seed tile may occur.

A solution is to partition the space using Robinson's aperiodic tile set into **"nested boards"**, each containing a piece of a valid tiling around a seed tile.

Recall **Robinson's tile set**. We add further labels and matching rules to draw the boards on tilings. Even after the added constraints the tiles still admit valid tilings.

First we add a binary color on all side arrows. So each side arrow is either **red** or **green**. The matching arrows in neighboring tiles must have the same color.

(1) On each cross, both side arrows have the same color.

(2) Side arrows of parity 1 crosses are both green.

(3) On each arm, possible side arrows at oppose edges of the tile have identical colors. (So the color runs unchanged through the arm tile.)

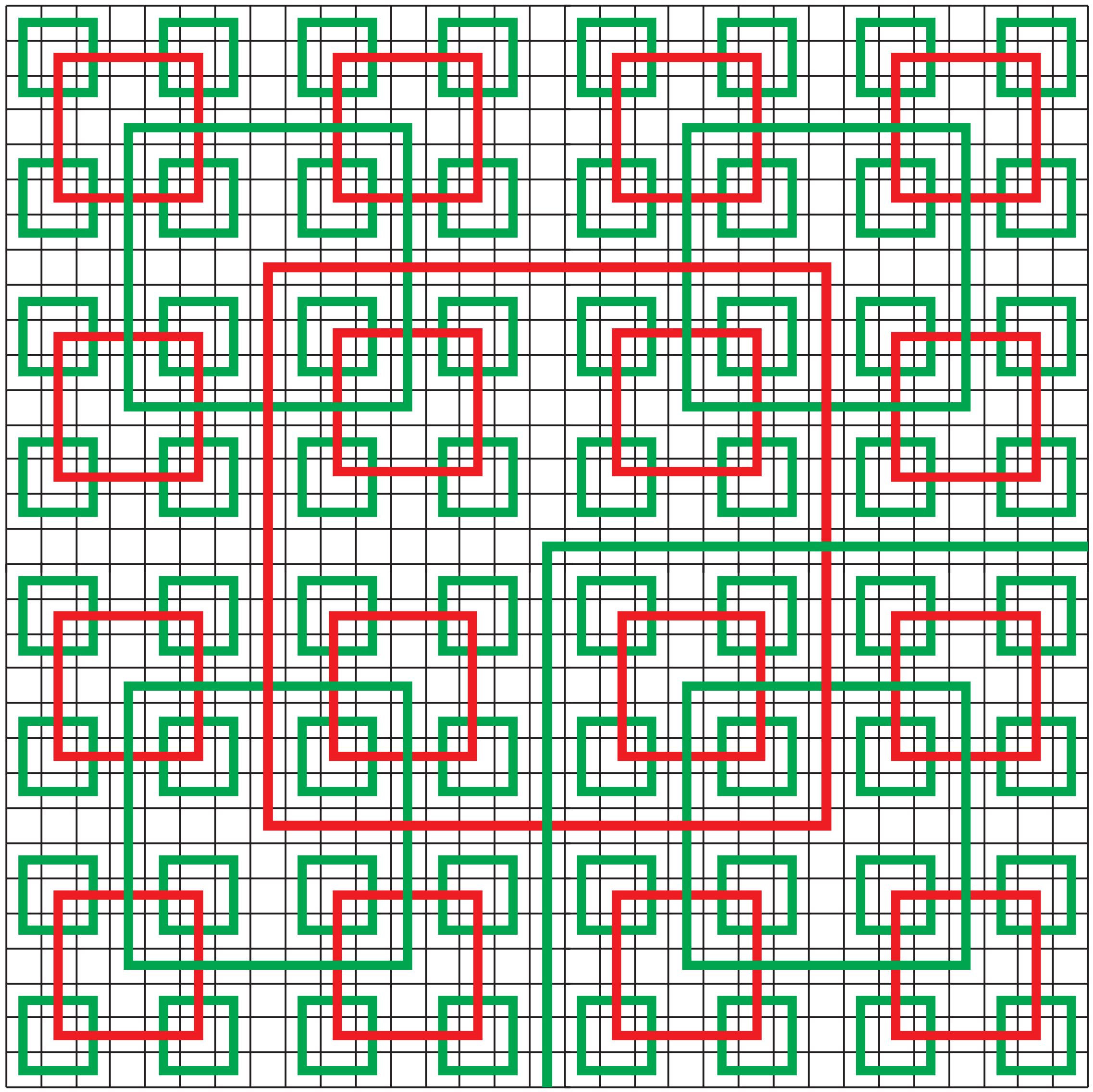
(4) On each arm, possible side arrows at adjacent edges of the tile have opposite colors.

By induction on n, we prove that the side arrows of the special $(2^n - 1) \times (2^n - 1)$ squares can be colored in a unique way so that the tiling remains valid. In this unique coloring, the colors of the (two) side arrows pointing out of the special square are

(a) **green** if n is **odd** and

(2) **red** if n is **even**.

In the special squares, sequences of matching side arrows define colored squares: the squares have corners at the crosses at the centers of the four quadrants of special squares, and arms transmit the color between the corners.



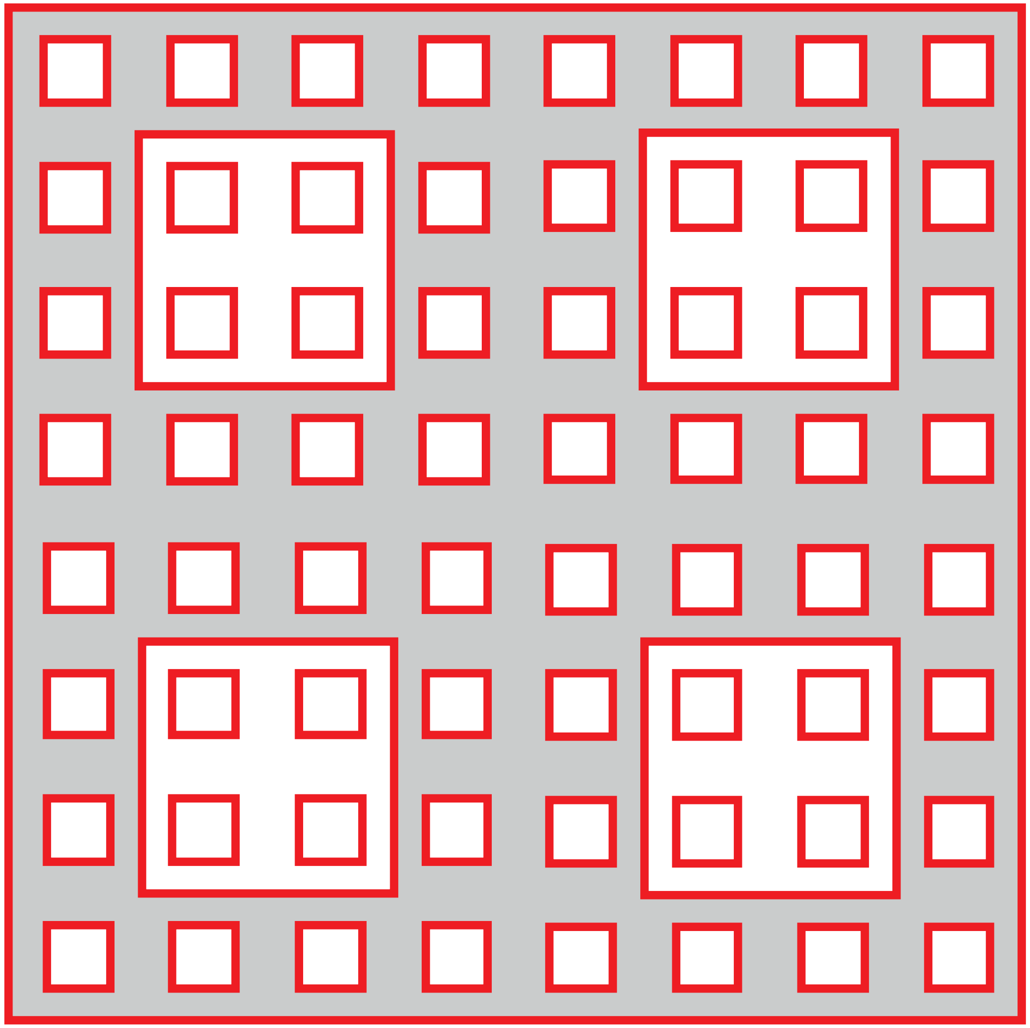
We focus on red squares. They do not intersect each other, but they may be nested inside each other.

The red squares have sizes $4^n \times 4^n$ for $n = 1, 2, \ldots$.

(This size is the distance between opposite edges. So there are $(4^n - 1) \times (4^n - 1)$ interior tiles inside the square.)

Each red square S defines a **board** that contains those tiles that are in the interior of S but are outside all nested red squares within S.

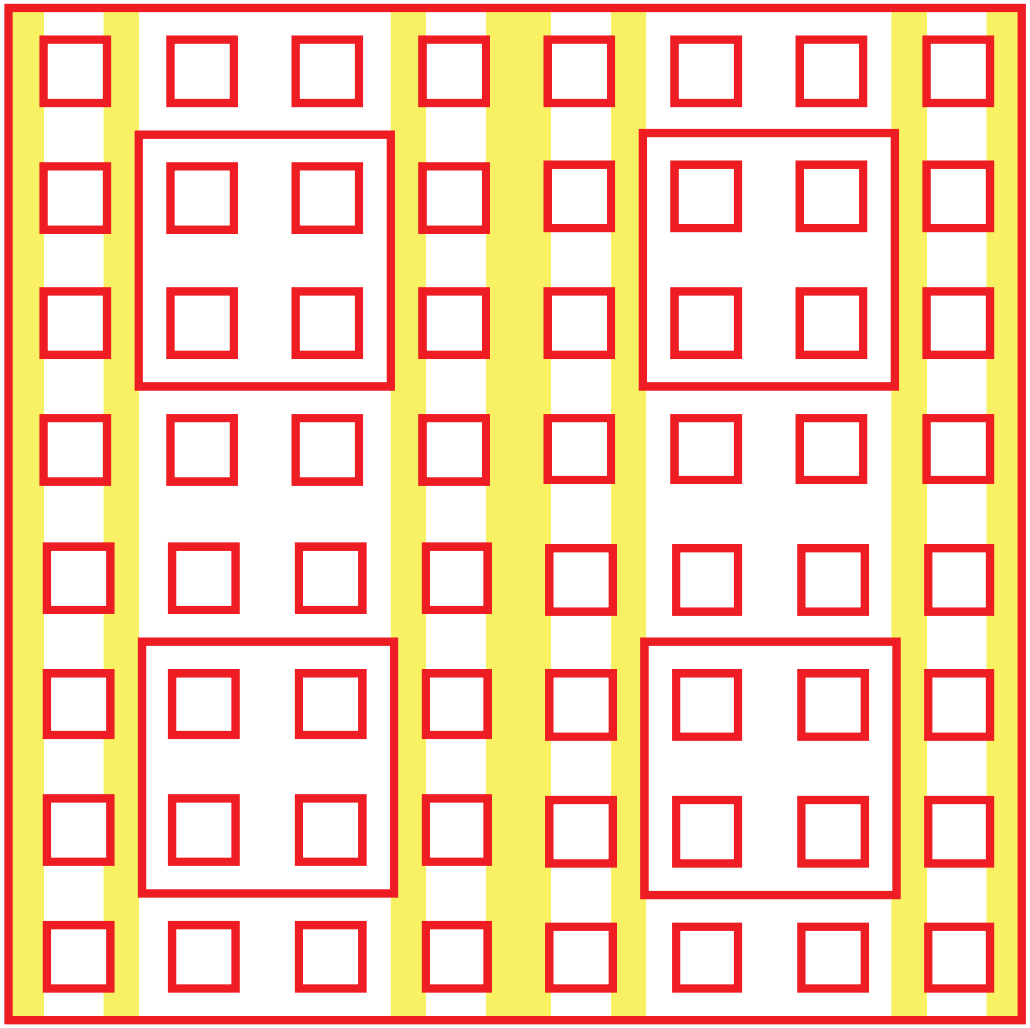
Note that there are nested boards within bigger boards.



- Each board has at the **center** a parity 4 cross with green side arrows. There are no other tiles like this in the board. Thus the center cell of the board can be identified.

- Tiles with red side arrows are on the boundary of a board. The position of the side arrow identifies the **inside** and the **outside** of the board. (The side arrow is on the outside half of the arm.)

A **free column** of a board is a vertical row of tiles inside the board that runs across the board between the top and the bottom boundary without being interrupted by another nested board.



Let us count the **number of free columns** inside a $4^n \times 4^n$ board. Let us denote this number by F_n. We need to know that $F_n \longrightarrow \infty$ as $n \longrightarrow \infty$.

We have a recurrence

$$F_{n+1} = 2F_n - 1,$$

with the starting value $F_1 = 3$. This easily leads to

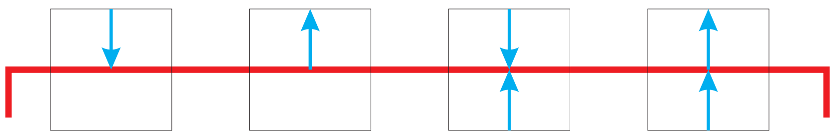
$$F_n = 2^n + 1.$$

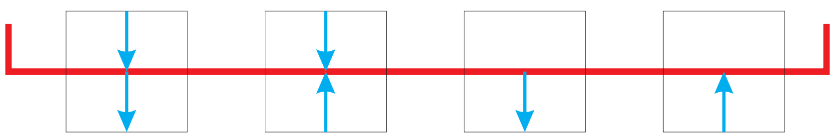
We need each tile to know whether it is on a free column or not. This is done using **obstruction signals**. Each tile may contain or may be without an obstruction signal.

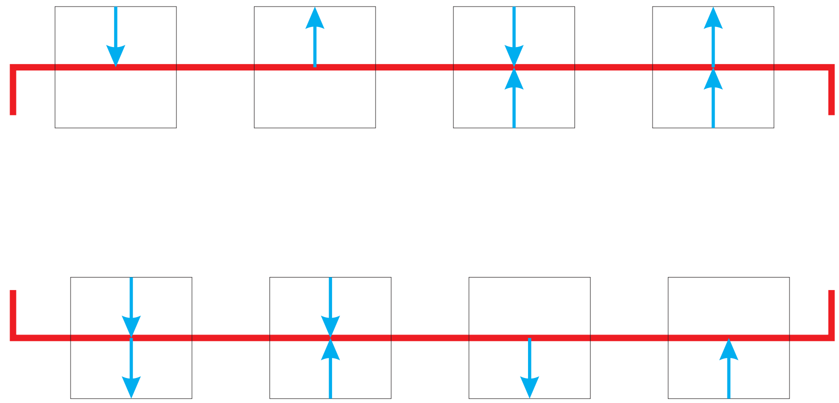
• An **outer edge** of a red border must emit or absorb a signal.

• The **inner edge** of the red borders may absorb but not emit a signal. Inner edges can also be without an obstruction signal.

Here are the four possible combinations of obstruction signals on the upper and the lower boundaries of a red square:



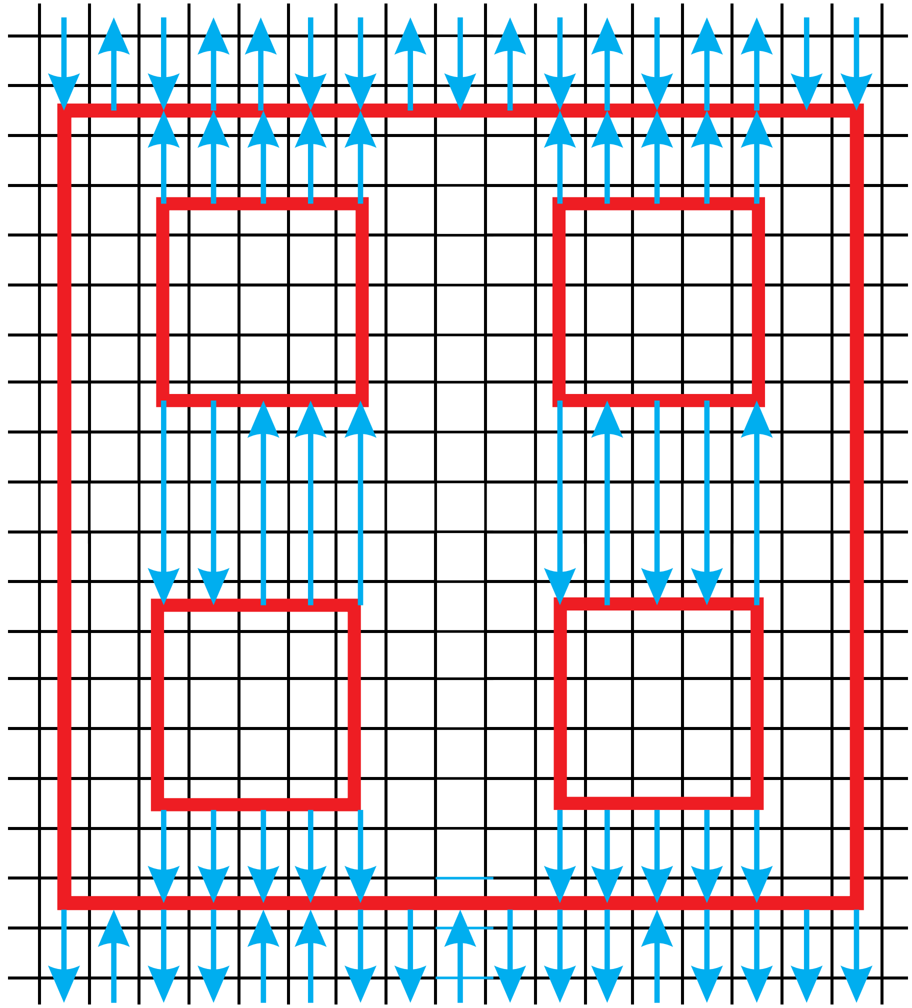




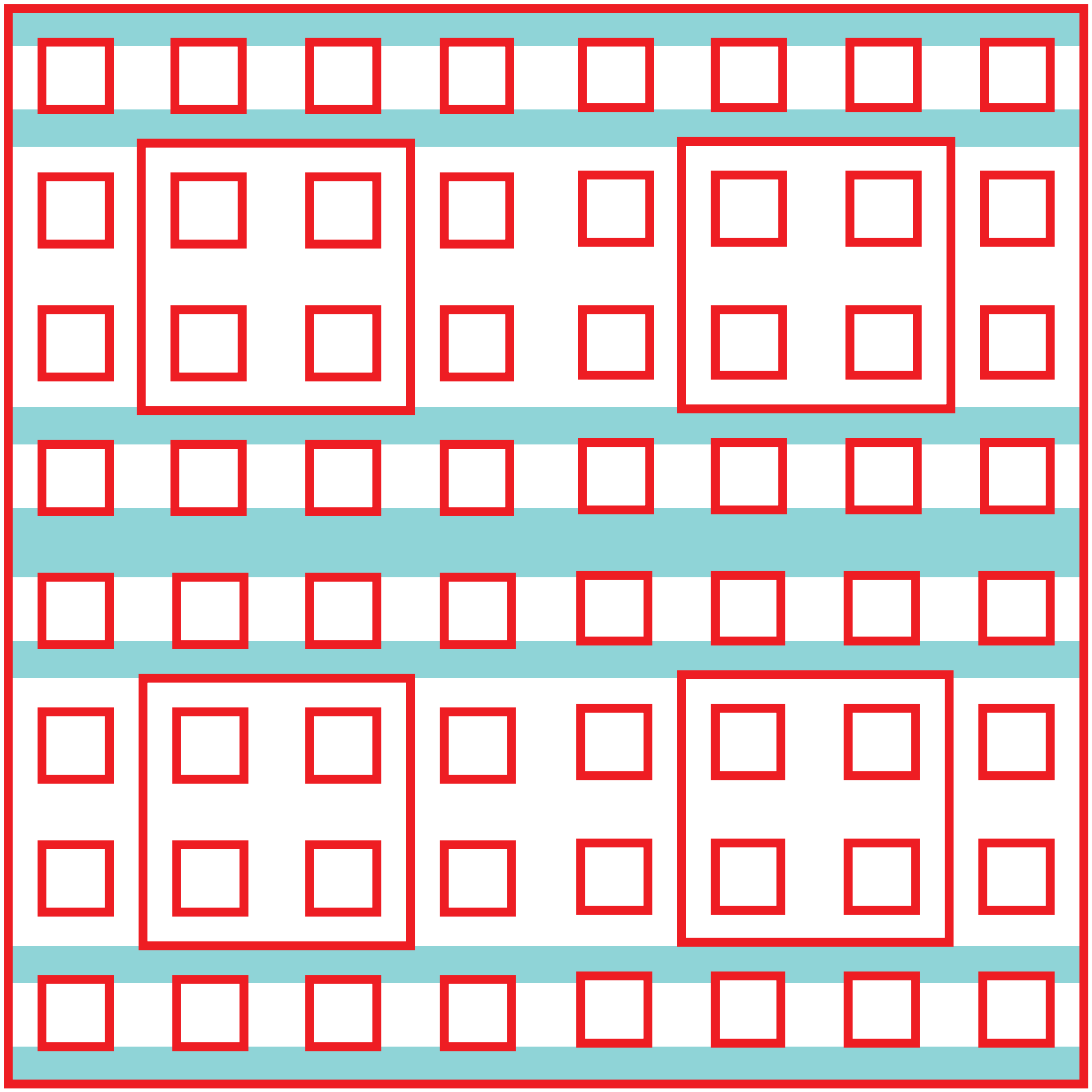
Obstruction signals are transmitted unchanged through the interior tiles of a board. Let us prove the following properties:

• Tiles of a board that are **on a free column** do not contain a vertical obstruction signal.

• Tiles of a board that are **not on a free column** contain a vertical obstruction signal.



Similar **horizontal obstruction signals** identify free horizontal rows of each board:



Based on the presence or absence of horizontal/vertical obstruction signals, tiles inside a board get classified into four classes:

(00) with horizontal and with vertical obstruction signal,

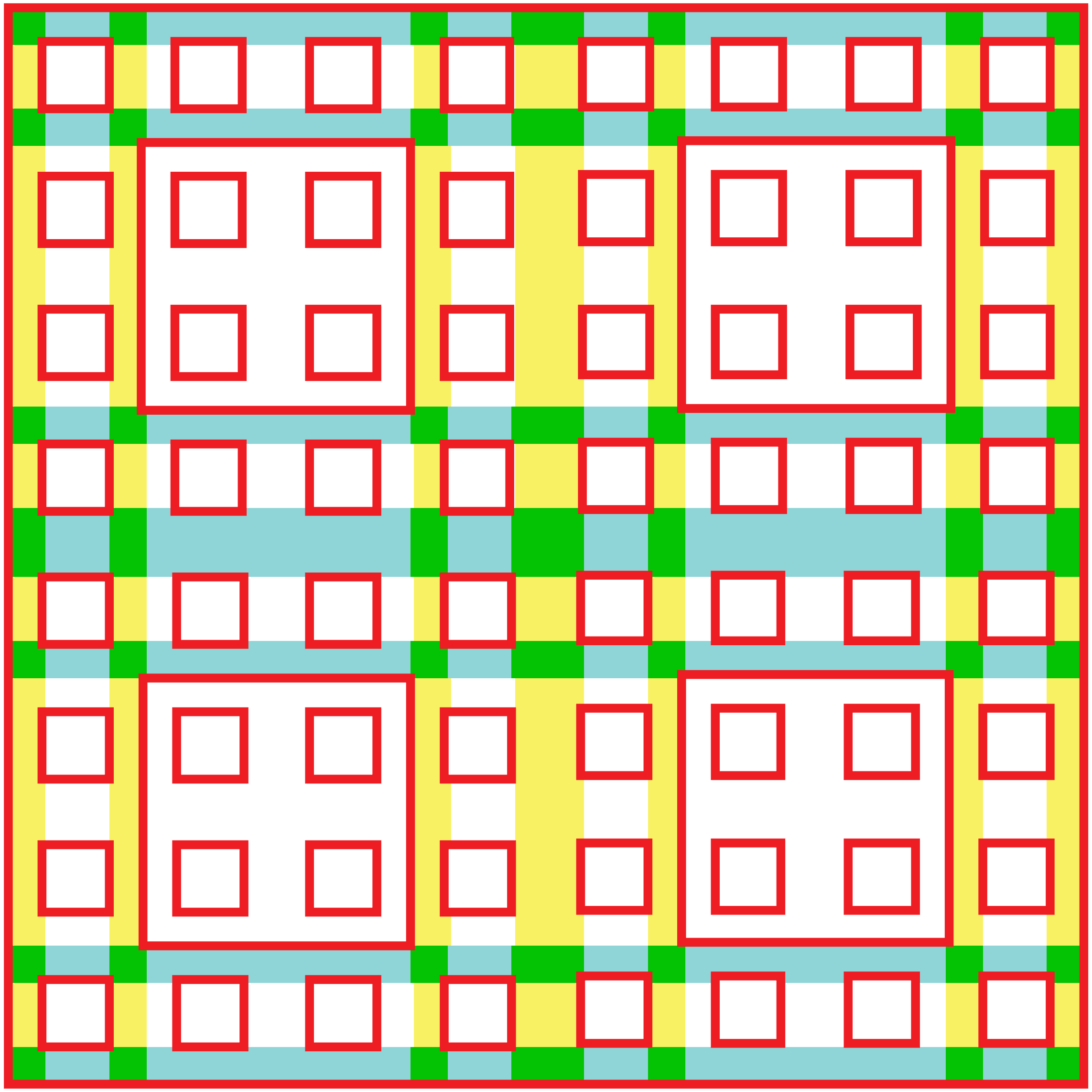
(01) with horizontal and without vertical obstruction signal,

(10) without horizontal and with vertical obstruction signal,

(11) without any kind of obstruction signal.

Each tile of board in a valid tiling knows its class within its board.

Tiles of type (11) are **free**, and they form a scattered $(2^n+1)\times(2^n+1)$ -square, whose disjoint parts are connected by tiles of types (01) and (10):



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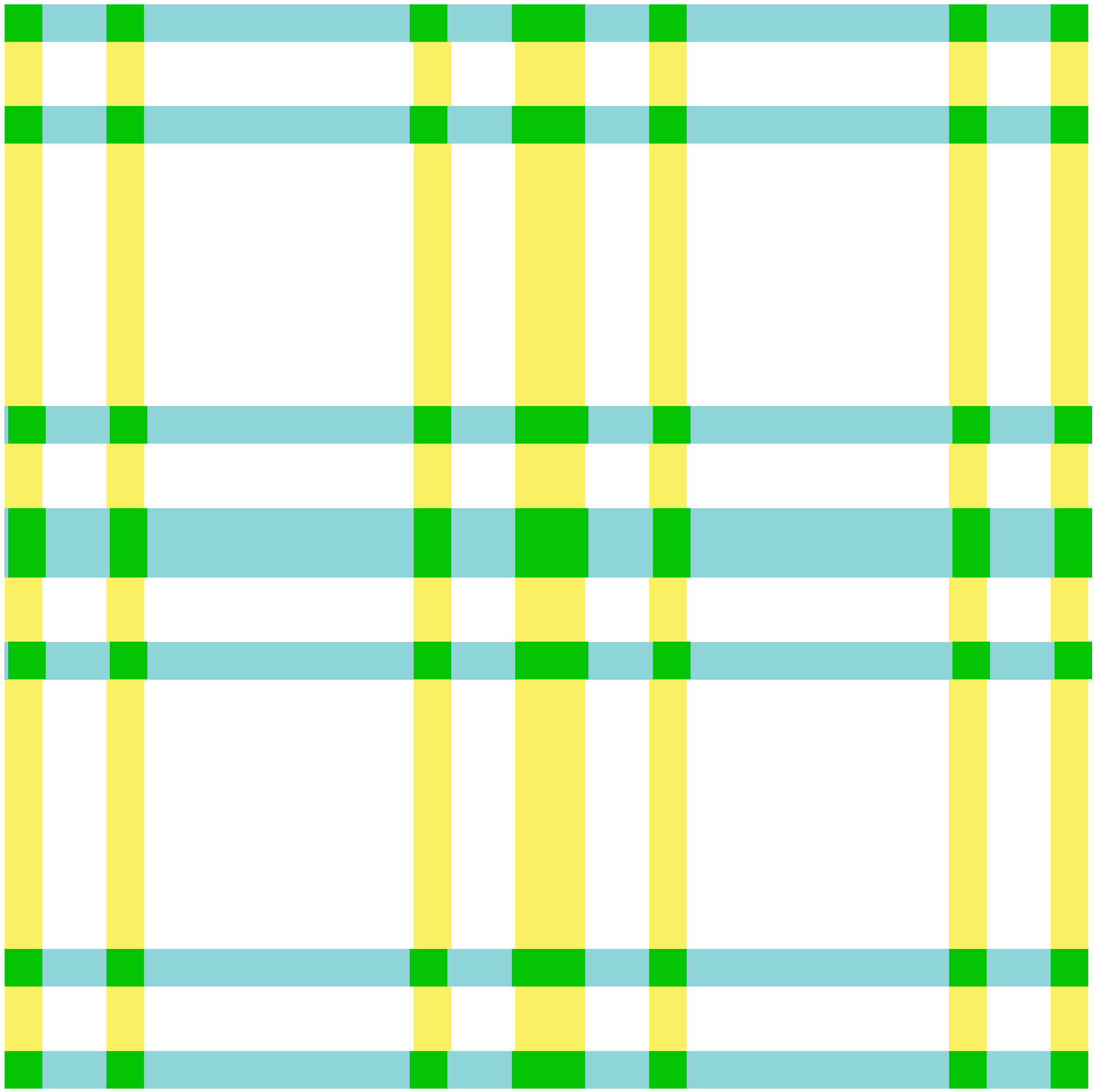


Let $\mathcal{R}$ be the fixed tile set constructed above, i.e., Robinson's tiles with

- red/green coloring of the side arrows, and
- horizontal/vertical obstruction signals.

Next we prove that **Tiling problem** is undecidable by reducing into it the **Seeded tiling problem**.

Let P be a given set of Wang tiles, and let $s \in P$ be the given seed tile. Let C be the set of colors used in P.

We construct an equivalent instance P' of the **Tiling problem** as follows: the tiles are pairs (=sandwich tiles) (r, p), whose

- first component $r \in \mathcal{R}$, and

- the second component p is a Wang tile over the color set $C \cup \{b\}$, where $b \notin C$ is a new "blank color".

The first layer tiles according to the local matching constraints of $\mathcal{R}$ described above. The second layer tiles under the color constraints, as in Wang tiles.

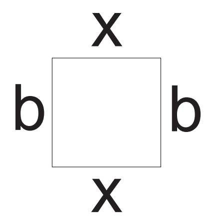
The two layers are tied together by the fact that only certain types of pairs (r, p) are allowed. We take in the new tile set P' only pairs (r, p) that satisfy the following:

(a) If r is a free tile (no obstruction signal present in r) then $p \in P$.

(=The free area simulates the tiling by P.)

The two layers are tied together by the fact that only certain types of pairs (r, p) are allowed. We take in the new tile set P' only pairs (r, p) that satisfy the following:

(b) If r is on a free column but not on a free row then p is a tile

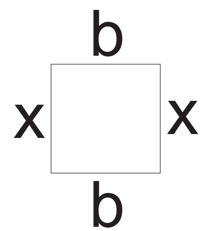


where $x \in C$ and b is the blank color.

(=These tiles transmit the color information between "neighboring" tiles in the scattered free area.)

The two layers are tied together by the fact that only certain types of pairs (r, p) are allowed. We take in the new tile set P' only pairs (r, p) that satisfy the following:

(c) If r is on a free row but not on a free column then p is a tile



where $x \in C$ and b is the blank color.

(=These tiles transmit the color information between "neighboring" tiles in the scattered free area.)

The two layers are tied together by the fact that only certain types of pairs (r, p) are allowed. We take in the new tile set P' only pairs (r, p) that satisfy the following:

(d) If r is not on a free row or column then p is arbitrary: any element of $(C \cup \{b\})^4$ is acceptable.

(=Different boards are independent of each other. Between them anything goes.)

The two layers are tied together by the fact that only certain types of pairs (r, p) are allowed. We take in the new tile set P' only pairs (r, p) that satisfy the following:

(e) If r is a parity 4 cross with green side arrows (=center tile of a board) then $p = s$, the seed tile.

(=The center of the board contains the seed tile.)

These conditions make the board behave as if the free areas were contiguous and the board then is like a connected square of size $(2^n + 1) \times (2^n + 1)$.

Let us prove that our sandwich tiles P' admit a tiling if and only if P admits a tiling that contains a copy of the seed tile s:

$\Longleftarrow$ Suppose first that P admits a tiling that contains s.

Let us prove that our sandwich tiles P' admit a tiling if and only if P admits a tiling that contains a copy of the seed tile s:

$\Longrightarrow$ For the converse direction, suppose that the sandwich tiles P' admit a tiling.

We have proved:

Theorem (Berger). The **Tiling problem** is undecidable.

Note: Classification of Wang tile sets into three parts:

A: Protosets that do not admit any tilings,

B: Protosets that admit some periodic tilings,

C: Aperiodic protosets.

We know that classes A and B are semi-decidable. Because of the undecidability of the classes we must have that class C is not semi-decidable.

$A \cup B$ is semi-decidable as a union of two semi-decidable sets.

$A \cup C$ (=tile sets that do not admit a periodic tiling) is not semi-decidable.

$B \cup C$ (=tile sets that admit a tiling) is not semi-decidable.