Metric on $A^{\mathbb{Z}^2}$

Define the distance of configurations $c \neq e$ as

$$d(c, e) = 2^{-\min\{\|(i,j)\| \mid c(i,j) \neq e(i,j)\}}$$

where we use the notation

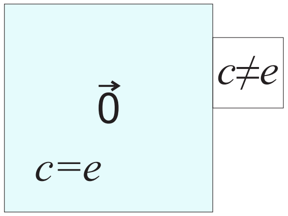
$$\|(i, j)\| = \max\{|i|, |j|\}.$$

(And for $c = e$ the distance $d(c, e) = 0$.)

This distance function is a metric on the set $A^{\mathbb{Z}^2}$.

$$d(c,e) = 2^{-\min\{\|(i,j)\| \ | \ c(i,j) \neq e(i,j)\}}$$

Two configurations c and e are close (i.e., $d(c,e)$ is small) if c and e agree on a large region around the origin.



$d(c,e)$ large if c and e differ close to $\vec{0}$.

$$d(c,e) = 2^{-\min\{\|(i,j)\| \mid c(i,j)\neq e(i,j)\}}$$

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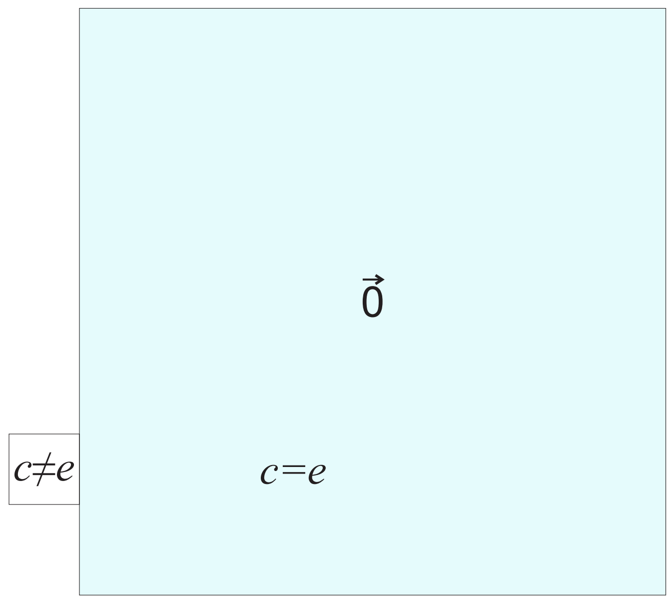
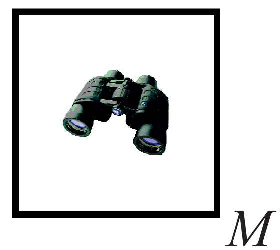


$d(c,e)$ small if c and e agree in a large region around $\vec{0}$.

Finite set $M \subseteq \mathbb{Z}^2$ is an observation window that corresponds to a "measuring device". Two configurations c and e seem identical through the measuring device if $e_{|M} = c_{|M}$.



Finite set $M \subseteq \mathbb{Z}^2$ is an observation window that corresponds to a "measuring device". Two configurations c and e seem identical through the measuring device if $e_{|M} = c_{|M}$.

M

Larger window M means better accuracy of observation.

Recall the definition of a metric space: (X, d) is a metric space if $X \neq \emptyset$ is a set and

$$d : X \times X \longrightarrow \mathbb{R}$$

is a distance function that satisfies the following three conditions:

(i) $d(x, y) > 0$ for $x \neq y$, and $d(x, y) = 0$ for $x = y$,

(ii) $d(x, y) = d(y, x)$,

(iii) $d(x, y) \leq d(x, z) + d(z, y)$.

For example: The set $X = \mathbb{R}^2$ with the usual **Euclidean metric**

$$d((x_1, y_1), (x_2, y_2)) = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$$

is a metric space.

Many essential properties of the space can be proved using the axioms (i)–(iii) only.

(i) $d(x, y) > 0$ for $x \neq y$, and $d(x, y) = 0$ for $x = y$,

(ii) $d(x, y) = d(y, x)$,

(iii) $d(x, y) \leq d(x, z) + d(z, y)$.

Let us prove that $X = A^{\mathbb{Z}^2}$ with the distance function

$$d(c, e) = 2^{-\min\{\|(i,j)\| \mid c(i,j) \neq e(i,j)\}}$$

is a metric space.

In fact: The space is an **ultrametric** as it satisfies the strong triangle inequality

(iii') $d(x, y) \leq \max\{d(x, z), d(z, y)\}$.

Let (X, d) be a metric space.

For every $\varepsilon > 0$ and $x \in X$ we denote

$$B_\varepsilon(x) = \{y \in X \mid d(x, y) < \varepsilon\}$$

and call $B_\varepsilon(x)$ the (open) **ε-ball** with center x.

A set $U \subseteq X$ is **open** if

$$\forall x \in U,\ \exists \varepsilon > 0\ :\ B_\varepsilon(x) \subseteq U.$$

A set is **closed** if its complement is open

A set is **clopen** if it is both open and closed.

$$U \text{ is open} \iff \forall x \in U,\ \exists \varepsilon > 0 \ :\ B_\varepsilon(x) \subseteq U.$$

Proposition. Let (X, d) be a metric space. Then

(i) $\emptyset$ and X are open,

(ii) arbitrary unions of open sets are open, and

(iii) intersections of <u>finitely</u> many open sets are open.

Proof.

Corollary. A set is open if and only if it is a union of open balls.

Proof.

Example. Let $X = \mathbb{R}$ and $d(x, y) = |x - y|$. This the the **usual metric** of real numbers.

- Open balls:

- Open sets:

- Closed intervals $[a, b]$ are examples of closed sets.

- Set $\mathbb{Q}$ of rational numbers is not open, not closed

- Clopen sets: $\emptyset$ and $\mathbb{R}$.

(i) $\emptyset$ and X are open,

(ii) arbitrary unions of open sets are open, and

(iii) intersections of finitely many open sets are open.

Many properties of metric spaces can be proved using properties (i), (ii) and (iii) only.

Further abstraction: A pair $(X, \mathcal{T})$ where X is a set and $\mathcal{T}$ is a family of subsets of X is a **topological space**, family $\mathcal{T}$ is called a **topology** on X, and sets in $\mathcal{T}$ are called **open** if axioms (i), (ii) and (iii) are satisfied.

Thus the family of open sets of a metric space (X, d) forms a topology on X. It is called a **metric topology**. There are also topologies that are not metrizable, i.e., not defined by any metric.

Example. For any X, let $\mathcal{T}$ contain all subsets of X. Then $\mathcal{T}$ is a topology, the **discrete topology** of X.

The discrete topology is metrizable as it is defined by the discrete metric

$$d(x, y) = \begin{cases} 1, & \text{if } x \neq y, \\ 0, & \text{if } x = y. \end{cases}$$

This metric satisfies the (strong) triangular inequality

$$d(x, y) \leq \max\{d(x, z), d(z, y)\}.$$

All singleton sets $\{x\}$ are open balls.

Example. For any set X let $\mathcal{T} = \{X, \emptyset\}$. Then $\mathcal{T}$ is a topology, the **trivial topology** of X.

If $|X| \geq 2$ then $\mathcal{T}$ is not defined by any metric:

Consistently with metric spaces we define:

A set is **closed** if its complement is open

A set is **clopen** if it is both open and closed.

By de Morgan's laws closed sets behave dually to open sets:

Proposition. Let $(X, \mathcal{T})$ be a topological space.

(i) $\emptyset$ and X are closed,

(ii) arbitrary intersections of closed sets are closed, and

(iii) unions of finitely many closed sets are closed.

Further terminology: Let $(X, \mathcal{T})$ be a top. space.

- $x \in X$ is **isolated** if $\{x\}$ is open. In the metric case:

- A space is **perfect** if it has no isolated points.

- Let $A \subseteq X$. The **closure** of A is

$$\overline{A} = \bigcap_{\substack{F closed \\ A \subseteq F}} F.$$

It is the smallest closed set that contains A:

$$F \text{ closed}, A \subseteq F \implies \overline{A} \subseteq F.$$

- Set $A \subseteq X$ is **dense** if $\overline{A} = X$.

• Dual to closure: The **interior** of A is

$$A^\circ = \bigcup_{\substack{V\, open \\ V \subseteq A}} V.$$

It is the largest open subset of A:

$$V \text{ open}, V \subseteq A \implies V \subseteq A^\circ.$$

• A set A is a **neighborhood** of point x if $x \in A^\circ$. Equivalently: there exists open U such that $x \in U \subseteq A$.

Example. Consider $\mathbb{R}$ and the **usual topology**.

• Every open ball contains infinitely many points so there are no isolated points. The space is perfect.

• The closure of $\mathbb{Q}$ is $\mathbb{R}$, so $\mathbb{Q}$ is dense in $\mathbb{R}$. The interior of $\mathbb{Q}$ is the empty set.

• The closure of $(0, 1)$ is $[0, 1]$.

• $\mathbb{Z}$ is closed, so it is its own closure.

Example. The **discrete topology** is far from perfect because every point is isolated.

Let $A \subseteq X$ and let d be a metric on X. Then d restricted to $A \times A$ is the **induced metric** on A.

Let $A \subseteq X$ and let $\mathcal{T}$ be a topology on X. Then

$$\{V \cap A \mid V \in \mathcal{T}\}$$

is a topology on A, the **induced topology**.

Let $\mathcal{T}$ be the metric topology defined by d on X. The topology that $\mathcal{T}$ induces on A is the same as the metric topology defined by the induced metric on A.

Always, when considering a subset of a topological (or metric) space, the default is that we assume the induced topology (metric) on A.

Example. The metric induced by the usual metric of $\mathbb{R}$ on subset $\mathbb{Z}$ is

$$d(n, m) = |n - m| \text{ for all } n, m \in \mathbb{Z}.$$

Then every singleton set $\{n\}$ is an open ball, and hence the induced topology on $\mathbb{Z}$ is the discrete topology. The discrete metric

$$d(n, m) = \begin{cases} 1, & \text{if } n \neq m, \\ 0, & \text{if } n = m \end{cases}$$

defines the same topology.

Convergence of sequences

A topological space $(X, \mathcal{T})$ is **Hausdorff** if for every $x \neq y$ there are open U_x and U_y such that $x \in U_x$, $y \in U_y$ and $U_x \cap U_y = \emptyset$. In other words, any two distinct points have non-intersecting neighborhoods:

Example. Every metric space is Hausdorff: For $x \neq y$ choose

$$\varepsilon = d(x, y)/2$$

and use

$$\begin{aligned} U_x &= B_\varepsilon(x), \\ U_y &= B_\varepsilon(y). \end{aligned}$$

$$\text{Metric} \implies \text{Hausdorff} \implies \text{Topology}$$

The trivial topology $\{\emptyset, X\}$ is not Hausdorff if $|X| \geq 2$.

In a Hausdorff space the singleton sets $\{x\}$ are closed: For every $y \neq x$ there exists an open set V_y such that $x \notin V_y$. The complement of $\{x\}$ is

$$\bigcup_{y \neq x} V_y,$$

thus open as a union of open sets.

A sequence $x_1, x_2, \ldots$ **converges** to x if for every open neighborhood U of x there is $n \in \mathbb{N}$ such that $x_i \in U$ for all $i \geq n$.

In the metric setting: For every $\varepsilon > 0$ there is $n \in \mathbb{N}$ such that $d(x_i, x) < \varepsilon$ for all $i \geq n$.

Example. Under the trivial topology $\{\emptyset, X\}$ every sequence converges to every point!

Proposition. In a Hausdorff topology every converging sequence converges to a unique point.

Proof.

We denote the unique limit by $\lim_{i \to \infty} x_i$.

Base of a topology

A family $\mathcal{B} \subseteq \mathcal{T}$ is a **base** of topology $\mathcal{T}$ iff every open set is a union of some members of $\mathcal{B}$.

Example. In a metric space (X, d) open sets are precisely unions of open balls. Thus the family

$$\{B_\varepsilon(x) \mid x \in X, \varepsilon > 0\}$$

of all open balls is a base.

Proposition. A family $\mathcal{B} \subseteq \mathcal{T}$ is a base of topology $\mathcal{T}$ if and only if

$$\forall U \in \mathcal{T}, \forall x \in U, \exists B \in \mathcal{B} \ : \ x \in B \subseteq U.$$

Proof.

Compactness

Let $\mathcal{T}$ be a topology on X, and let $A \subseteq X$.

A family $\mathcal{U} \subseteq \mathcal{T}$ is called an **open cover** of A if

$$A \subseteq \bigcup_{V \in \mathcal{U}} V.$$

A subfamily $\mathcal{U}' \subseteq \mathcal{U}$ of $\mathcal{U}$ is called a **subcover** if it is also a cover of A.

Set $A \subseteq X$ is called **compact** if every open cover of A has a finite subcover of A. The topology is called compact if the whole space X is compact.

In other words: a topology is compact iff every family of open sets whose union is X has a finite subfamily whose union is X.

Example. In the usual topology of $\mathbb{R}$

$$A = \{0\} \cup \{\frac{1}{n} \mid n \in \mathbb{Z}_+\}$$

is compact:

On the other hand,

$$B = \{\frac{1}{n} \mid n \in \mathbb{Z}_+\}$$

is not compact:

Compactness of X could as well be defined using a dual concept:

Proposition. Topology of X is compact if and only if every family of closed sets whose intersection is empty has a finite subfamily whose intersection is empty.

Corollary. Let

$$F_1 \supseteq F_2 \supseteq F_3 \supseteq \dots$$

be an infinite chain of closed sets in a compact space X. If $F_i \neq \emptyset$ for all i then

$$\bigcap_{i=1}^{\infty} F_i \neq \emptyset.$$