Compactness in metric spaces is equivalent to **sequential compactness**:

Proposition. Let (X, d) be a metric space. Set $A \subseteq X$ is compact if and only if every sequence of elements of A has a subsequence that converges to an element of A

(A **subsequence** of a sequence $x_1, x_2, \ldots$ is a sequence $x_{i_1}, x_{i_2}, \ldots$ for some $i_1 < i_2 < \ldots$.)

Proof

In compact metric spaces compact sets are exactly the closed sets:

Proposition A. Let X be a **compact** topological space. For $A \subseteq X$

$$A \text{ closed} \implies A \text{ compact.}$$

Proposition B. Let X be a **Hausdorff** topological space. For $A \subseteq X$

$$A \text{ compact} \implies A \text{ closed.}$$

Proofs.

A quick note on metric balls

- **An open ball**
$$B_\varepsilon(x) = \{y \mid d(x,y) < \varepsilon\}$$
is open in the topology.
- **A closed ball**
$$\overline{B}_\varepsilon(x) = \{y \mid d(x,y) \leq \varepsilon\}$$
is closed in the topology.

But $\overline{B}_\varepsilon(x)$ is not always the closure of $B_\varepsilon(x)$. (Think of the radius-1 balls in the discrete metric!)

Countability

Proposition. A compact metric space has a countable base and a countable dense set of points.

Proof.

Perfect sets

Recall that $x \in X$ is isolated if $\{x\}$ is open. On subsets $A \subset X$ we use the induced topology, so we have that $x \in A$ is **isolated in A** if $\{x\} = A \cap U$ for some open $U \subseteq X$.

A non-empty $A \subseteq X$ is **perfect** if it is closed and has no isolated points.

Proposition. Let (X, d) be a compact metric space. Then every perfect $A \subseteq X$ is uncountable.

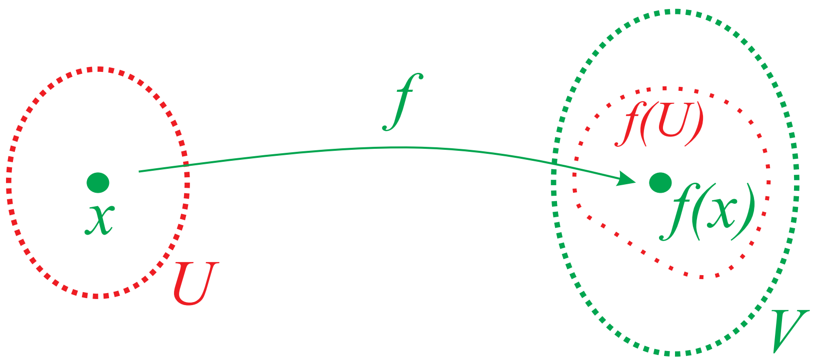
Proof.

Continuity

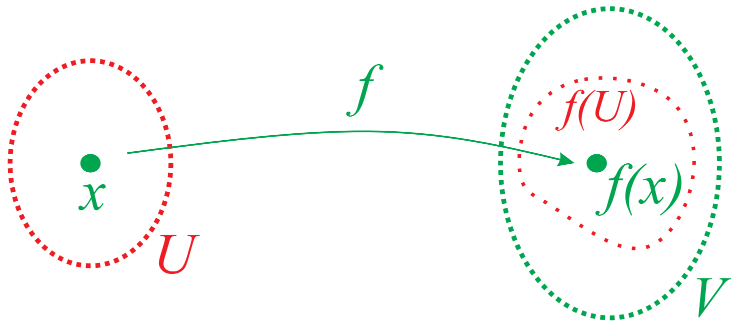
Let X, Y be topological spaces. Function $f : X \longrightarrow Y$ is **continuous at point** $x \in X$ if

$$V \subseteq Y \text{ open, } f(x) \in V \\ \implies \exists \text{ open } U \subseteq X : x \in U \text{ and } f(U) \subseteq V.$$

(For every open neighborhood V of $f(x)$ there exists an open neighborhood U of x such that $f(U) \subseteq V$.



Function $f : X \longrightarrow Y$ is **continuous** if it is continuous at every $x \in X$.



Examples.

• If X has the discrete topology the every $f : X \longrightarrow Y$ is continuous. (Choose $U = \{x\}$.)

• If Y has the trivial topology the every $f : X \longrightarrow Y$ is continuous. (Choose $U = X$: works because $V = Y$.)

• A constant function ($\forall x \in X : f(x) = a$ for some fixed $a \in Y$) is continuous. (Choose $U = X$.)

• If X has the trivial topology and Y the discrete topology then constant functions are the only continuous functions:

Proposition. The following conditions are equivalent:

(i) Function $f : X \longrightarrow Y$ is continuous,

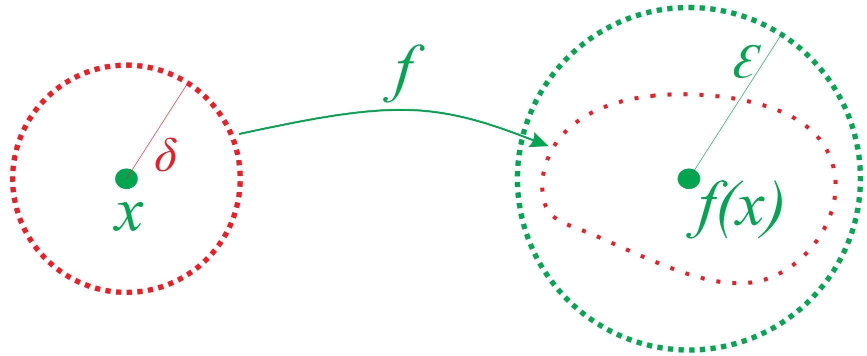
(ii) pre-image $f^{-1}(V)$ is open for each open $V \subseteq Y$,

(iii) pre-image $f^{-1}(F)$ is closed for each closed $F \subseteq Y$.

Proof.

In the metric case: f is continuous if

$$\forall \varepsilon > 0,\ \forall x \in X,\ \exists \delta > 0\ :\ f(B_\delta(x)) \subseteq B_\varepsilon(f(x)).$$



Proposition. Let $f : X \longrightarrow Y$ be continuous. For every compact A the set $f(A)$ is compact.

Proof.

Proposition. Let $f : X \longrightarrow Y$ be a continuous bijection where X is a compact and Y is a Hausdorff topological space. Then the inverse function $f^{-1} : Y \longrightarrow X$ is also continuous.

Proof.

If $f : X \longrightarrow Y$ is a bijection and both f and f^{-1} are continuous then f is a **homeomorphism** and spaces X and Y are **homeomorphic**. This is the "isomorphism" of topological structures.

Corollary. Continuous bijection between compact metric spaces is a homeomorphism.