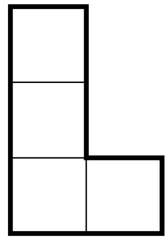
Tilings and Patterns: Homework 13 (4.12.2023)

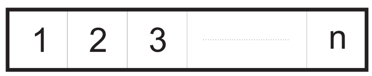
1. The L-tetrominoe is the 4-polyominoe consisting of four unit squares glued together in the shape of letter L:



   Prove that it is possible to tile a $w \times h$ rectangle by L-tetrominoes if $w, h > 1$ and eight divides the product wh.

2. Prove the converse to problem 6 above: Show that if L-tetrominoes can tile a $w \times h$ rectangle then $w, h > 1$ and eight divides wh.

3. Let a, b and n be positive integers. In this problem we consider tilings of the $a \times b$ rectangle using copies of the $1 \times n$ rectangle, i.e. the straight n-polyominoe

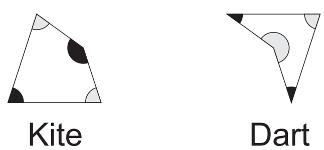


   (a) Let us color the unit cells of the $a \times b$ board by n colors $1, 2, \dots n$ so that the cell in position (i, j) gets the unique color $c \in \{1, 2, \dots, n\}$ that is congruent to $i + j$ modulo n. Prove: if $0 < a < n$ and $0 < b < n$ then for some c_1 and c_2 there are more cells colored with c_1 than there are cells colored with c_2.

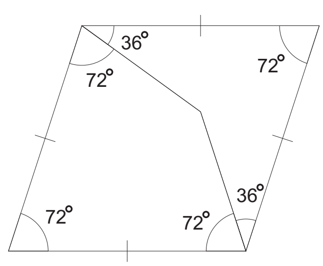
   (b) Prove: a tiling of the $a \times b$ rectangle by the straight n-polyominoe is possible if and only if a or b is divisible by n.

4. Suppose that an $A \times B$ rectangle (where A and B are real numbers) is exactly filled with rectangular tiles, where the tiles may have different sizes, but each tile has at least one side whose length is an integer. Prove that A or B must be an integer. (Hint: Use checkerboard coloring where cells are of sizes $\frac{1}{2} \times \frac{1}{2}$.)

In the following three problems we consider the two quadrilateral tiles known as Penrose's kite and dart:

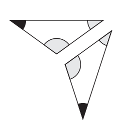


The two shapes are obtained by cutting a rhombus into two parts, and coloring the vertices with two colors (dark and light):

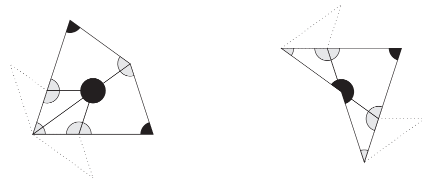


In addition to the geometric tiling requirement (=tiles cover the plane without overlaps) we also impose the constraint that tilings by kites and darts are edge-to-edge, and the meeting vertices of neighboring tiles must have the same color.

We sometimes also refer to half-darts, the triangular tiles obtained by cutting the dart along its diagonal:



The following two patches of tiles (and any rotated versions of them) are called macro kites and macro darts, respectively:



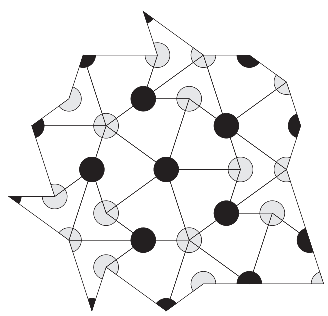
Macro-kite Macro-dart

The macro kite consists of two kites and two half-darts, and the macro dart consists of one kite and two half-darts. (The missing halves of the darts are illustrated above in dashed lines.)

5. (a) Prove that if the shorter edges of a kite or a dart have length 1, then the length of the longer edges is the golden ratio $\varphi = (1 + \sqrt{5})/2$, i.e., the positive solution of the equation

$$\varphi^2 = \varphi + 1.$$

   (b) Identify all macro kites and macro darts in the following patch of a tiling:



6. Suppose we have a valid tiling t of the plane by kites and darts.
   (a) Prove that every kite and every half-dart belongs into a unique macro kite or macro dart.
   (b) Prove that if we replace each macro kite (and macro dart) with a kite (dart, respectively) and shrink the tiling by a factor φ, we obtain a valid tiling t' of the plane by kites and darts.
   (c) Prove that if t is symmetric under the translation by a vector $\vec{x}$, then t' is symmetric under the translation by the vector $\frac{1}{\varphi}\vec{x}$.
   (d) Prove that kites and darts do not admit any tiling with a non-trivial translational symmetry.

7. In the following we consider correctly tiled finite patches of tiles. We define the following operation on a patch, executed in four phases:

   (i) Re-scale the patch by the factor φ, that is, apply the transformation $\vec{x} \mapsto \varphi\vec{x}$ of the plane.
   (ii) Replace all re-scaled kites and darts by macro kites and macro darts, respectively.
   (iii) If two neighboring half-darts together form a full dart, merge them into a dart.
   (iv) Remove all remaining (non-merged) half-darts.

   (a) Prove that the operation defined above provides a correctly tiled patch of kites and darts, if applied to a correctly tile patch.
   (b) Prove that there exist two valid tilings with a five-fold rotational symmetry. Hint: do the substitution operation above repeatedly, starting from the following initial patch. What happens in the first two iterations?