Subshifts

A set $\Sigma \subseteq A^{\mathbb{Z}^2}$ is **translation invariant** if

$$\tau(\Sigma) = \Sigma$$

for every $\tau \in \mathbb{T}$. It is enough to verify that $\sigma_e(\Sigma) = \Sigma$ and $\sigma_n(\Sigma) = \Sigma$.

A topologically closed, translation invariant set is a (two-dimensional) **subshift**. The entire configuration space $A^{\mathbb{Z}^2}$ is also called the (two-dimensional) **full shift** over A.

The system $(\Sigma, \mathbb{T})$ is also a dynamical system, a **subsystem** of $(A^{\mathbb{Z}^2}, \mathbb{T})$.

A finite **pattern** over A is a pair (D, p) where $D \subseteq \mathbb{Z}^2$ is finite, the domain of the pattern, and $p : D \longrightarrow A$.

(We often omit D from the notation and talk about a pattern $p \in A^D$, the domain D being implicitly assumed.)

Let us denote by $P(A)$ the set of **all finite patterns** over A. Clearly $P(A)$ is countable as the number of finite subsets of $\mathbb{Z}^2$ is countable.

Pattern $p \in A^D$ is a **subpattern** of a configuration $c \in A^{\mathbb{Z}^2}$ if $c|_D = p$.

Configurations that have p as a subpattern form the cylinder

$$[p] = \{c \in A^{\mathbb{Z}^2} \mid c|_D = p\}.$$

This is of course the same cylinder as $\mathrm{Cyl}(c, D)$ for any configuration c in the cylinder.

We say that the pattern $p \in A^D$ **appears** in c if p is a subpattern of $\tau(c)$ for some translation $\tau \in \mathbb{T}$:

For any configuration c let $\mathbf{Patt}(c)$ be the set of all finite patterns that appear in c:

$$\mathrm{Patt}(c) = \{p \in A^D \mid \exists \tau \in \mathbb{T} \ : \ \tau(c)|_D = p\}.$$

For any set $S \subseteq A^{\mathbb{Z}^2}$ of configurations we denote

$$\mathrm{Patt}(S) = \bigcup_{c \in S} \mathrm{Patt}(c)$$

for the set of finite patterns that appear in some elements of S.

For any set P of finite patterns we define the set

$$\Sigma(P) = \{c \in A^{\mathbb{Z}^2} \mid \mathrm{Patt}(c) \cap P = \emptyset\}$$

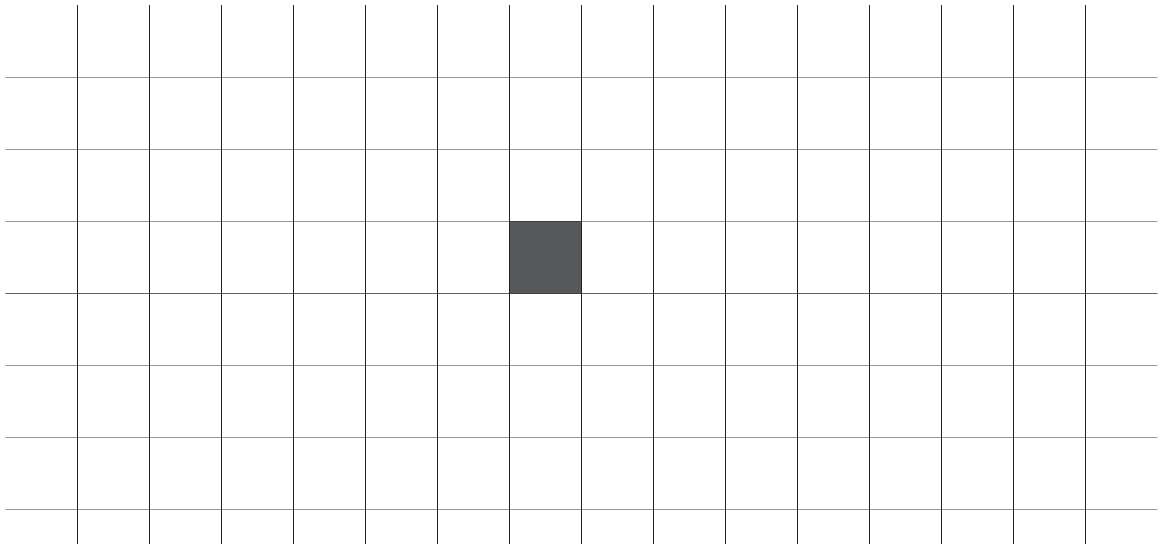
of configurations in which no element of P appears.

$$\Sigma(P) = \{c \in A^{\mathbb{Z}^2} \mid \mathrm{Patt}(c) \cap P = \emptyset\}$$

Theorem. Σ is a subshift if and only if $\Sigma = \Sigma(P)$ for some set P of finite patterns.

Proof.

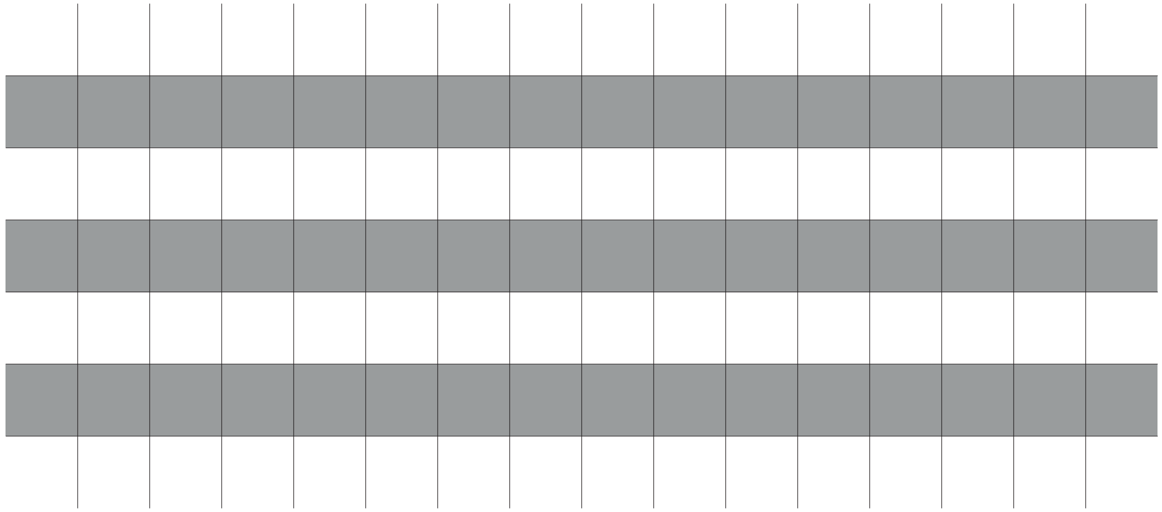
Example. The set consisting of the configrations with a single black cell on a white background, and the all white configuration:



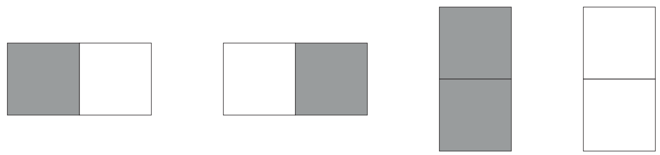
This is translation invariant and topologically closed. So it is a subshift (the **sunny-side-up** subshift).

It is defined by forbidding all patterns $p \in \{0,1\}^D$ where $|D| = 2$ and $p(i,j) = 1$ for both $(i,j) \in D$.

Example. The set of the two configurations where black and white horizontal rows alternate:

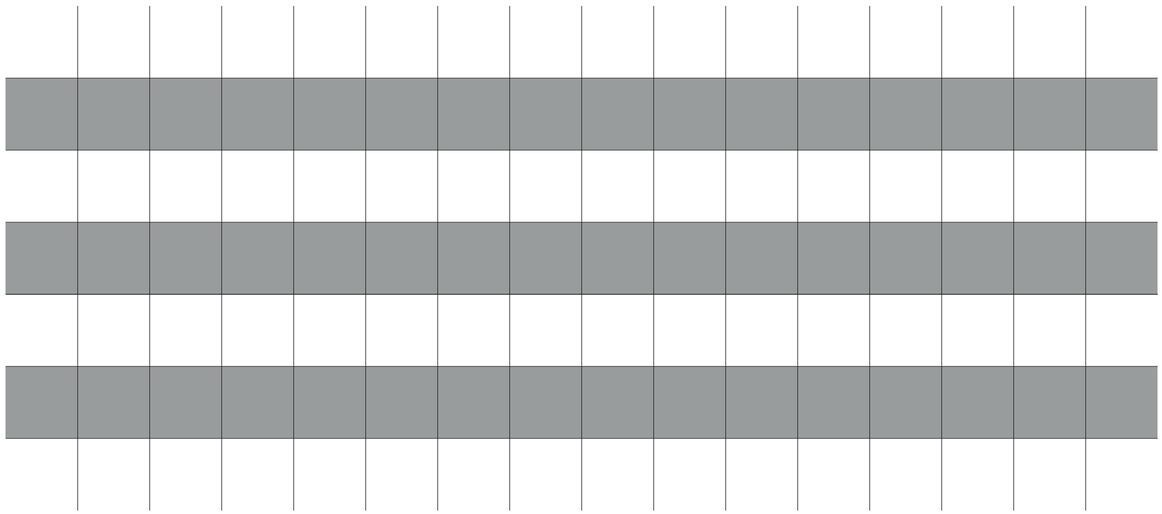


This is the subshift $\Sigma(P)$ determined by forbidding the patterns

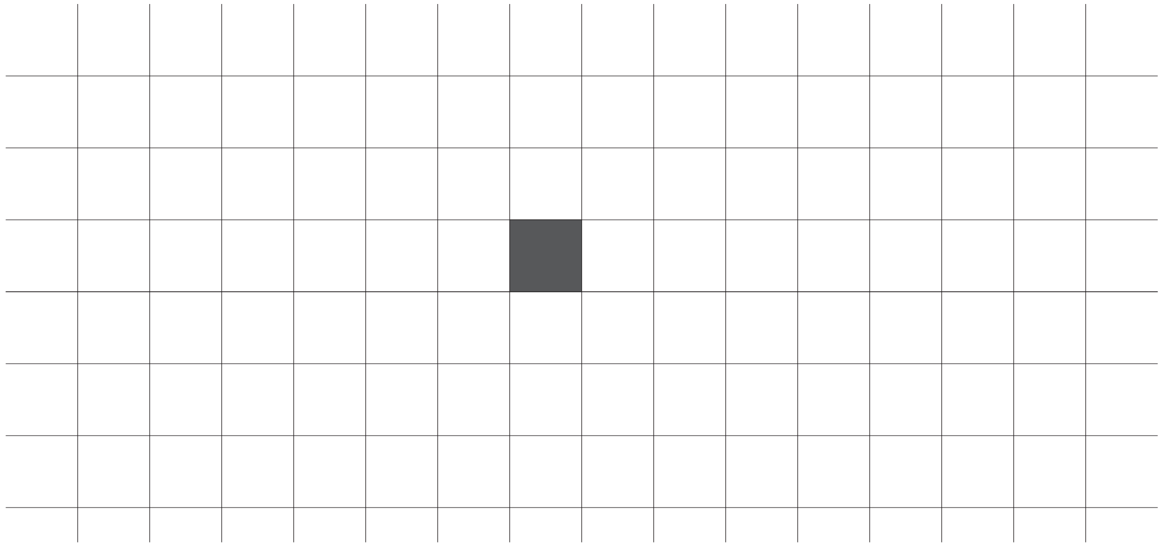


Subshifts $\Sigma(P)$ for finite P are called **subshifts of finite type (SFT)**. So a SFT can be specified by giving a finite collection P of forbidden patterns.

Example. The alternating stripes subshift is an SFT.



Example. The sunny-side-up subshift is **not** an SFT.



The set $V(T)$ of valid tilings by a Wang tile set T is an SFT.

Conversely, we have seen a method of turning any SFT Σ into an "equivalent" set T of Wang tiles. In this construction the Wang tiling corresponding to a configuration $c \in \Sigma$ is obtained by **sliding an $n \times n$ window** across c and reading the patterns (which are the tiles) inside the window.

The "sliding window" function

$$h : \Sigma \longrightarrow V(T)$$

is a continuous bijection. It also commutes with all translations:

$$h \circ \tau = \tau \circ h$$

for all $\tau \in \mathbb{T}$.

A translation commuting continuous bijection is called a **conjugacy** between the dynamical systems $(\Sigma, \mathbb{T})$ and $(V(T), \mathbb{T})$. Conjugacies are the isomorphisms between dynamical systems. Conjugate dynamical systems are "the same".