Orbit closure

For any $c \in A^{\mathbb{Z}^2}$ the set

$$\mathcal{O}(c) = \{\tau(c) \mid \tau \in \mathbb{T}\}$$

is the **orbit** of c. The set $\mathcal{O}(c)$ is translation invariant.

The **orbit closure**

$$\overline{\mathcal{O}(c)}$$

of c is the topological closure of the orbit.

Lemma. The orbit closure $\overline{\mathcal{O}(c)}$ is a subshift, for every $c \in A^{\mathbb{Z}^2}$.

Proof.

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Lemma. The orbit closure $\overline{\mathcal{O}(c)}$ is a subshift, for every $c \in A^{\mathbb{Z}^2}$.

Proof. We only need to prove that the orbit closure is translation invariant:

Let $e \in \overline{\mathcal{O}(c)}$ and $\tau \in \mathbb{T}$. For any open neighborhood U of $\tau(e)$, the open set $\tau^{-1}(U)$ is an open neighborhood of e. Hence $\tau^{-1}(U) \cap \mathcal{O}(c) \neq \emptyset$, so that $U \cap \mathcal{O}(c) \neq \emptyset$. This means that $\tau(e) \in \overline{\mathcal{O}(c)}$.

The orbit closure $\overline{\mathcal{O}(c)}$ is the subshift **generated** by c: it is the intersection of all subshifts that contain c.

Indeed, if Σ is a subshift and $c \in \Sigma$ then

$\mathcal{O}(c) \subseteq \Sigma$ (because Σ is translation invariant) and

$\overline{\mathcal{O}(c)} \subseteq \Sigma$ (because Σ is closed).

Remark: If $P_1 \subseteq P_2$ then $\Sigma(P_2) \subseteq \Sigma(P_1)$.

Follows directly from the definition

$$\Sigma(P) = \{c \in A^{\mathbb{Z}^2} \mid \mathrm{Patt}(c) \cap P = \emptyset\}$$

Let P be the complement of $\text{Patt}(c)$, that is, the set of all finite patters that do not appear in c:

$$P = \{p \mid p \notin \text{Patt}(c)\}.$$

Then $\overline{\mathcal{O}(c)} = \Sigma(P)$.

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Then $\overline{\mathcal{O}(c)} = \Sigma(P)$.

Proof. $\overline{\mathcal{O}(c)} = \Sigma(P')$ for some P'.

- As $\Sigma(P)$ is a subshift containing c, we have $\overline{\mathcal{O}(c)} \subseteq \Sigma(P)$.
- As $P' \subseteq P$, by the remark, $\Sigma(P) \subseteq \Sigma(P') = \overline{\mathcal{O}(c)}$.

$$\overline{\mathcal{O}(c)} = \Sigma(P) \text{ for } P = \{p \mid p \notin \mathrm{Patt}(c)\}$$

Lemma. $e \in \overline{\mathcal{O}(c)}$ if and only if $\mathrm{Patt}(e) \subseteq \mathrm{Patt}(c)$.

Proof.

$$\overline{\mathcal{O}(c)} = \Sigma(P) \text{ for } P = \{p \mid p \notin \mathrm{Patt}(c)\}$$

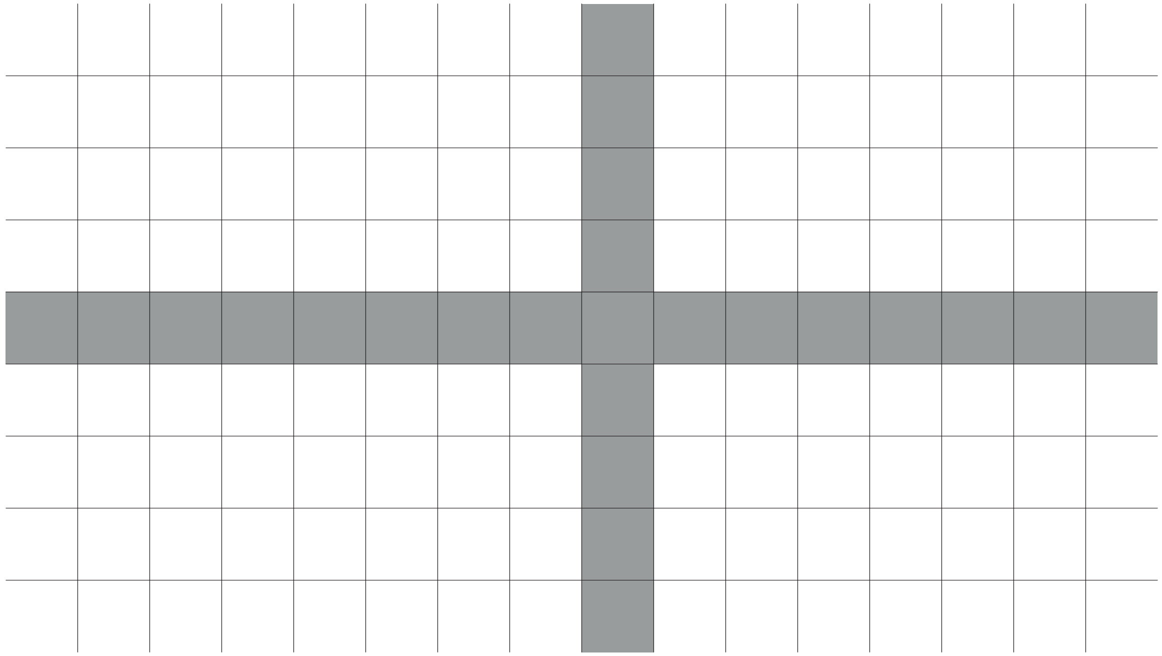
Lemma. $e \in \overline{\mathcal{O}(c)}$ if and only if $\mathrm{Patt}(e) \subseteq \mathrm{Patt}(c)$.

Proof.

$$e \in \overline{\mathcal{O}(c)} = \Sigma(P) \iff \mathrm{Patt}(e) \cap P = \emptyset \iff \mathrm{Patt}(e) \subseteq \mathrm{Patt}(c).$$

Example. Let $A = \{0, 1\}$ and let $c \in A^{\mathbb{Z}^2}$ be the infinite cross:

- $c(i, 0) = c(0, i) = 1$ for all $i \in \mathbb{Z}$, and
- $c(i, j) = 0$ if $i, j \neq 0$.



The orbit closure $\overline{\mathcal{O}(c)}$ contains the following configurations:

Transitivity

A non-empty subshift Σ is called **transitive** if for every $p_1, p_2 \in \mathrm{Patt}(\Sigma)$ there exists $c \in \Sigma$ such that

$$p_1, p_2 \in \mathrm{Patt}(c).$$

In other words, any two patterns that appear in some elements of Σ, appear in the same element of Σ.

Transitive subshifts are exactly the orbit closures of configurations:

Theorem. Subshift Σ is transitive if and only if

$$\Sigma = \overline{\mathcal{O}(c)}$$

for some $c \in \Sigma$.

Proof.

An element $c \in \Sigma$ is **transitive** in Σ if $\Sigma = \overline{\mathcal{O}(c)}$.

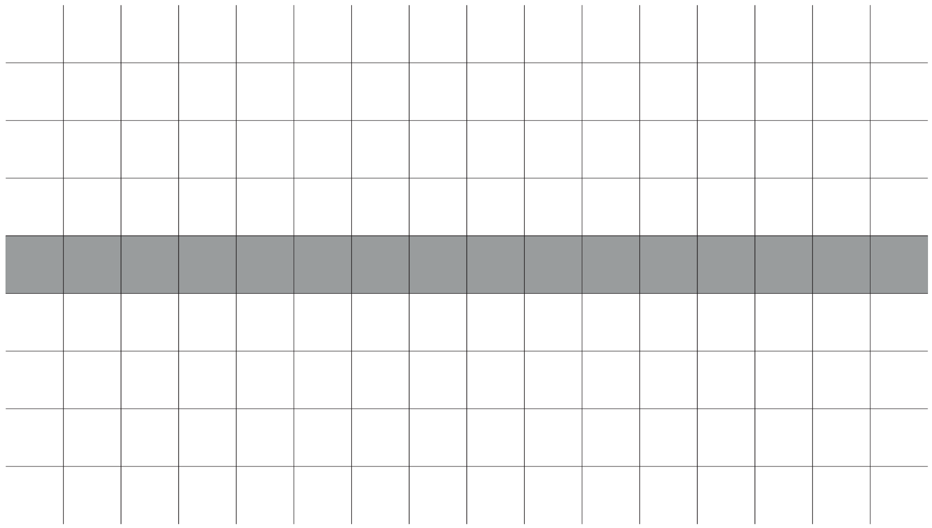
In other words, a transitive configuration contains all finite patterns that appear in any configuration of the subshift.

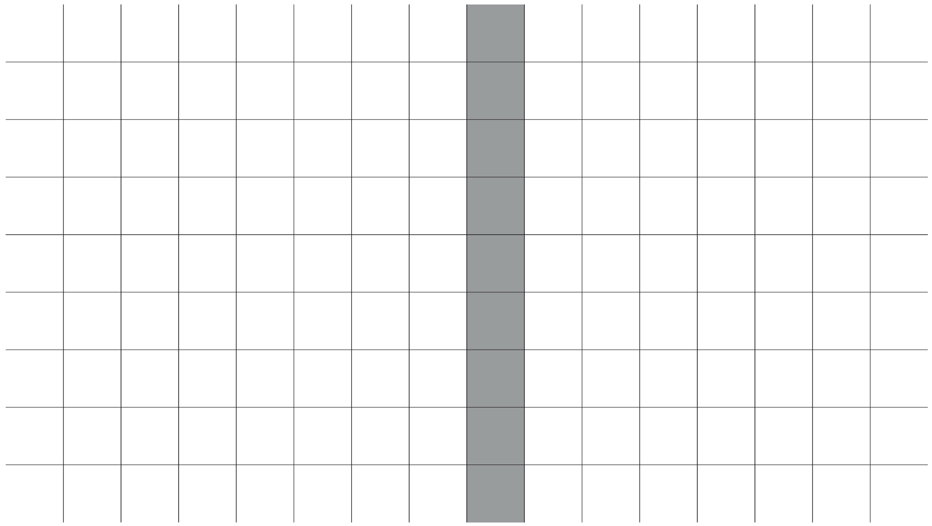
Corollary. A subshift has a transitive element if and only if the subshift is transitive.

Example. The infinite cross is transitive in its orbit closure. The orbit closure contains a non-transitive subset

$$\Sigma = \overline{\mathcal{O}(c_v)} \cup \overline{\mathcal{O}(c_h)}$$

generated by the horizontal and vertical black rows.





Minimality

A non-empty subshift Σ is called **minimal** if the only subshifts contained in Σ are $\emptyset$ and Σ.

By the following theorem Σ is minimal if and only if the orbits of all its elements are dense in Σ:

Theorem. Let Σ be a non-empty subshift. The following are equivalent:

(i) Σ is minimal.

(ii) All elements of Σ are transitive in Σ.

(iii) $\mathrm{Patt}(e) = \mathrm{Patt}(c)$ for all $e, c \in \Sigma$.

Proof.

Theorem. Every non-empty subshift Σ has a subset that is a minimal subshift.

Proof.