Periodicity and recurrence properties

Recall: A configuration $c \in A^{\mathbb{Z}^2}$ is (one-way) **periodic** if there exists $\vec{n} \in \mathbb{Z}^2 \setminus \vec{0}$ such that $c = \tau_{\vec{n}}(c)$.

It is **strongly** (or two-way) **periodic** if it is periodic with two linearly independent periods $\vec{n}_1$ and $\vec{n}_2$. A strongly periodic configuration is always periodic with horizontal and vertical periods $(0, n)$ and $(n, 0)$ for some $n > 0$.

If a **subshift of finite type** contains a one-way periodic element then it contains a strongly periodic element as well.

(This was show for Wang tilings. All SFT are conjugate to Wang tilings. Conjugacy preserves periods.)

Remark. The orbit $\mathcal{O}(c)$ of c is finite if and only if c is strongly periodic. Also, the orbit is closed if and only if c is strongly periodic (homework).

Uniform recurrence

A configuration $c \in A^{\mathbb{Z}^2}$ is **uniformly recurrent** if for every finite pattern $p \in \mathrm{Patt}(c)$ there exists n such that p appears in c inside every $n \times n$ square.

More precisely: c is uniformly recurrent iff

$$(\forall p \in \mathrm{Patt}(c))$$
$$(\exists_{\text{finite}}\ \mathbb{T}' \subseteq \mathbb{T})$$
$$(\forall \tau \in \mathbb{T})$$
$$(\exists \tau' \in \mathbb{T}')$$
$$\tau'(\tau(c)) \in [p].$$

Uniformly recurrent configurations generate minimal subshifts:

Theorem. Subshift $\overline{\mathcal{O}(c)}$ is minimal if and only if c is uniformly recurrent.

Proof.

Thus all configurations of a minimal subshift are uniformly recurrent.

There are two possibilities in this case:

- either all of them are strongly periodic, in which case the subshift is their finite orbit,
- or all elements are non-periodic (but uniformly recurrent) configurations that contain exactly the same finite patterns.

In the second case the subshift contains an uncountable number of elements:

Theorem. A minimal subshift is either finite or uncountably infinite.

Proof.

A summary, in terms of Wang tilings:

Corollary. If a tile set admits a valid tiling then it admits a uniformly recurrent tiling. If it admits a uniformly recurrent tiling that is not strongly periodic, then it admits uncountably many different tilings. In particular, every aperiodic tile set admits uncountably many valid, uniformly recurrent tilings.