A weaker recurrence property

A configuration is **recurrent** if every finite pattern that appears in c appears more than once in c.

More precisely: c is recurrent iff for all finite patterns p:

$$(\forall p \in \mathrm{Patt}(c))\ (\exists \vec{n} \neq \vec{m}) \qquad \tau_{\vec{n}}(c), \tau_{\vec{m}}(c) \in [p].$$

Then, in fact, patterns that appear in c appear infinitely many times in c:

Lemma. Configuration c is recurrent if and only if for every $p \in \mathrm{Patt}(c)$ there are infinitely many translations $\tau \in \mathbb{T}$ such that $\tau(c) \in [p]$.

Proof.

Two stronger recurrence properties

Configuration c is **quasi-periodic** if every occurrence of a finite pattern in c is part of a two-way periodic repetition of the pattern (but the period may be different for different patterns).

More precisely: for every finite $D \subseteq \mathbb{Z}^2$ there exist linearly independent $\vec{a}, \vec{b} \in \mathbb{Z}^2$ such that

$$(\forall i, j \in \mathbb{Z}) \qquad \tau_{i\vec{a}+j\vec{b}}(c)|_D = c|_D.$$

Configuration c is **isochronous** if every finite pattern of c appears two-way periodically in c.

More precisely: for every finite $D \subseteq \mathbb{Z}^2$ there exists an offset vector $\vec{c} \in \mathbb{Z}^2$ and two linearly independent $\vec{a}, \vec{b} \in \mathbb{Z}^2$ such that

$$(\forall i, j \in \mathbb{Z}) \qquad \tau_{i\vec{a}+j\vec{b}+\vec{c}}(c)|_D = c|_D.$$

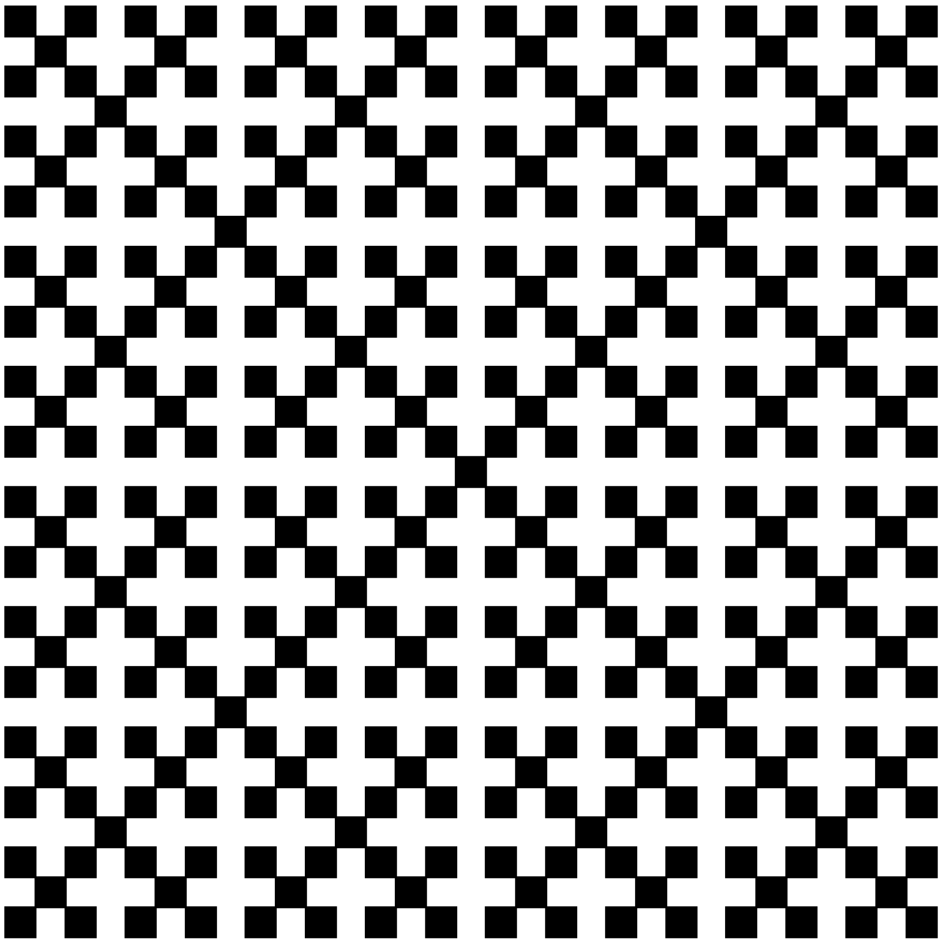
The following implications are obvious from the definitions:

$$\text{strongly periodic} \Longrightarrow \text{quasi-periodic} \Longrightarrow \text{isochronous} \\ \Longrightarrow \text{uniformly recurrent} \Longrightarrow \text{recurrent}$$

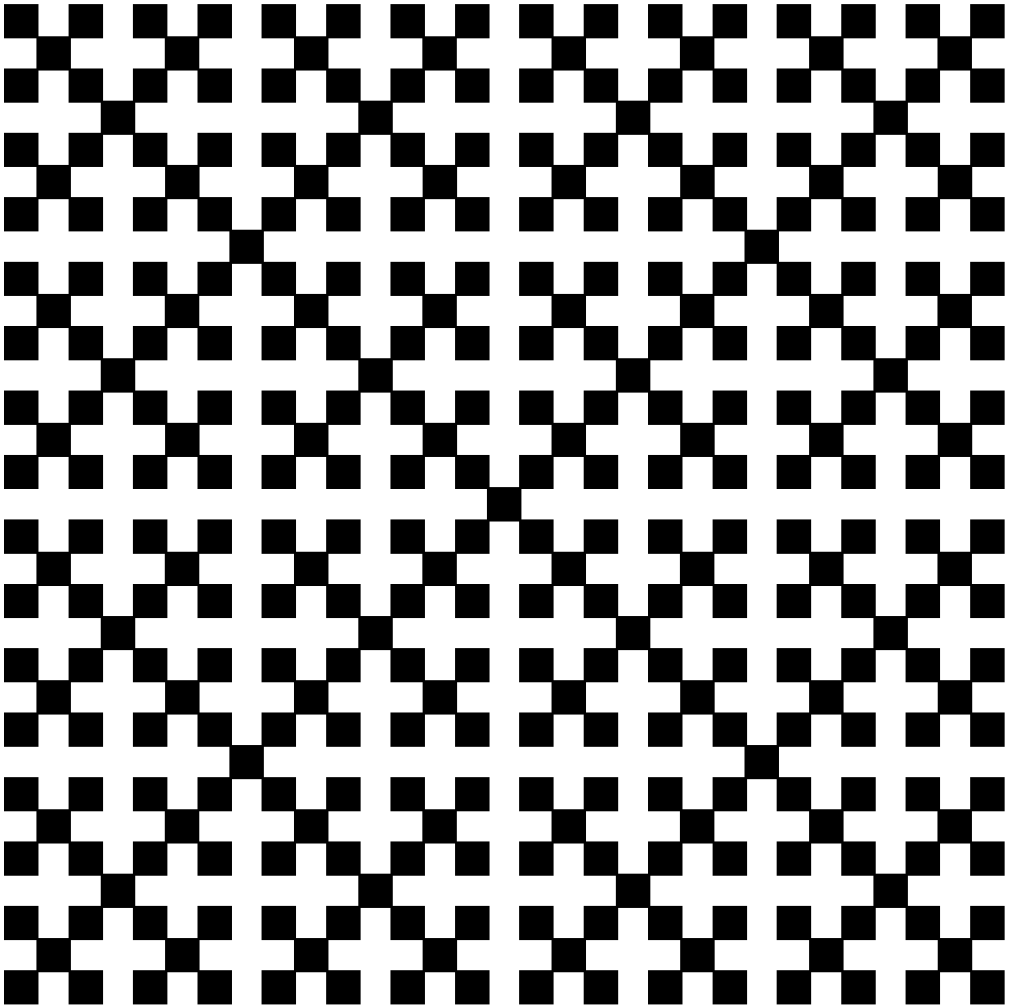
Example. Let $\deg_2(n)$ denote the largest power of 2 that divides integer n. (And set $\deg_2(0) = \infty$.)

Define configuration $c \in \{0,1\}^{\mathbb{Z}^2}$ as follows:

$$c(i,j) = 1 \text{ if and only if } \deg_2(i) = \deg_2(j).$$



$$c(i,j) = 1 \text{ if and only if } \deg_2(i) = \deg_2(j).$$



(Black squares are as the crosses in a tiling by Robinson's tile set.)
This configuration is **isochronous** but **not quasi-periodic**.

Isolated points

Configuration $c \in \Sigma$ is **isolated** in Σ if for some finite pattern p

$$[p] \cap \Sigma = \{c\}.$$

Lemma. Let Σ be a subshift. All $c \in \Sigma$ are isolated in Σ if and only if Σ is finite.

Proof.

Finite subshifts are exactly the ones whose elements are all two-way periodic:

Theorem. A subshift Σ is finite if and only if every $c \in \Sigma$ is two-way periodic.

Proof.

Sensitivity concepts from topolgical dynamics

Configuration $c \in \Sigma$ is an **equicontinuity point** if

$$(\forall \varepsilon > 0)(\exists \delta > 0)(\forall e \in \Sigma)(\forall \tau \in \mathbb{T})$$

$$d(c, e) < \delta \Longrightarrow d(\tau(c), \tau(e)) < \varepsilon.$$

In other words, if e is chosen sufficiently close to c then all translates $\tau(e)$ and $\tau(c)$ are close to each other.

Lemma. c is an equicontinuity point in Σ if and only if it is isolated in Σ.

Proof.

The subshift Σ is called **equicontinuous** if all $c \in \Sigma$ are equicontinuity points.

Corollary. A subshift is equicontinuous if and only if it is finite.

A subshift Σ is called **sensitive** if there exists $\varepsilon > 0$, called the **sensitivity constant**, such that

$$(\forall c \in \Sigma)(\forall \delta > 0)(\exists e \in \Sigma)(\exists \tau \in \mathbb{T})$$

$$d(c, e) < \delta \text{ and } d(\tau(c), \tau(e)) > \varepsilon.$$

In other words, arbitrarily close to each c there is another configuration e such that for a suitable translation τ the configurations $\tau(c)$ and $\tau(e)$ are not close to each other.

Lemma. Σ is sensitive if and only if it has no isolated points.

Proof.

A subshift is **expansive** if

$$(\exists \varepsilon > 0)(\forall c, e \in \Sigma)$$

$$c \neq e \Longrightarrow (\exists \tau \in \mathbb{T})\ d(\tau(c), \tau(e)) > \varepsilon.$$

Lemma. All subshifts are expansive.

Proof.

Theorem. Let Σ be a subshift.

(i) Σ is expansive.

(ii) Σ is sensitive if and only if it has no isolated points.

(ii) Σ is equicontinuous if and only all its elements are isolated, i.e., the subshift is finite.