Aperiodic geometric (polygonal) tile sets

Recall: Any Wang tile set can be converted into prototiles that are polygons (bumps and dents construction).

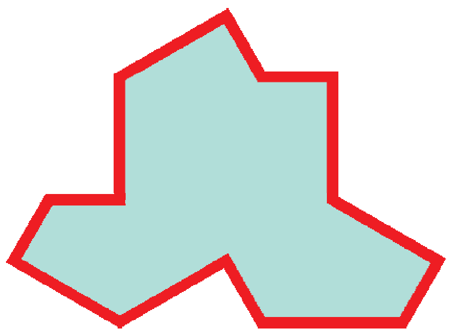
If the Wang tile set is aperiodic then the corresponding polygons admit a tiling but do not admit a periodic tiling (=tiling with a translational symmetry). We say the geometric tile set is also **aperiodic**.

In the exercises we had an example of an aperiodic set of six tiles, similar to Robinson's Wang tile set.

Theorem. There exists a protoset of polygons that admits a valid tiling but does not admit a valid tiling with a non-trivial symmetry.

The smallest aperiodic Wang protoset contains 11 tiles, but with geometric tiles a single tile is enough to force non-periodicity!

A polygonal prototile (the **hat**) was recently discovered that is alone aperiodic: there exist monohedral tilings of $\mathbb{R}^2$ using the hat but none of these tilings has a translational symmetry.



We discuss the hat in more details later in the class.

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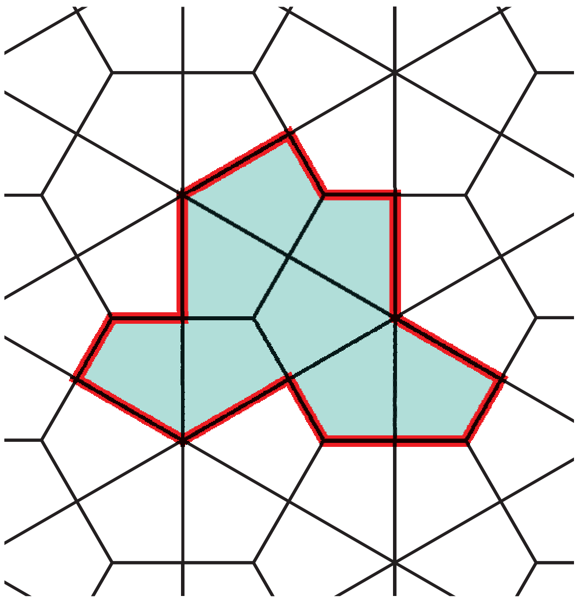

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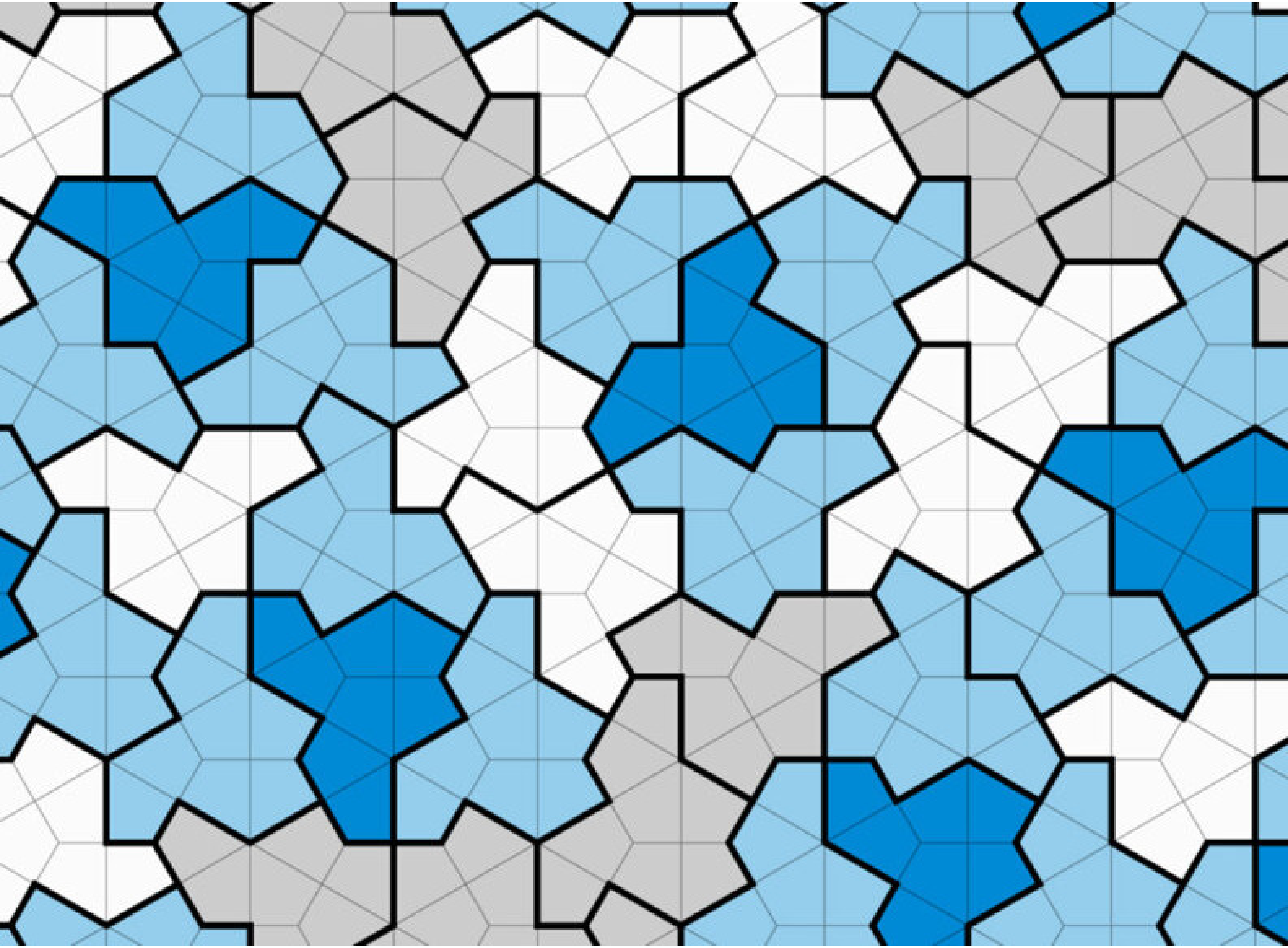
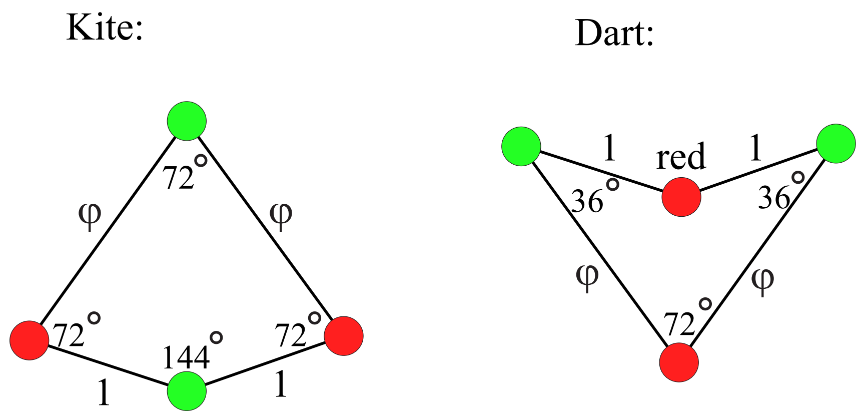


In 1974 Roger Penrose presented a famous aperiodic pair of polygonal tiles: the **kite** and the **dart**.



The Penrose tiles are obtained by cutting in two a rhombus that has a 72° angle. The resulting quadrilaterals have edges in the ratio

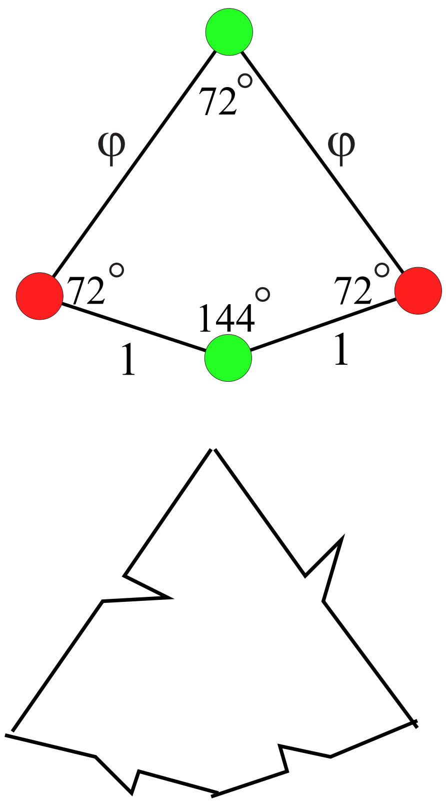
$$\varphi = (1 + \sqrt{5})/2 = 1.618\ldots,$$

the golden ratio.

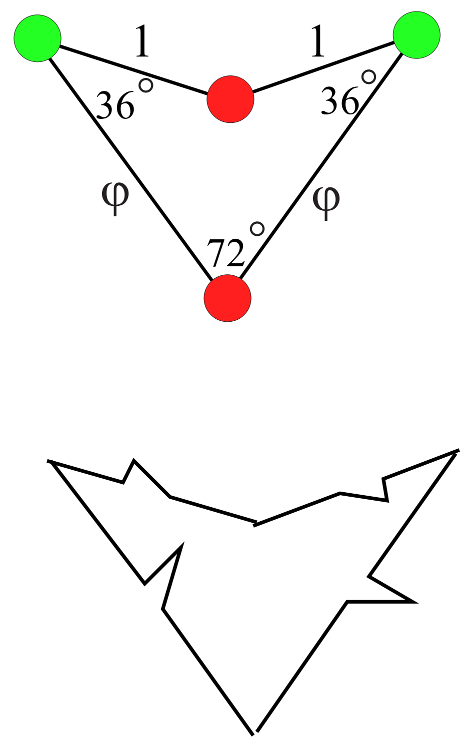
The vertices are colored red and green. Valid tilings are **edge-to-edge** and the **colors at the vertices** must match. (This condition prevents one from gluing the kite and dart back together to form the rhombus.)

These matching rules can be easily enforced using geometric constraints using bumps and dents:

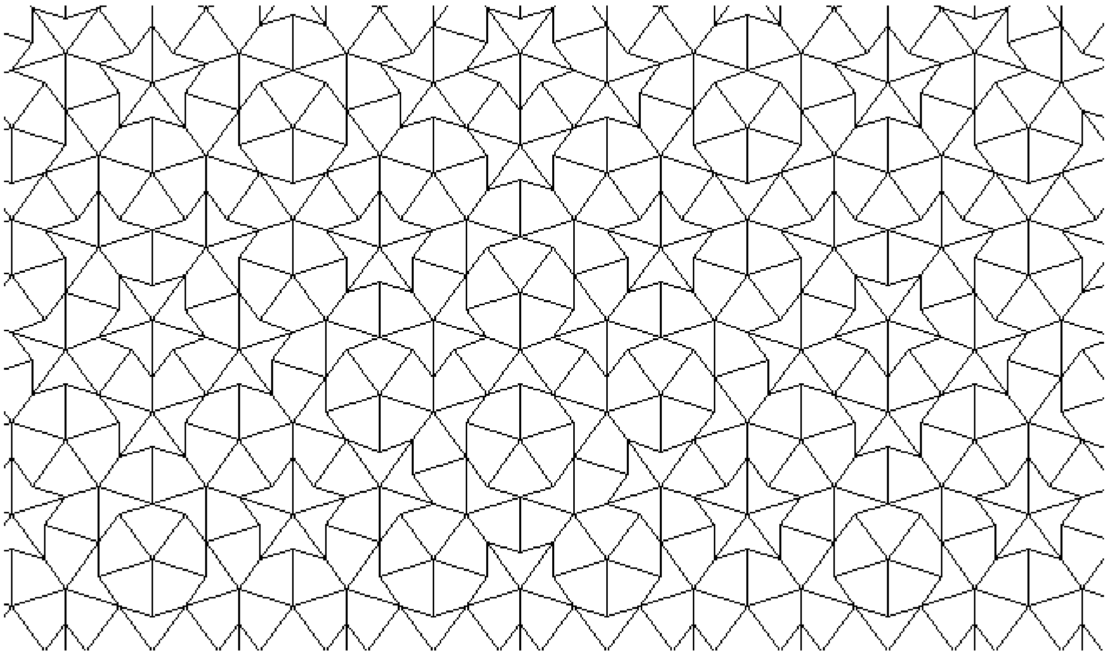
Kite:



Dart:



Here is a part of a tiling using kites and darts. (For clarity, the bumps and the dents are not shown):



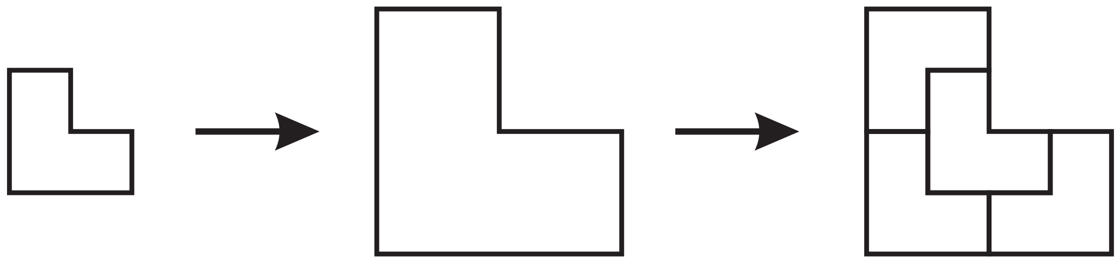
The following result is proved in the homeworks:

Theorem. Penrose kite and dart are an aperiodic pair of prototiles. They admit valid tilings with a 5-fold rotational symmetry.

Substitutions

One can generate valid tilings by Penrose kites and darts using **substitutions**, as seen at the exercises.

With substitutions one can easily generate hierarchical, non-periodic tilings. We illustrate this with a simple example, the **chair substitution**, where an ***L*-tromino** is expanded by factor two and cut into four smaller L-trominoes:



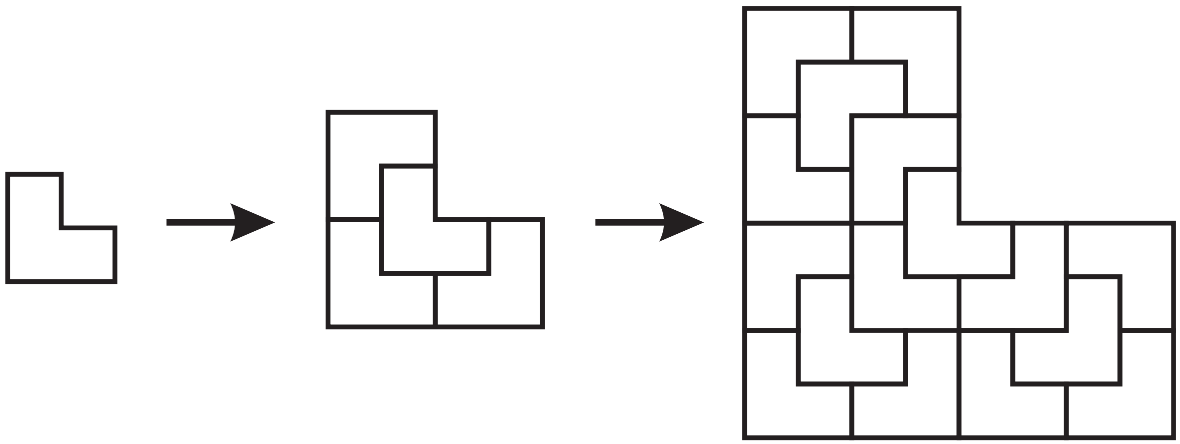
Let us call this patch of four tiles a **macro tile**.

So, starting from a single tile, we repeat the following operations:

(i) **Magnify** your patch of tiles by factor two horizontally and vertically.

(ii) **Substitute** each magnified tile by four smaller copies as above.

Here are the first two iterations:

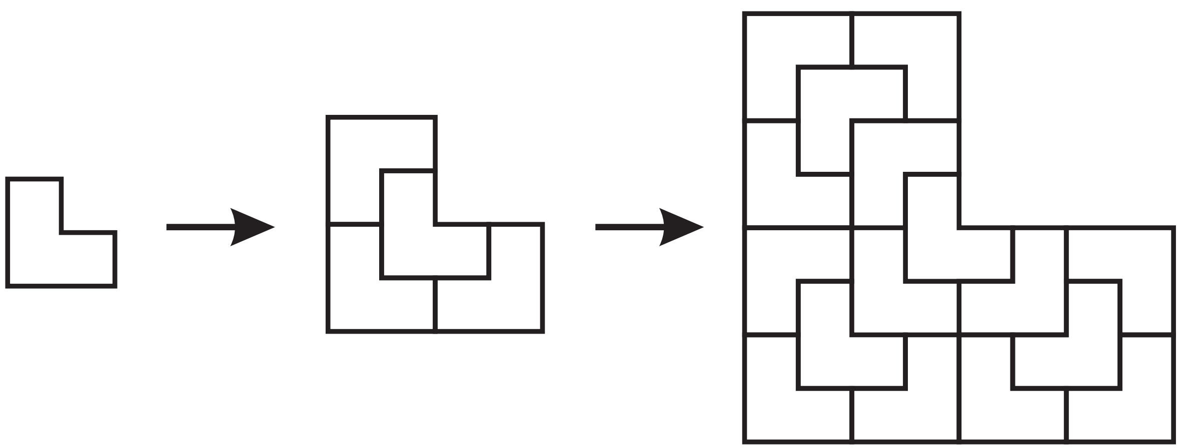


The k'th iteration results in a patch of 4^k tiles.

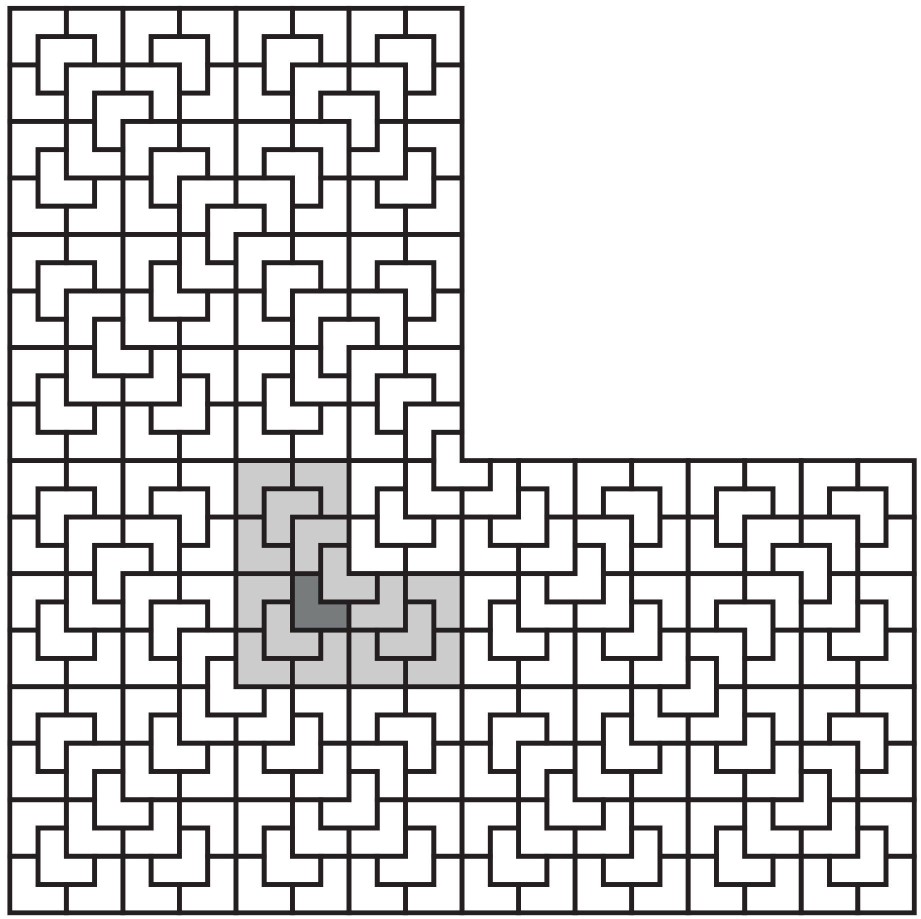
Iterating the substitution provide ever larger areas covered by the tiles. To obtain a tiling of the infinite plane, we suitably **position** the obtained patches on the plane so that the next patch **extends** around the previous one, and take the **limit** of the process.

We do the positioning in such a way that the patches grow in all directions, so that each point of the plane gets eventually covered by a tile.

For example, we can align the second iteration over the initial tile so that it is fully surrounded in the patch, and repeat this positioning at all even rounds.



Because the grey tile is not on the boundary, it is guaranteed to get surrounded by more and more tiles on all sides. Here's the fourth iterate:



The dark tile is the seed, the light grey part is the patch after two iterations.

In the limit we obtain a **tiling** t of the infinite plane.

In t each tile belongs to a macro tile, which in turn belongs to a macro tile of macro tiles, etc.

To prove that the tiling t is not periodic, we need to know that every tile belongs to a **unique macro tile**.

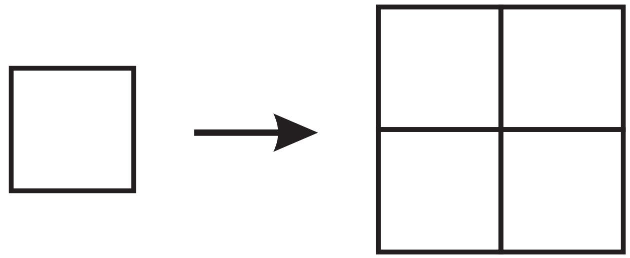
Indeed, two macro tiles cannot overlap:

We conclude that tiling t has a unique partitioning into a tiling by the macro tiles. The same reasoning then applies to the next levels, so that tiling t can be partitioned in a unique way into a tiling by k'th level macro tiles, for every k.

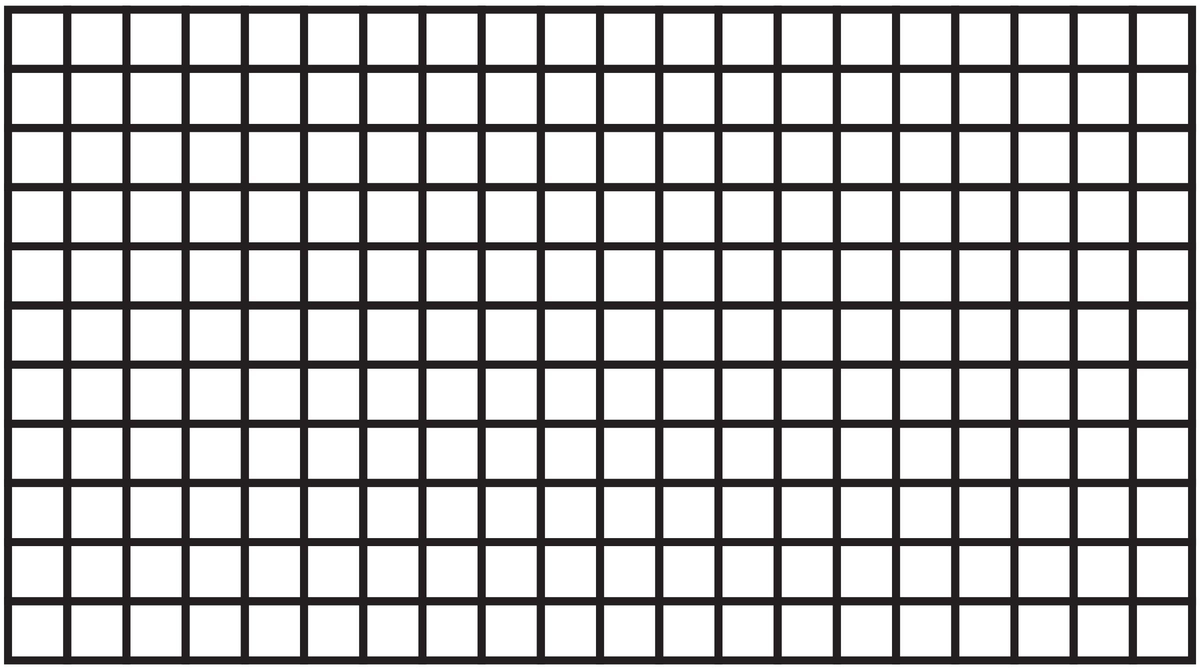
This implies that tiling t is **not periodic**:

We have proved that the tiling obtained by iterating the chair substitution is non-periodic. It was essential in the proof that the partitioning of the tiling into macro tiles is **unique**.

Example. Also the **square substitution**



can be iterated to generate a tiling such that each tile belongs a macro tile, macro-macro tile, etc.:



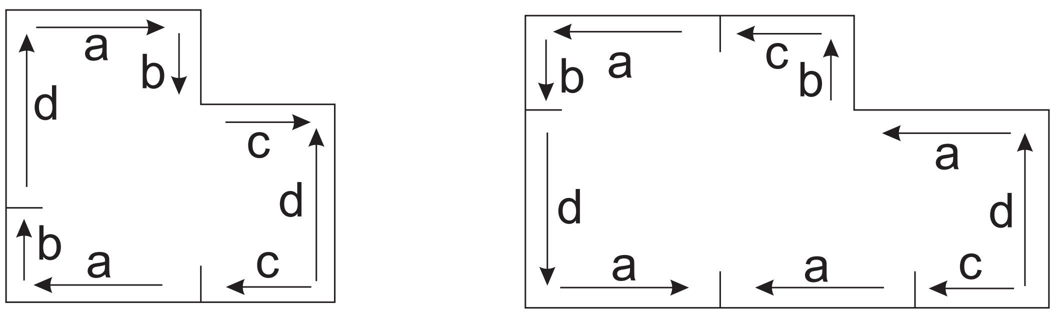
How come this tiling then is periodic??

Remark: the L-tromino is of course not aperiodic.

There are general methods to decorate tiles with color constraints in substitutions so that non-periodicity is forced. This increases the total number of tiles.

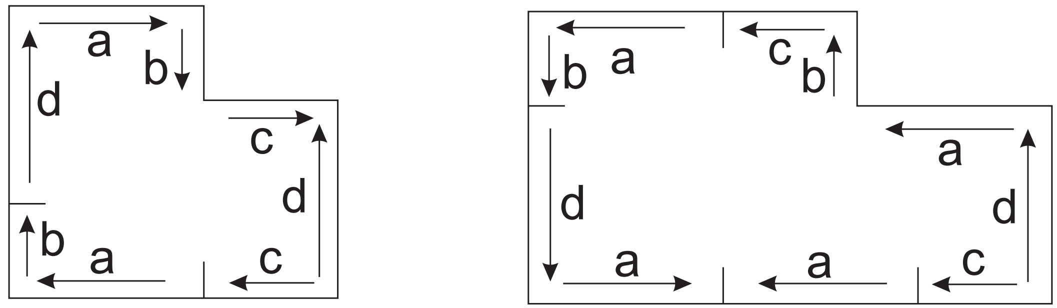
Amman's aperiodic tile set

The following pair of tiles, due to R.Amman in 1977, forms an aperiodic tile set.



The tiles may be rotated and flipped in any orientation. Let us call these the ***A*-tile** and the ***B*-tile**.

The **labeled arrows** along the edges give a matching rule: each arrow must fit against an arrow with the same label and orientation.

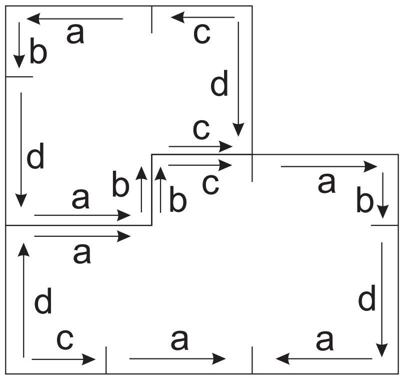


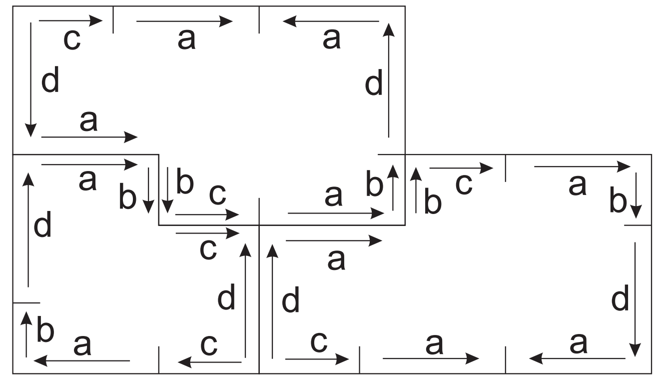
The lengths of the arrows are arbitrary (positive), but all arrows with the same label have also the same length.

Remark: this matching rule can not be implemented geometrically with bumps and dents. The reason is that some tiles are flipped, in which case the bumps of the matching arrows would line up against each other.

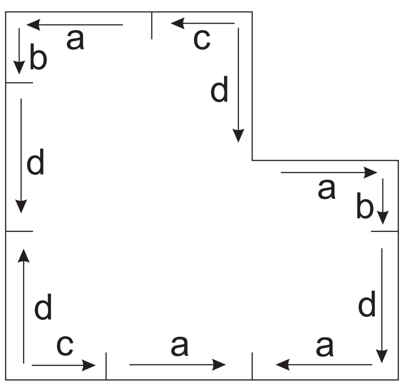
The tiles can be implemented geometrically, though, by not using bumps but only dents and a new key tile that fits between lined up dents. This way the tiles can be converted into a aperiodic set of **three geometric tiles**.

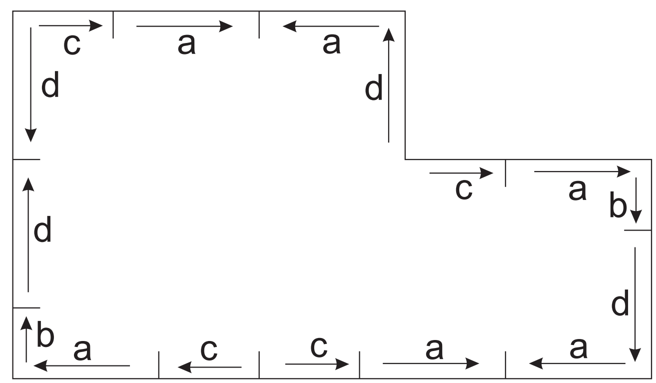
An A-tile and a B-tile fit together into the macro tile on the left, and an A-tile and two B-tiles form also the macro tile on the right:





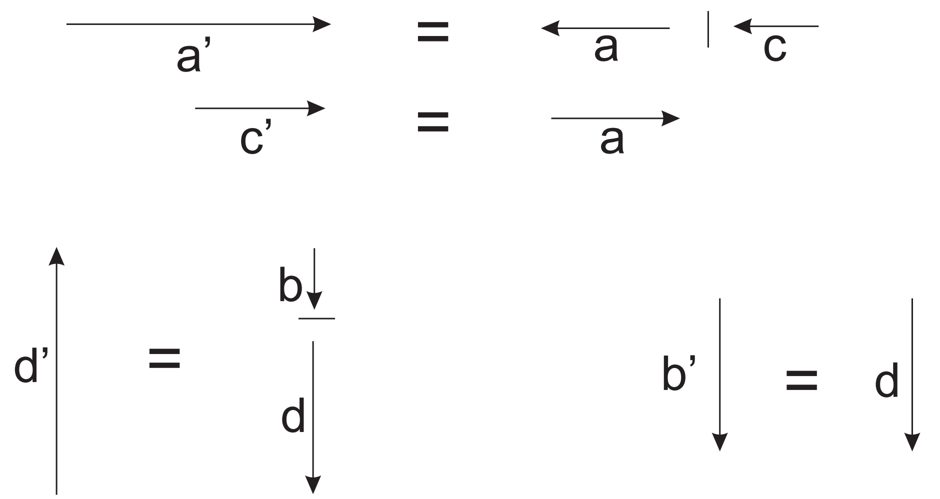
Any tiling by these **super-A** and **super-B** tiles



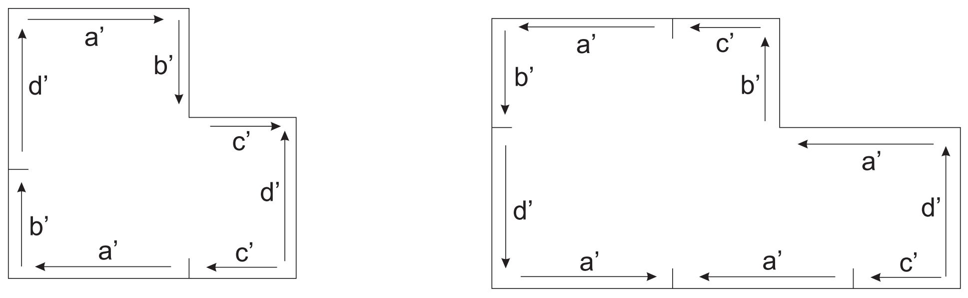


can be broken into a tiling by the original tiles.

Let us redecorate the supertiles with arrows labeled a', b', c' and d', where the new arrows represent combinations of old arrows as follows:



The redecorated supertiles



are called **expanded** A and **expanded** B, respectively.

- In any tiling by the expanded tiles, the tiles can be replaced by the corresponding supertiles, and the tiling remains valid.
- Conversely, in any tiling by the supertiles, replacing the supertiles by the corresponding expanded tiles yields a valid tiling.

(The latter fact follows from the fact that in the supertiles each c-arrow is always immediately followed by an a-arrow in the same direction, and every b-arrow is immediately followed by a d-arrow.)

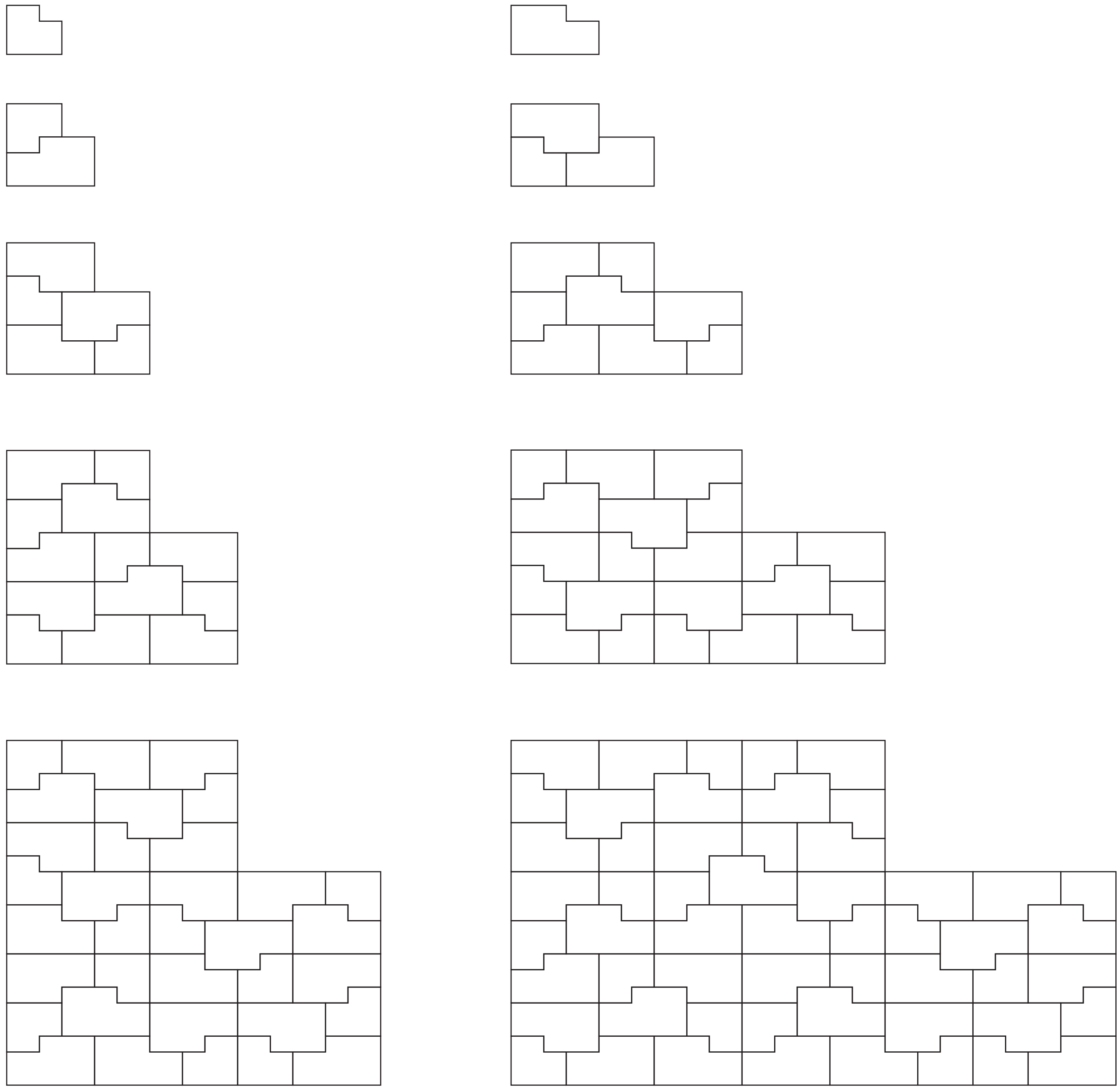
We conclude that the **supertiles** and the **expanded tiles** admit exactly the **equivalent tilings**.

The expanded tiles are **isomorphic** to the original tiles, where the arrows with labels a', b', c' and d' correspond to the arrows a, b, c and d, respectively. (However, the arrow lengths and their ratios may change, so the shapes of the expanded tiles are not necessarily similar to the original tiles.)

We can now build supertiles of **level two** by combining the expanded tiles the same way we combined the original tiles to build the first level supertiles.

Iterating the process allows us to build supertiles and expanded tiles of levels two, three, four and so on. These provide tilings of larger and larger regions of the plane by the original A- and B-tiles.

We can take a **limit**, which yields a valid, hierarchical **tiling** of the infinite plane.



Next we prove that there are **no periodic tilings**.

Consider an arbitrary tiling t of the plane by A and B.

Claim 1: Every tile belongs to a supertile.

Claim 2: The A- and B-tiles can be grouped into non-overlapping supertiles.

Claim 3: The grouping into supertiles is unique.

Theorem. The A- and B-tiles form an aperiodic pair of tiles.

Proof.