The extension theorem

Taking a limit, in all our substitutions we were able to position the patches so that level k patch is a sub-patch of the level $k+1$ patch that extends to all directions. Then it is clear that there is a "limit" tiling that has all the levels as sub-patches.

But it can be shown that even if the obtained patches do not contain previous ones as sub-patches, a valid tiling exists as long as arbitrarily large disks can be covered. This extends the **compactness argument** of Wang tiles to geometric tiles.

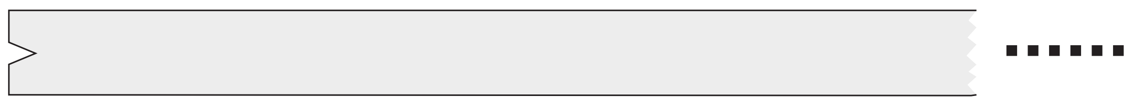
Theorem (extension theorem). A finite protoset $\mathcal{P}$ of topological disks admits a tiling if and only if, for every $r > 0$, a disk of radius r can be covered by copies of the prototiles. More precisely: there is a collection of tiles, all congruent to elements of $\mathcal{P}$, such that

(i) the interiors of the tiles are pairwise disjoint, and

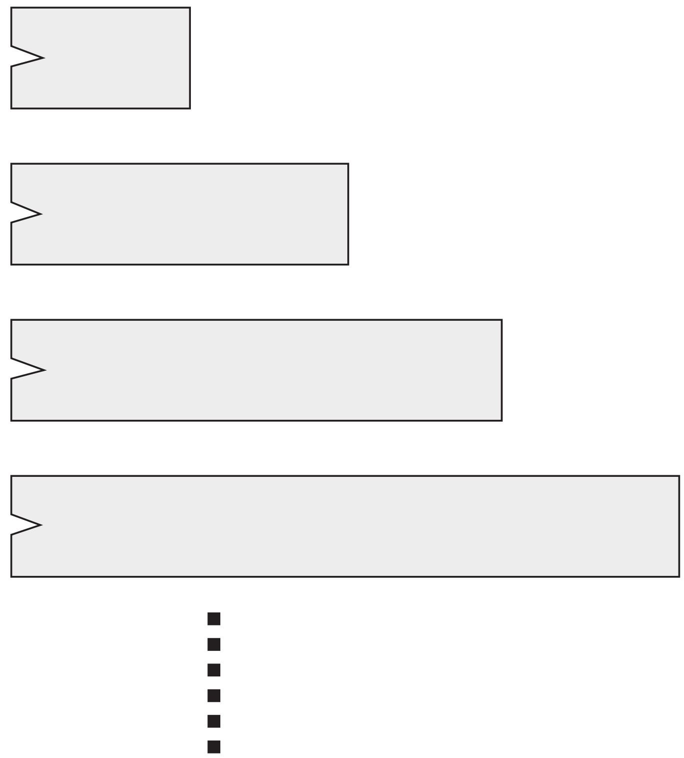
(ii) a disk of radius r is included in the union of the tiles.

Remark: The theorem only considers **finite** protosets of **topological disks**.

Example. The following single "tile" tiles arbitrarily large disks but does not tile the plane. (The tile is not a topological disk.)



Example. The following infinite prototile set (all topological disks) tiles arbitrarily large disks but does not tile the plane.



The periodicity theorem

For Wang tiles we have proved that **"One-periodic $\Longrightarrow$ two-periodic"**:

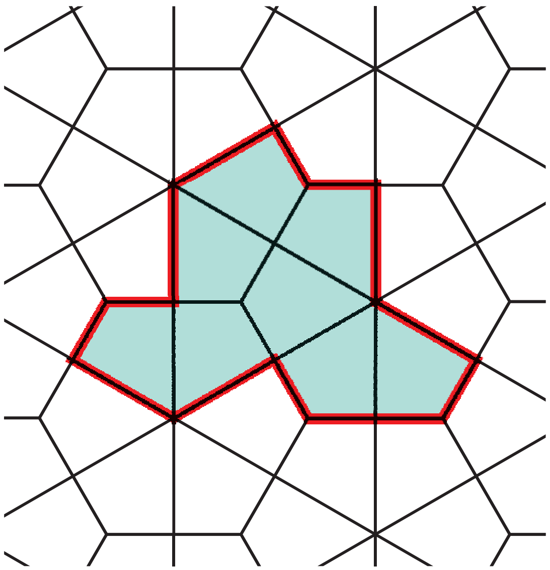
Theorem. If a Wang tile set admits a tiling with a period, then it also admits a tiling with two periods in non-parallel directions.

Also this can be generalized to geometric tiles. What we require is that:

- Tiles are polygons,
- Considered tilings are edge-to-edge,
- The prototile set is finite.

Theorem (periodicity theorem). Let $\mathcal{P}$ be a finite set of polygons. Assume that there exists an edge-to-edge tiling by the protoset $\mathcal{P}$ that is one-way periodic (=invariant under some translation). Then there also exists an edge-to-edge tiling by $\mathcal{P}$ that is two-way periodic (=invariant under translations by two linearly independent vectors).

Hat: an aperiodic monotile

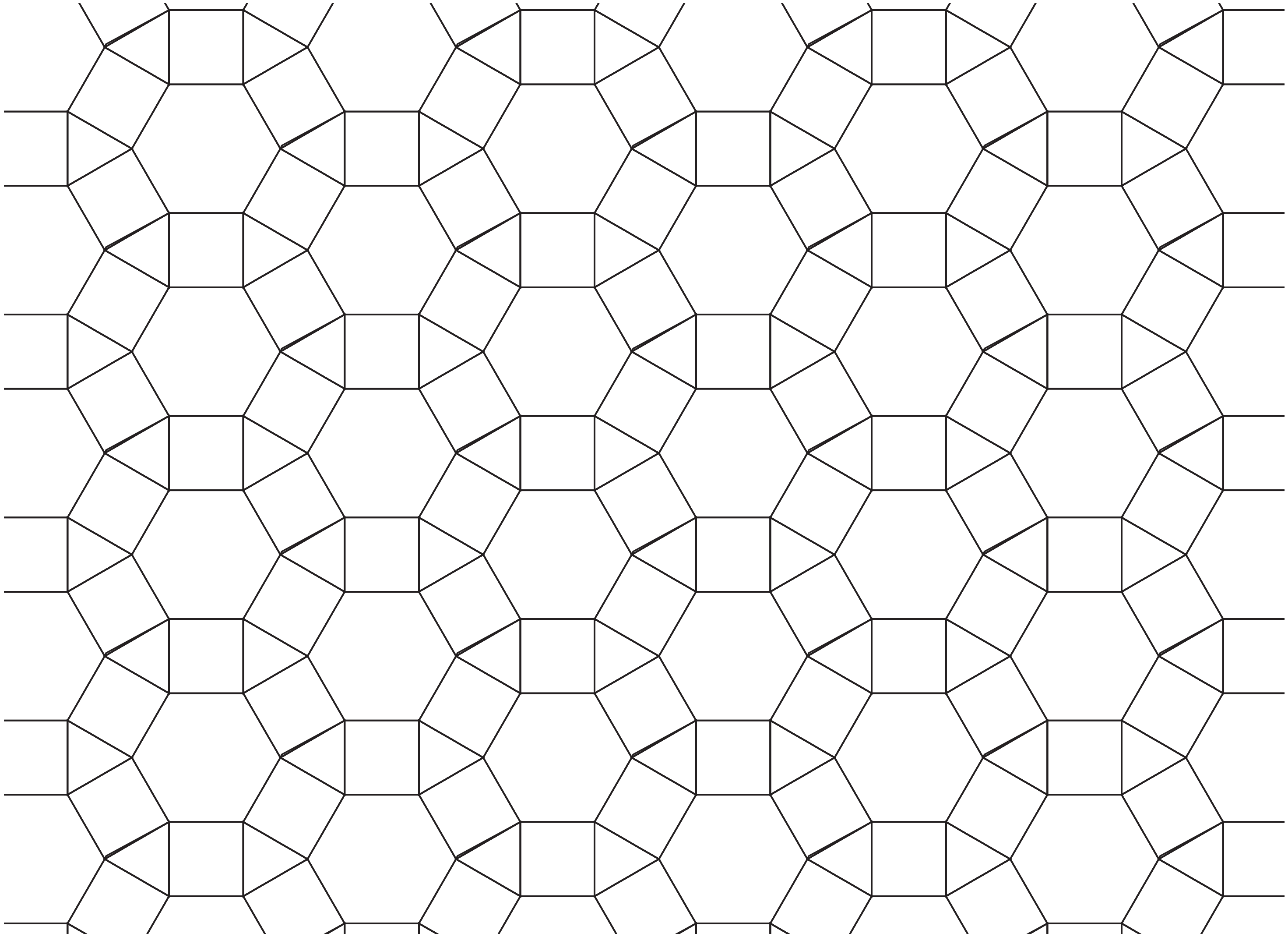


The hat is an **aperiodic monotile**: there exists monohedral tilings of the plane where all tiles are congruent to the hat, but none of these tilings are periodic (=invariant under a non-zero translation).

In the tilings both even and odd isometric copes of the hat are used (=the hat may be flipped upside down).

The hat "lives" in the grid that is formed by overlapping the regular tilings by equilateral triangles and regular hexagons.

The grid is the dual of the $3 \cdot 4 \cdot 6 \cdot 4$ Archimedean tiling, obtained by joining the centers of adjacent tiles of the Archimedean tiling.



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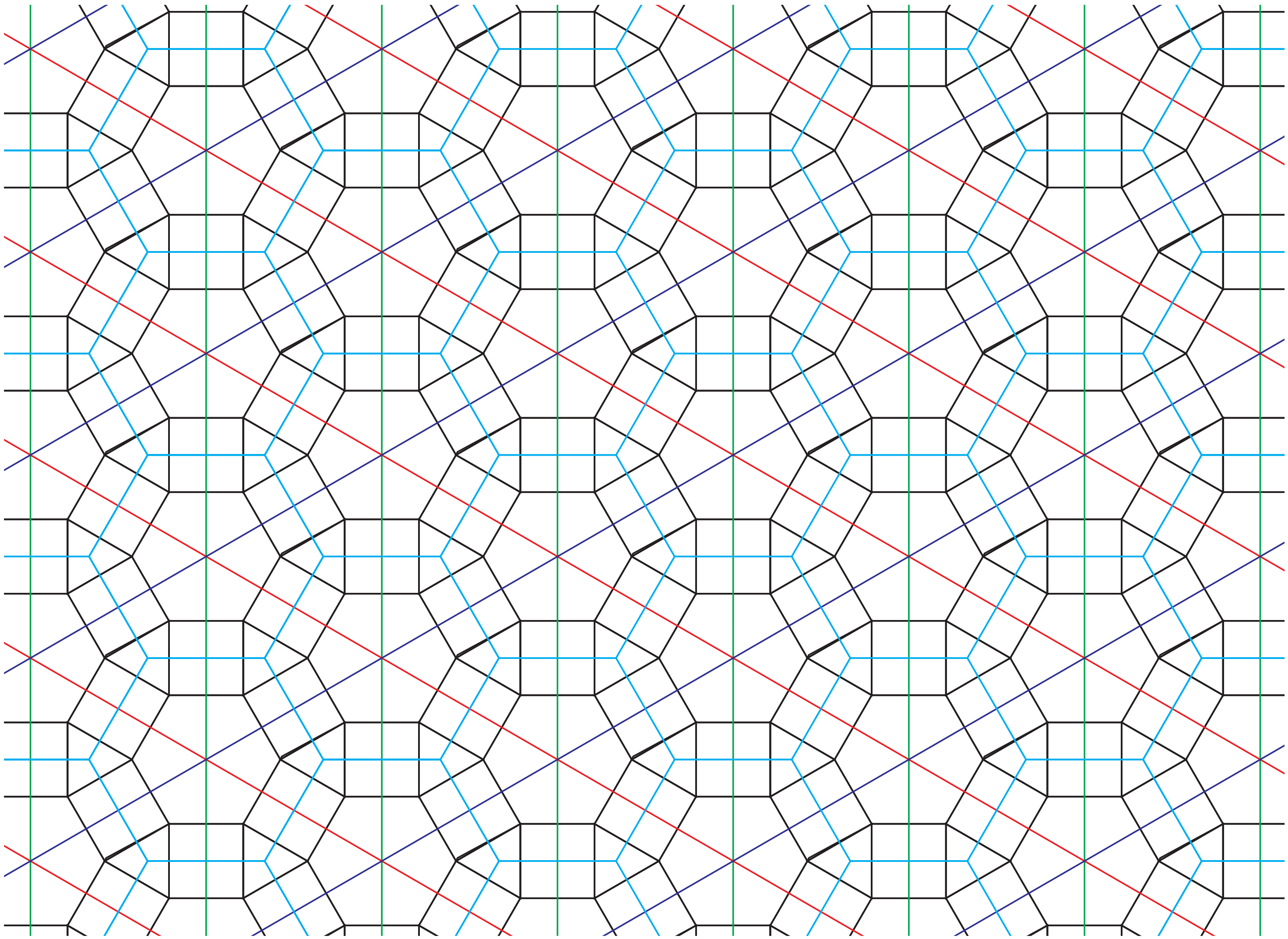

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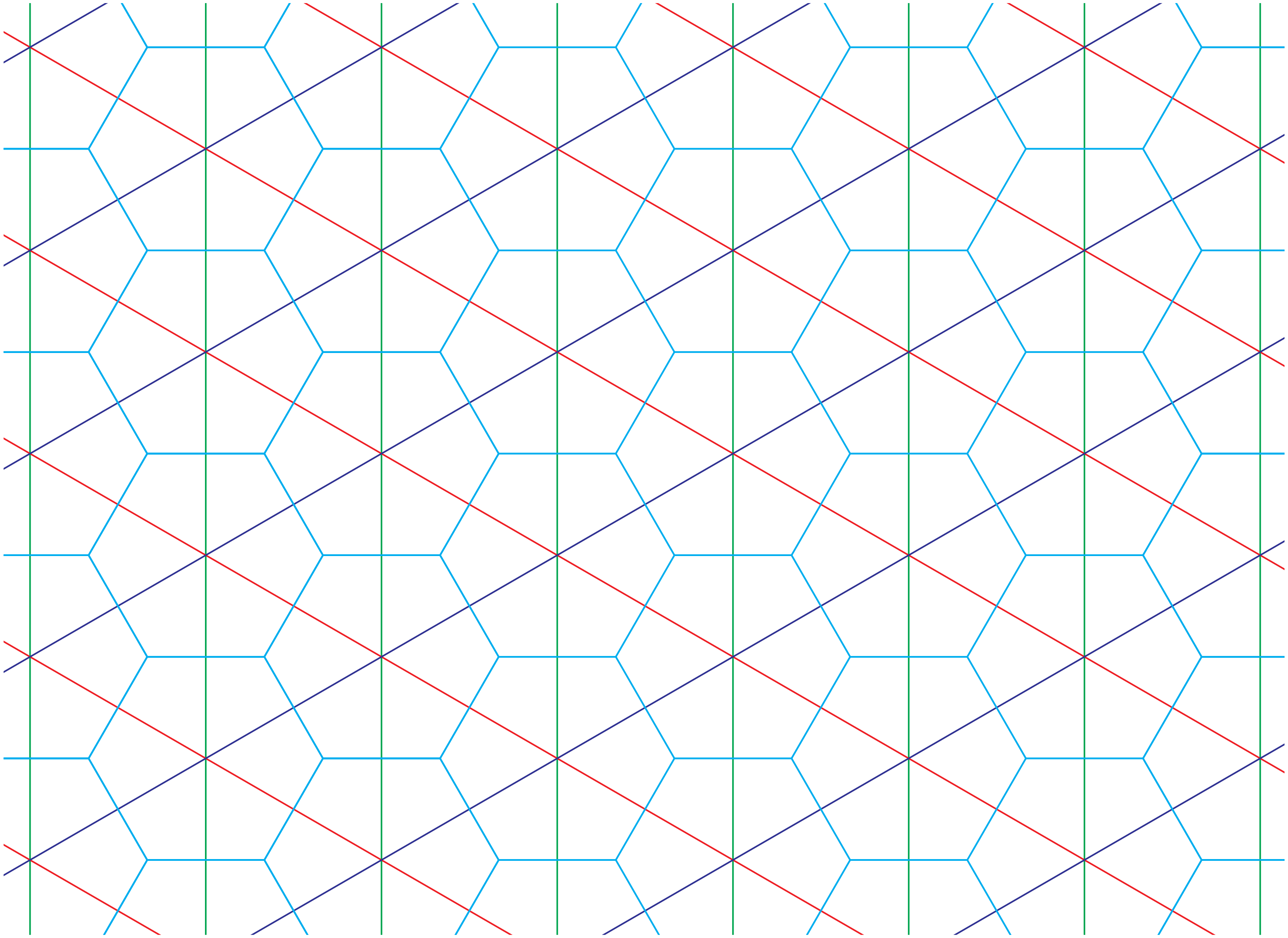
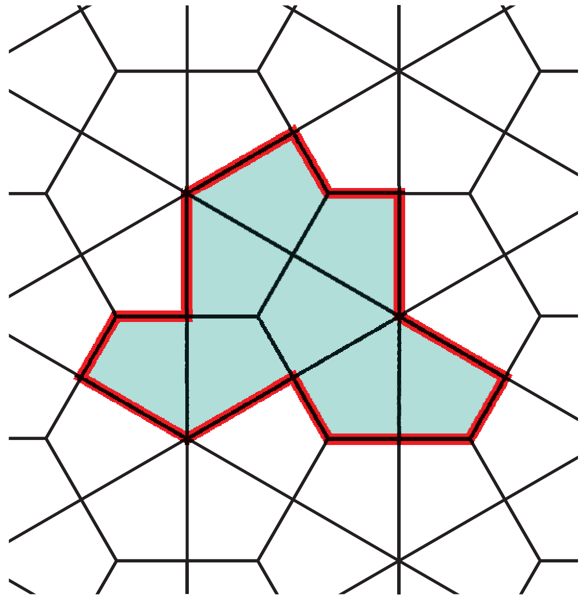


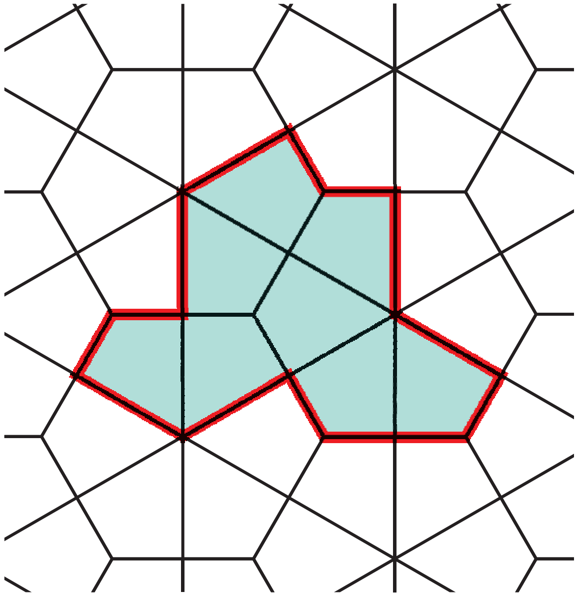


A kite in the grid has

- two short edges (length 1),
- two long edges (length $\sqrt{3}$).

The hat inherits these edge lengths. As a polygon the hat has also one edge of length 2 (formed by two consecutive parallel short edges of kites), but this segment is considered as two edges of length 1.

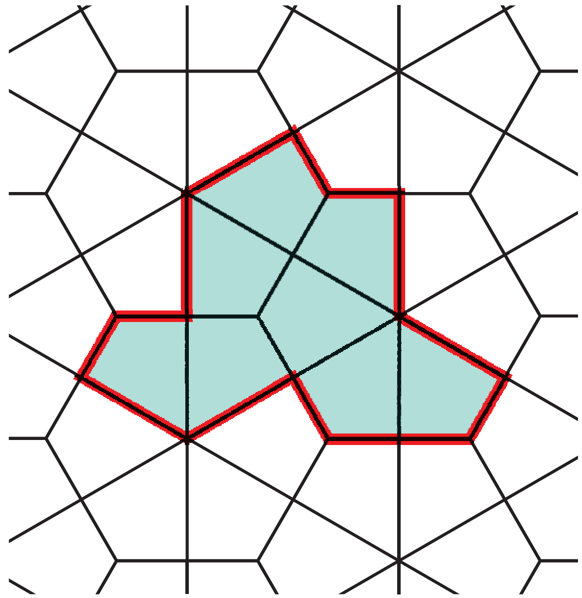
$\Longrightarrow$ The hat has 6 long edges and 8 short edges.



Reading the edges as vectors, going around the hat clockwise, gives 6 long vectors and 8 short vectors that sum up to $\vec{0}$.

But: also the sum of the long vectors is $\vec{0}$ and the sum f the short vectors is $\vec{0}$.

Reason: The six long vectors are three pairs of opposite vectors.



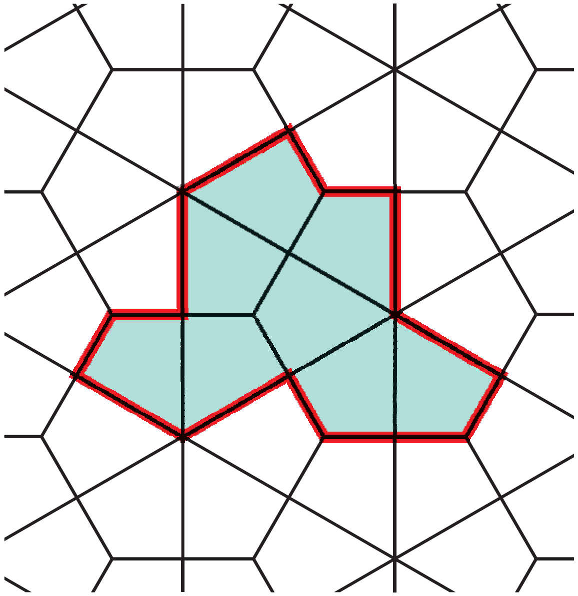
Thus one can **deform** the hat by scaling the lengths of

- all long edges by some $a \geq 0$ and
- all short edges by some $b \geq 0$.

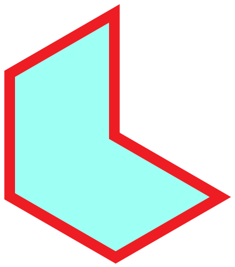
(but not $a = b = 0$)

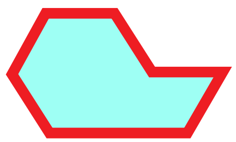
The scaled long edge vectors sum up to zero and and the scaled short edge vectors sum up to zero.

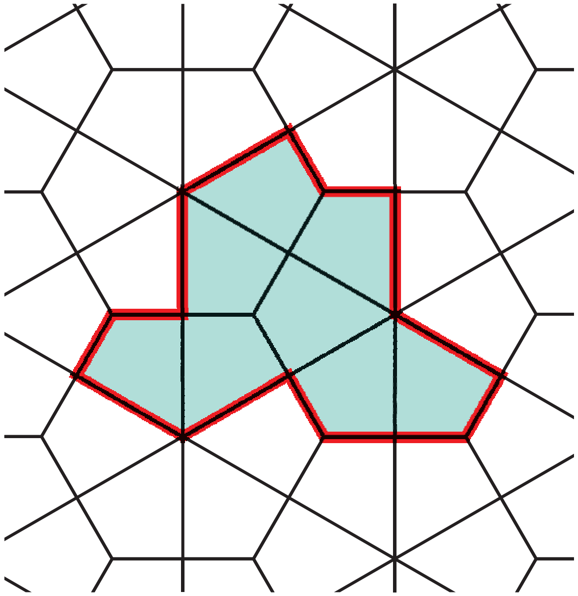
$\Longrightarrow$ The scaled edges define a boundary of a deformed tile.



We especially need the extremal cases where $a = 1, b = 0$; or $a = 0, b = 1$:







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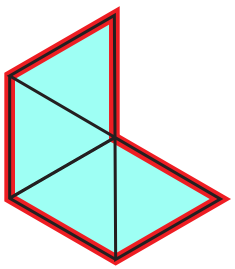
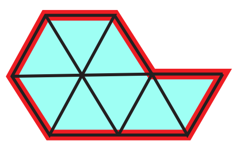




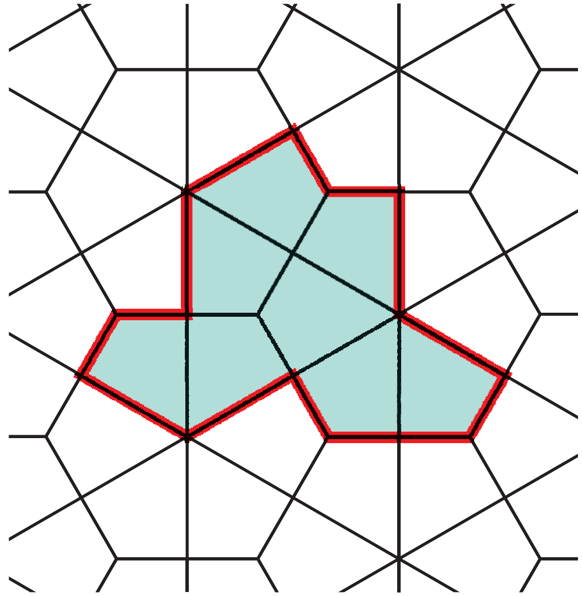
In fact we only use the left one, **chevron**.

Deforming the tiles in a valid hat tiling produces a corresponding **valid tiling by deformed tiles**:

Let $\mathcal{T}$ be a hat tiling, and let $\mathcal{V} \subseteq \mathbb{R}^2$ be its vertex set.
Assume w.l.o.g. that $\vec{0} \in \mathcal{V}$.

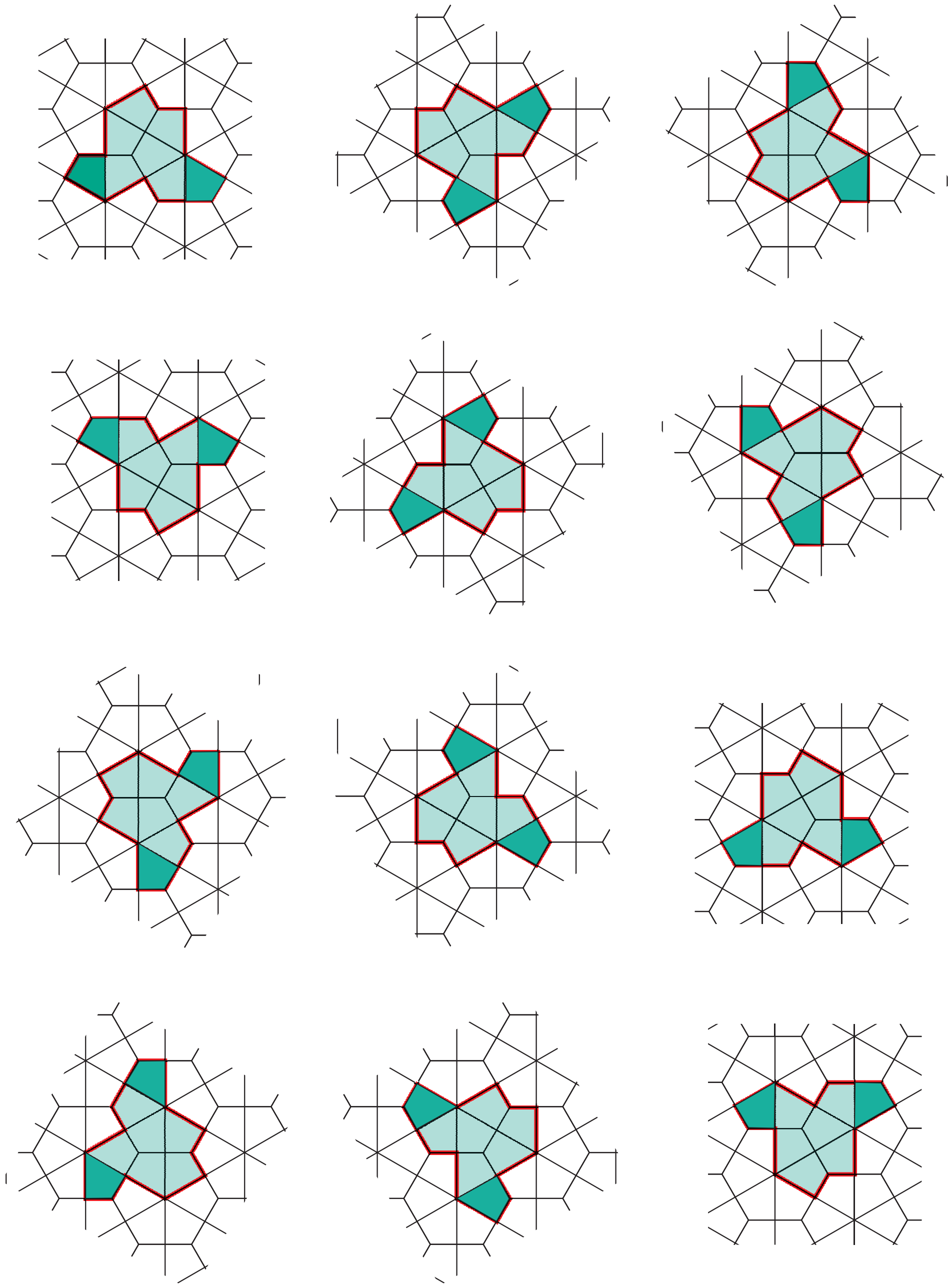
• Along any cycle that follows the edges in the tiling: the sum of the long vectors is $\vec{0}$ and the sum of the short vectors is $\vec{0}$.

• In the deformed tiling, vertex $V \in \mathcal{V}$ will be moved to position $V' \in \mathbb{R}^2$ as follows: Take any path along the edges from $\vec{0}$ to V. Scale the long and short edges along this path by factors a and b. The path still starts at $\vec{0}$ but the new end point will be V'. The position of V' does not depend on the choice of the path!



It turns out (case analysis, details skipped) that hats in any valid tiling are aligned on the underlying grid of kites. (It is enough to prove that surrounding hats of any given hat are aligned in the same grid: by induction then all hats are aligned.)

$\Longrightarrow$ there are **12 available orientations of the hat**: six by rotation, and another six by rotating the flipped over hat.



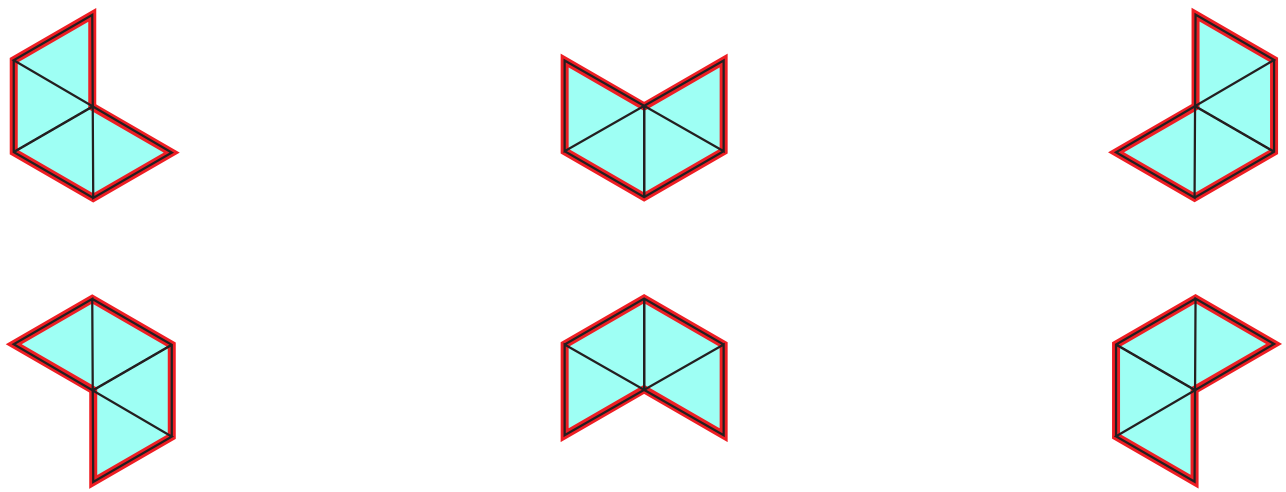
In the hat, one pair of oppositely oriented kites is covered twice – all other kite orientations are covered once.

The four hat orientations in each column cover the same kite orientation twice.

Because all kite orientations appear with equal proportions in the underlying grid, a valid tiling must have one third of its tiles from each of the three columns.

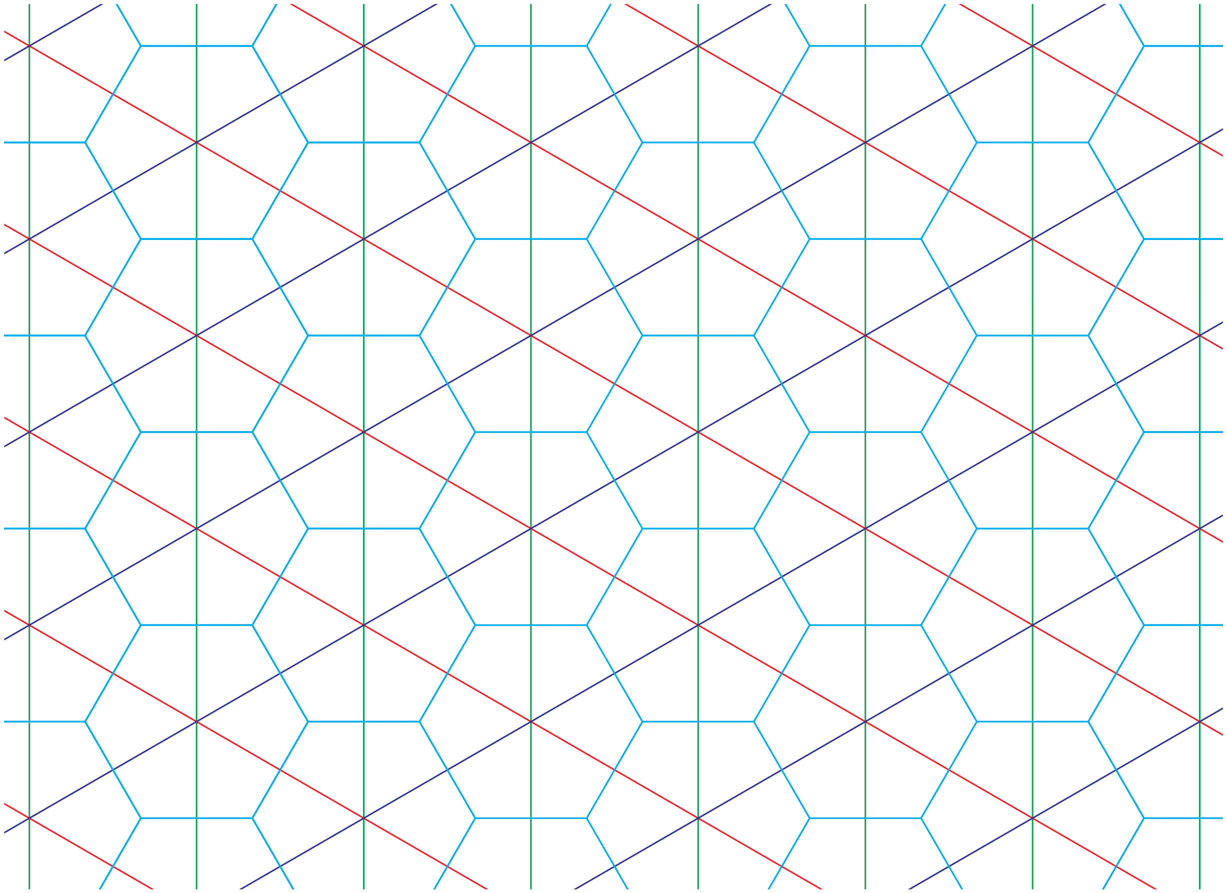
Hat $\longrightarrow$ chevron deformation preserves the orientations of the long edges.

From the 12 orientations of the hat we obtain six different orientation of the chevron:



A chevron tiling that is obtained by deforming a hat tiling has one third of its chevrons from each of the three columns.

The triangular part of the kite grid is formed by three families of parallel lines.



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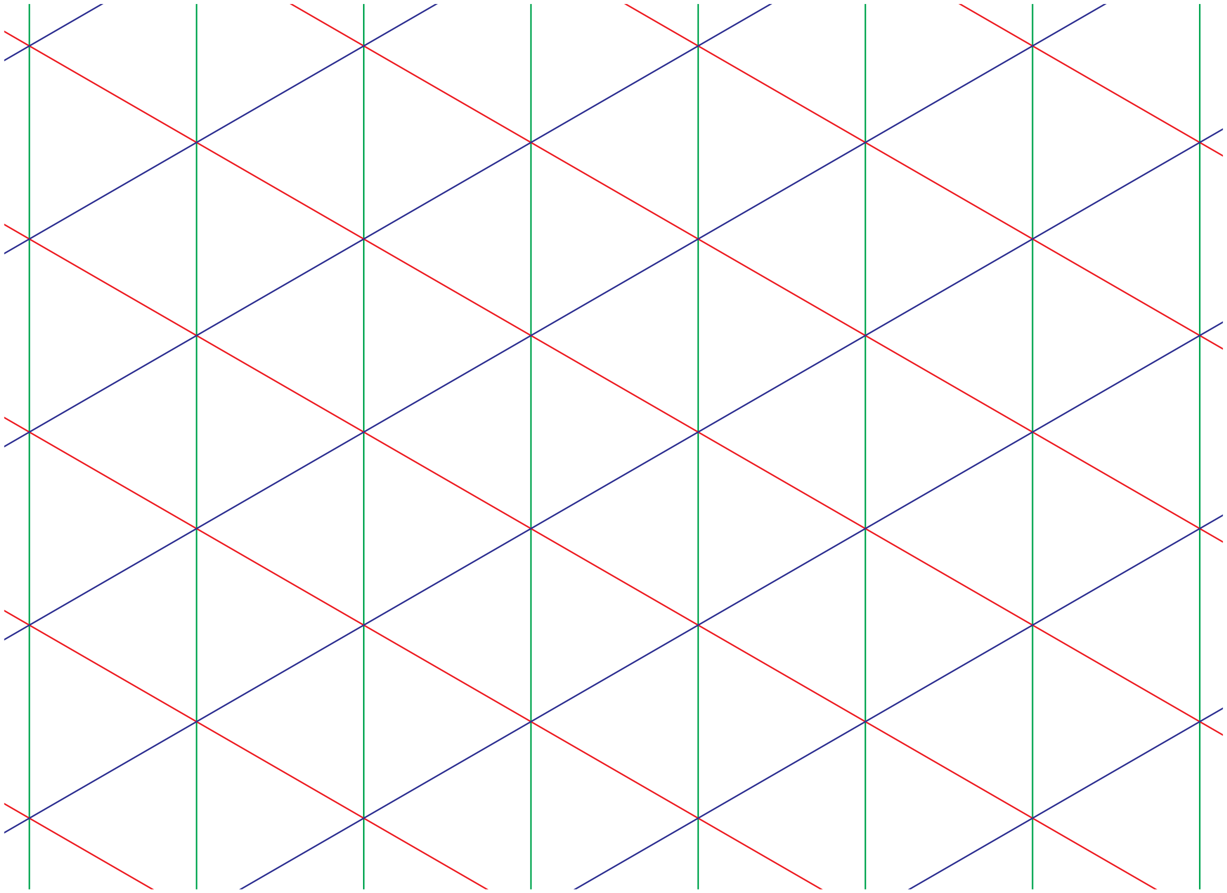
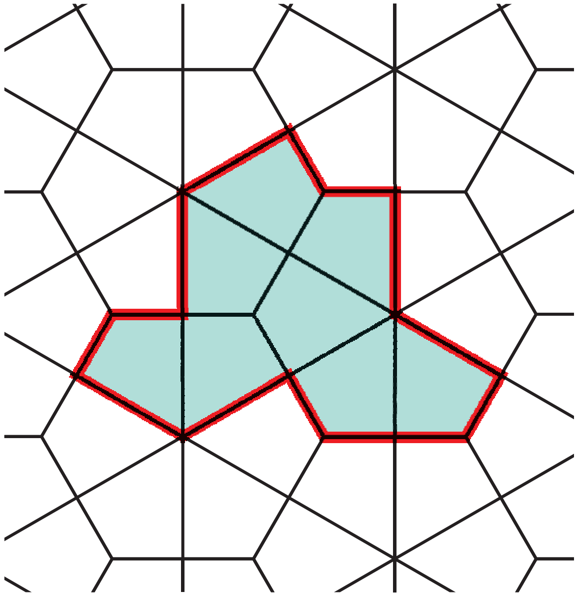


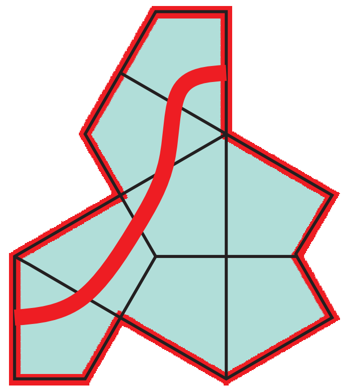
We number the directions 1,2 and 3, and call the lines in direction $i \in \{1, 2, 3\}$ the ***i*-lines** of the grid.

The distance between consecutive i-lines is 3.



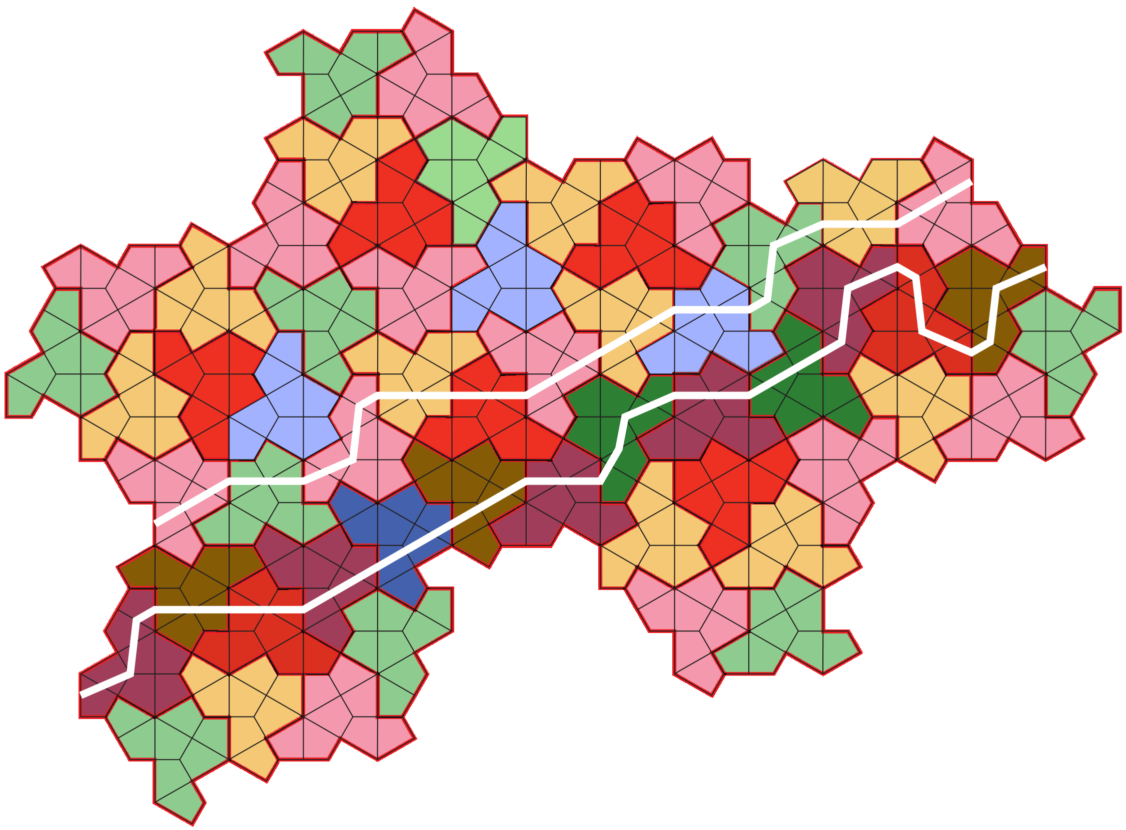
The pairs of parallel long edges of the hat are always on consecutive i-lines.

A **de Bruijn segment** on a hat tile in direction i is a line drawn inside the tile connecting the centers of the two long edges of the hat that are parallel to i-lines.



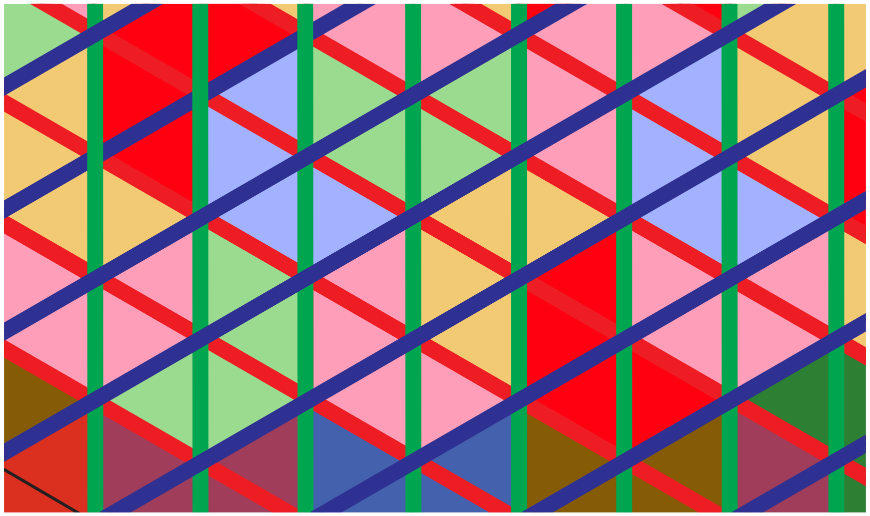
In a hat tiling, de Bruijn segments continue across edges, defining infinite **de Bruijn lines**.

Each tile is crossed by a unique de Bruijn line in each direction i, and the lines in the same direction do not cross each other.

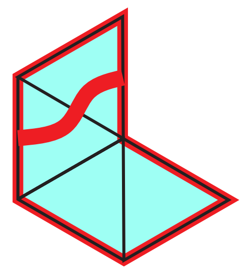


The set of tiles along a de Bruijn line in direction i is called an **i-strip**.

A deformed tiling by chevrons is also aligned on a triangular grid with the same three directions. In this grid the i-lines are at distance $\frac{3}{2}$ from each other, that is, twice as dense as in the kite grid.

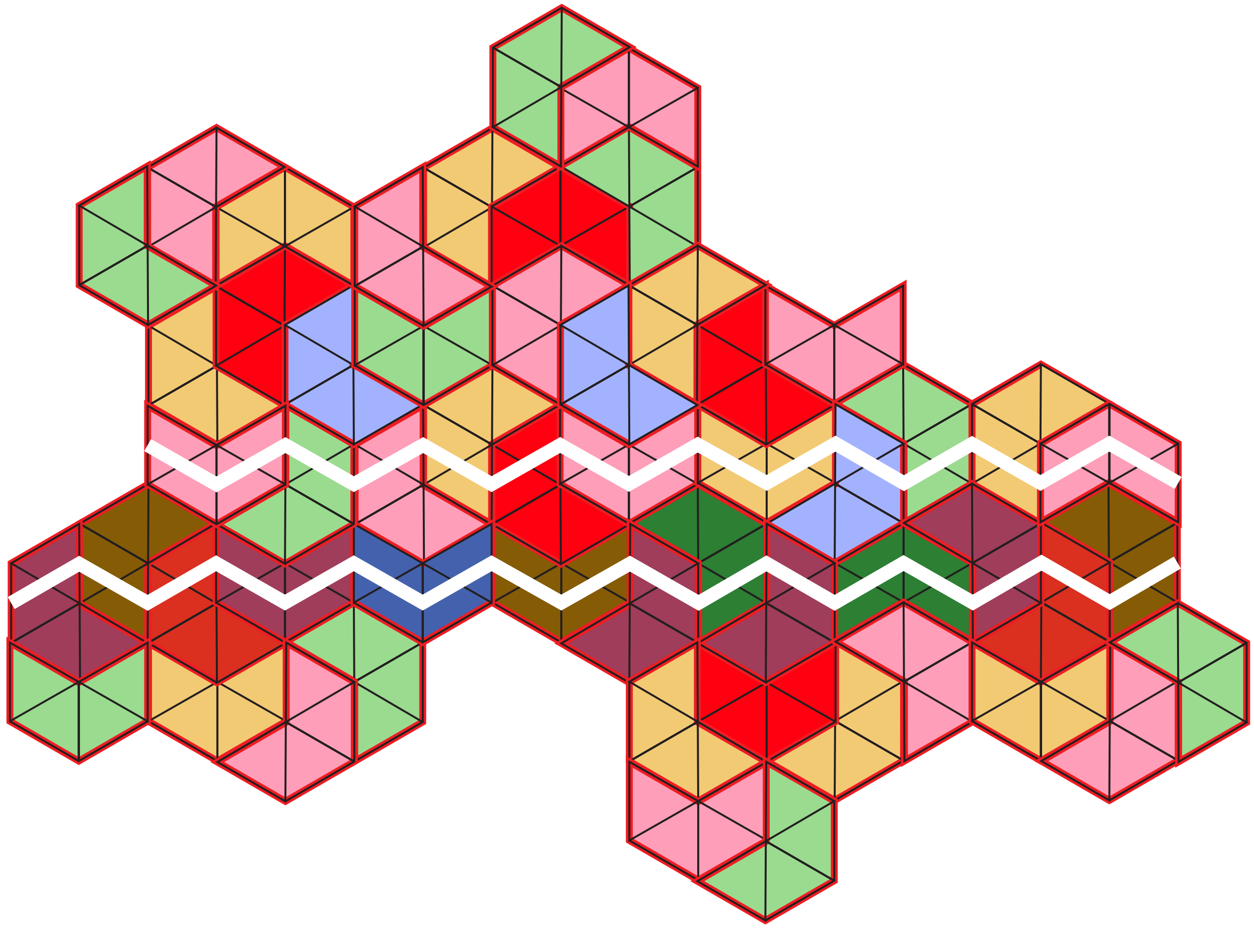


Also the chevron has a pair of parallel edges in each direction. Joining the centers of parallel sides by a line segment gives a **de Bruijn segment**:

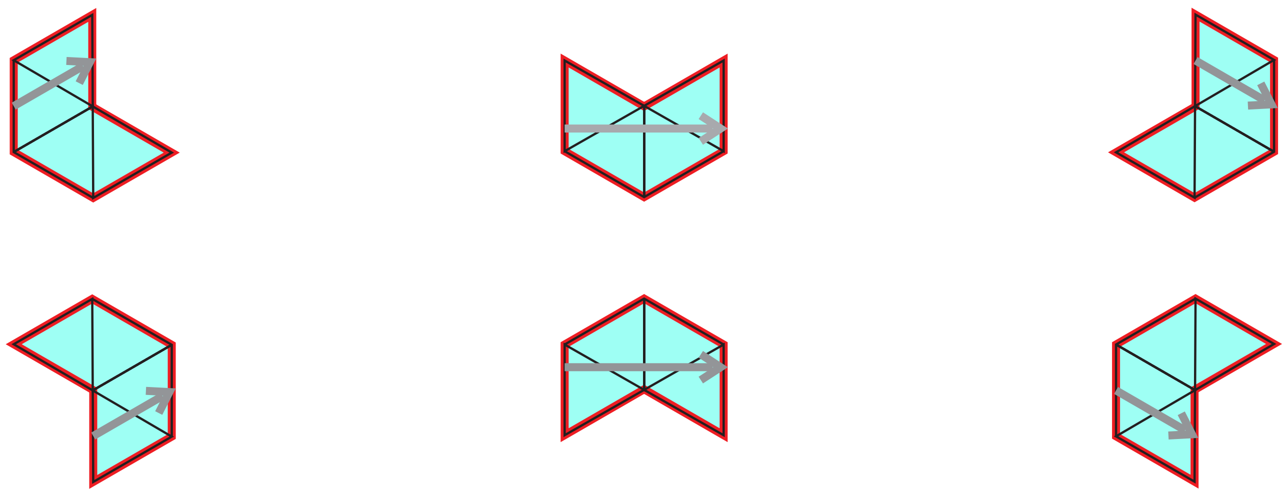


Depending on the orientation of a chevron, the de Bruijn segment either connects consecutive i-lines or it skips over one line.

In a valid tiling the segments define infinite **de Bruijn paths** in directions $i \in \{1, 2, 3\}$, and the tiles along each such path is called an ***i*-strip**.



Recall that (in a tiling by chevrons that is obtained by deforming a hat tiling) one third of the chevrons come from each of the three columns in



Consider, for example, the direction of vertical lines. In one third of the tiles the de Bruijn segment

- moves to the next i-line but goes a step higher (first column),
- moves to the next i-line but goes a step lower (last column),
- skips over one i-line but stays on the same height (middle column).

On the average, the vertical hight remains the same, and the horizontal movement is by $\frac{4}{3}$ lines per tile crossed (=two units since the distance between consecutive lines is $\frac{3}{2}$).

Assume that there exists a two-periodic tiling $\mathcal{T}$ by the hats. Construct the corresponding deformed tiling $\mathcal{T}'$ by chevrons.

Let $\vec{p}$ and $\vec{q}$ be generators of the periods of $\mathcal{T}$, meaning that

$$\mathcal{P} = \mathbb{Z}\vec{p} + \mathbb{Z}\vec{q}$$

is set of the periods of $\mathcal{T}$.

Let $\mathcal{V} \subseteq \mathbb{R}^2$ be the set of vertices of $\mathcal{T}$, and assume that $\vec{0} \in \mathcal{V}$.

Then $\mathcal{P} \subseteq \mathcal{V}$.

Let $\mathcal{V}' \subseteq \mathbb{R}^2$ be the set of vertices of $\mathcal{T}'$, and let

$$f : \mathcal{V} \longrightarrow \mathcal{V}'$$

assign to each vertex $\vec{v}$ of $\mathcal{T}$ the corresponding vertex $f(\vec{v})$ of $\mathcal{T}'$.
(Corresponding means: For any path along edges of $\mathcal{T}$ from $\vec{0}$ to $\vec{v}$ the path obtained by erasing all short edges leads from $\vec{0}$ to $f(\vec{v}.)$

Claim: For any $\vec{v} \in \mathcal{P}$ and $\vec{x} \in \mathcal{V}$ we have that

$$f(\vec{x} + \vec{v}) = f(\vec{x}) + f(\vec{v}).$$

Proof.

Claim: For any $\vec{v} \in \mathcal{P}$ and $\vec{x} \in \mathcal{V}$ we have that

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In particular:

- $f(\vec{v})$ is a period of $\mathcal{T}'$ for any $\vec{v} \in \mathcal{P}$, and
- f is a linear on $\mathcal{P}$: $f(i\vec{p} + j\vec{q}) = if(\vec{p}) + jf(\vec{q})$ for all $i, j \in \mathbb{Z}$

Denote

$$\mathcal{P}' = f(\mathcal{P}) = \mathbb{Z}f(\vec{p}) + \mathbb{Z}f(\vec{q}).$$

Elements of $\mathcal{P}'$ are periods of $\mathcal{T}'$ (but there might exist also other periods).

Let $\hat{f} : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ be the unique linear function that maps

$$\vec{p} \mapsto f(\vec{p}) \text{ and } \vec{q} \mapsto f(\vec{q}).$$

Then $\hat{f}$ and f are identical on $\mathcal{P}$ (but may differ on $\mathcal{V} \setminus \mathcal{P}$).

Consider i-strips of $\mathcal{T}$ for directions $i \in \{1, 2, 3\}$.

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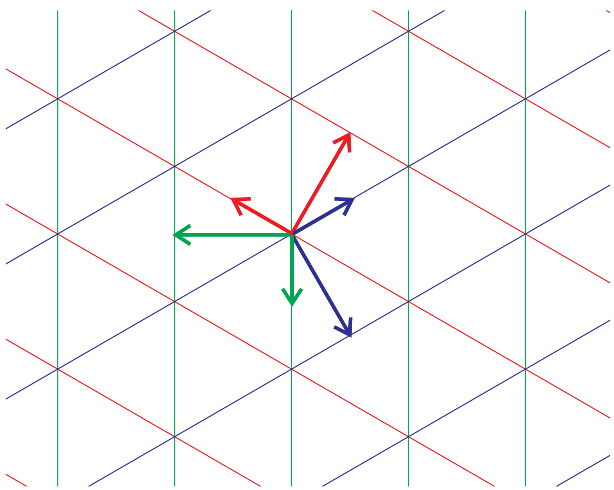
• For each i, let $\vec{v}_i$ be a vector of length 3 perpendicular to i-lines, and let $\vec{u}_i$ be a unit vector parallel to i-lines. We can choose these so that

$$\vec{v}_1 + \vec{v}_2 + \vec{v}_2 = \vec{0} \text{ and } \vec{u}_1 + \vec{u}_2 + \vec{u}_3 = \vec{0}.$$



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• Let τ_i be by vector $\vec{p}_i = k_i\vec{v}_i + y_i\vec{u}_i$. Here $k_i \in \mathbb{Z}$ is the number of tiles that τ_i moves forward on an i-strip, and $y_i \in \mathbb{R}$.

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- We can replace $\vec{p}_i$ by its multiples so we may assume that $k_1 = k_2 = k_3 = k$:

$$\vec{p}_i = k\vec{v}_i + y_i\vec{u}_i$$

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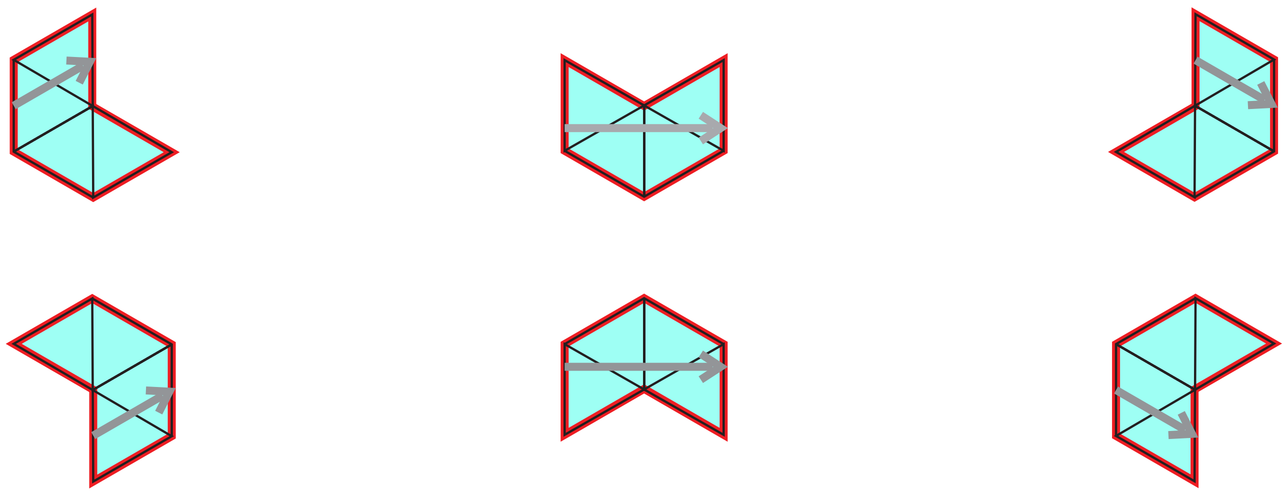

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$$\hat{f}(\vec{p_1} + \vec{p_2} + \vec{p_3}) = \vec{0},$$

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- Finally, as $\vec{p_i} = k\vec{v_i} + y_i\vec{u_i}$ and $\vec{v_1} + \vec{v_2} + \vec{v_2} = \vec{0}$, we have that

$$y_1\vec{u_1} + y_2\vec{u_2} + y_3\vec{u_3} = \vec{0}.$$

This further implies that $y_1 = y_2 = y_3$.

Conclusion:

- p_i are of equal length and at angles 120° to each other,
- $\hat{f}(p_i)$ are of equal length and at angles 120° to each other.

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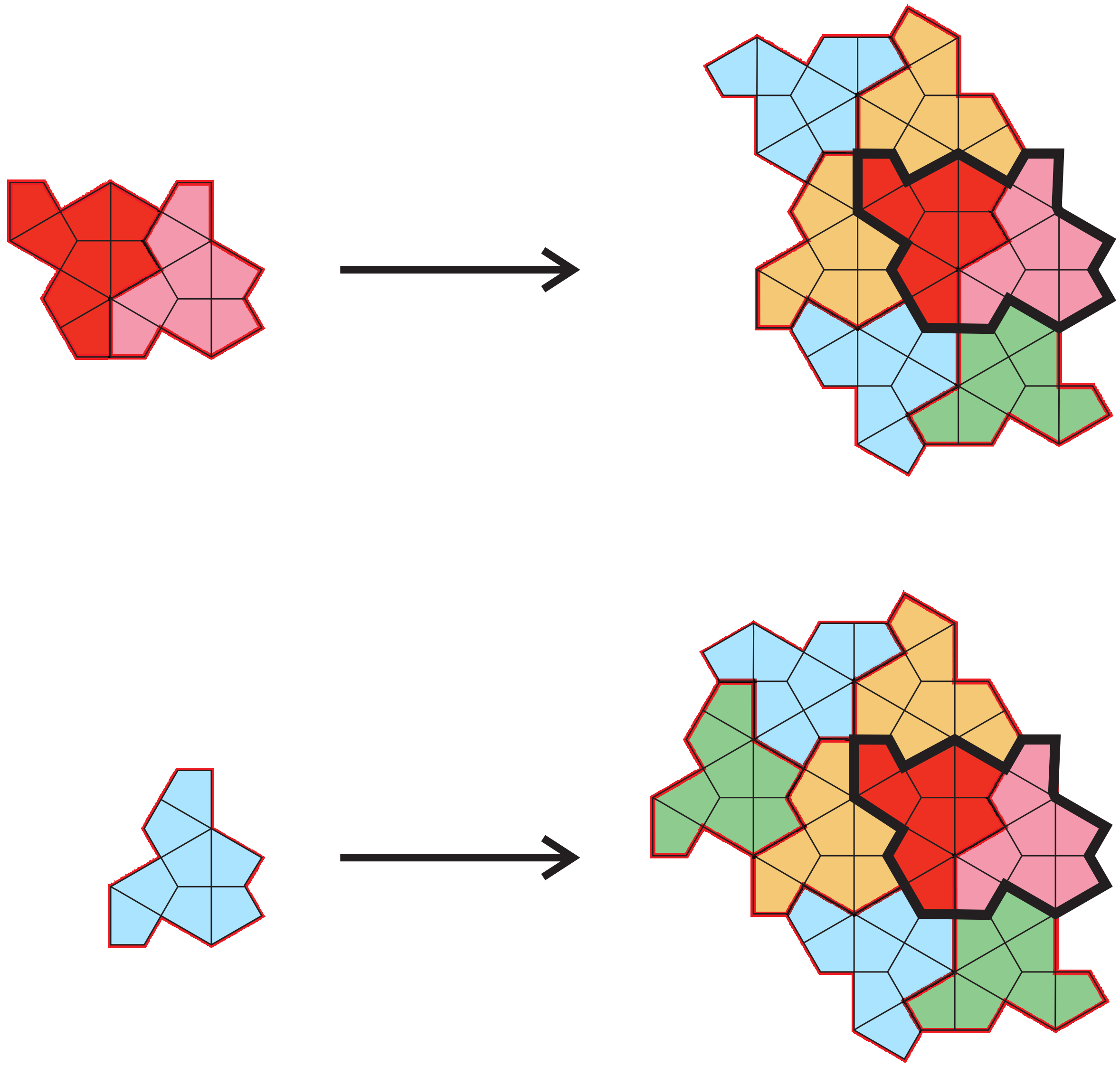
Because the area of the hat is twice the area of the chevron, the value of the similarity scaling factor is $c = \sqrt{2}$.

(All distances between points get divided by $\sqrt{2}$.)

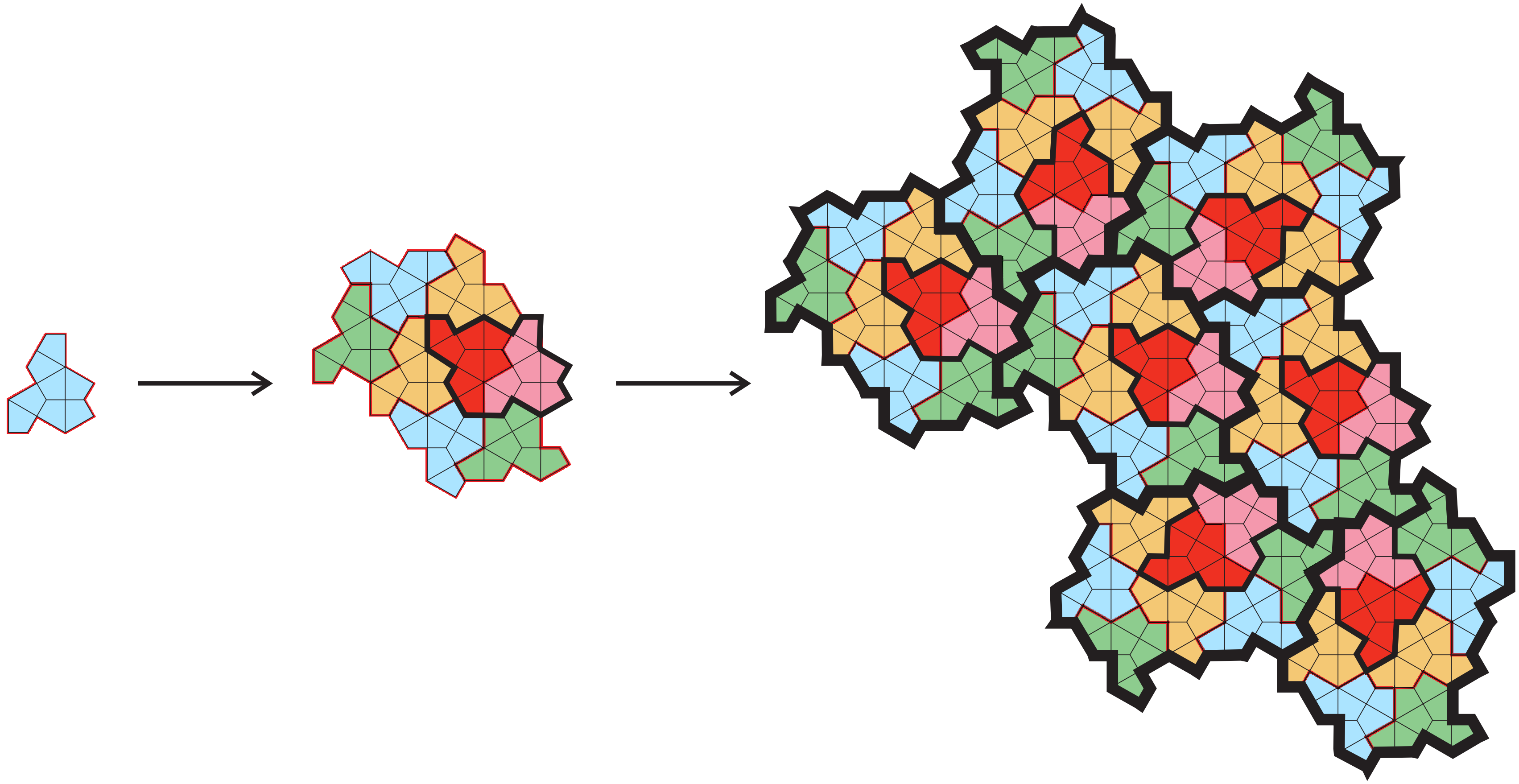
But: Periodicity vectors $\vec{p}$ and $\hat{f}(\vec{p})$ are between vertices of the same triangle lattice. In such a lattice the distances cannot have ratio $\sqrt{2}$.

Theorem. The hat tile does not admit a two-way periodic tiling.

A valid tiling can be generated using a substitution:



First two iterations:



Deformed tiles are also aperiodic monotiles:

Deforming the hat with scaling factors $a \geq 0$ and $b \geq 0$ produces **equivalent** tiles that are also aperiodic, except when

(i) $a = 0$, or

(ii) $b = 0$, or

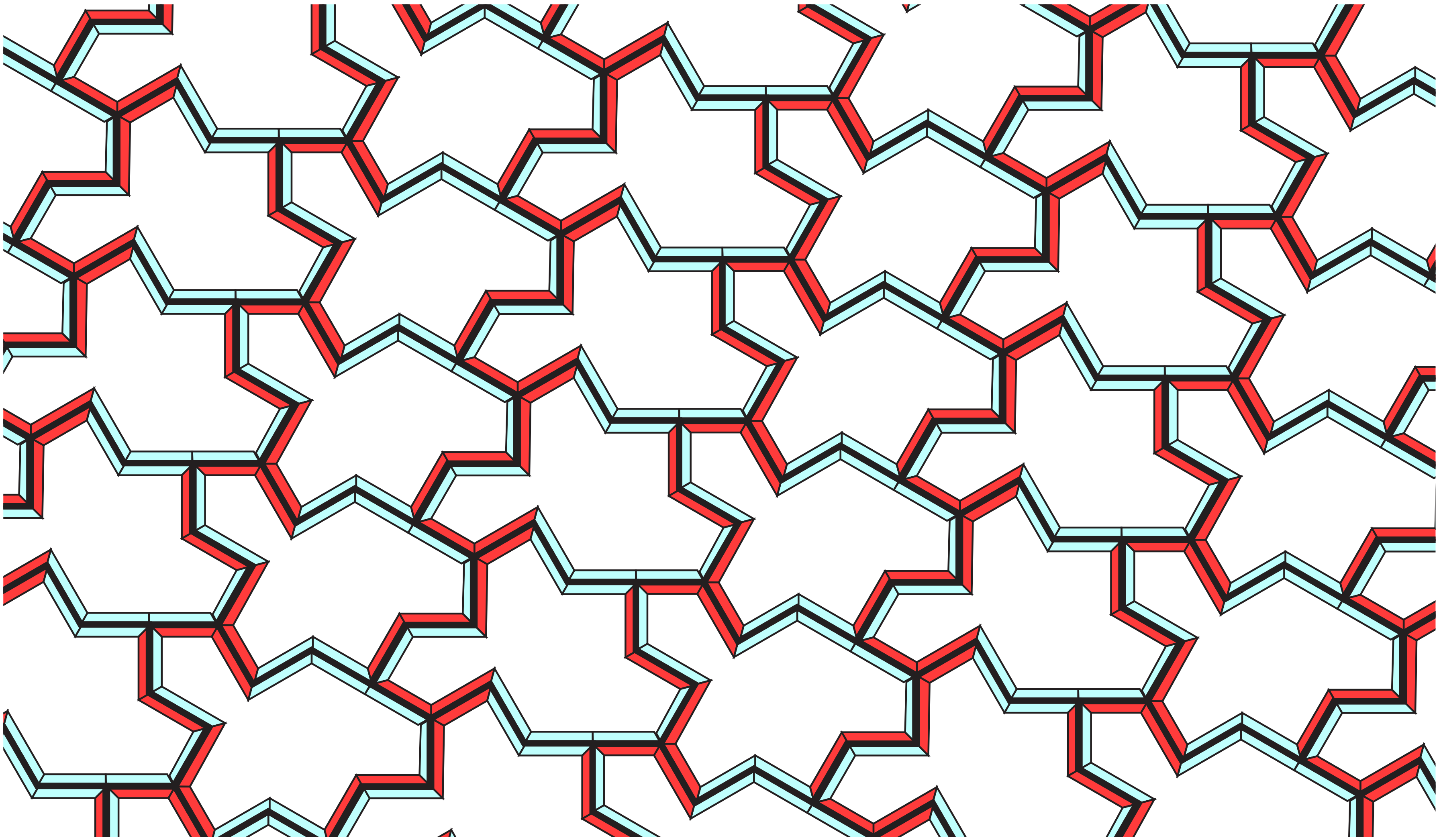
(iii) $b/a = \sqrt{3}$.

In the cases (i) and (ii) the long and short edges vanish, respectively. In the case (iii) the long and short edges become equally long.

• In all cases deforming a tiling by the hat becomes a tiling by the deformed tile.
$\Longrightarrow$ all deformed tiles admit monohedral tilings.

• Conversely, in all cases except (i), (ii) and (iii) in a valid tiling by the deformed tiles long edges meet long edges and short edges meet short edges. Then the tiling can be **inversely deformed** back to a tiling by the hat.
$\Longrightarrow$ the deformed tile does not admit a periodic tiling.

A periodic tiling when the long edges (red) and the short edges (blue) are equally long:



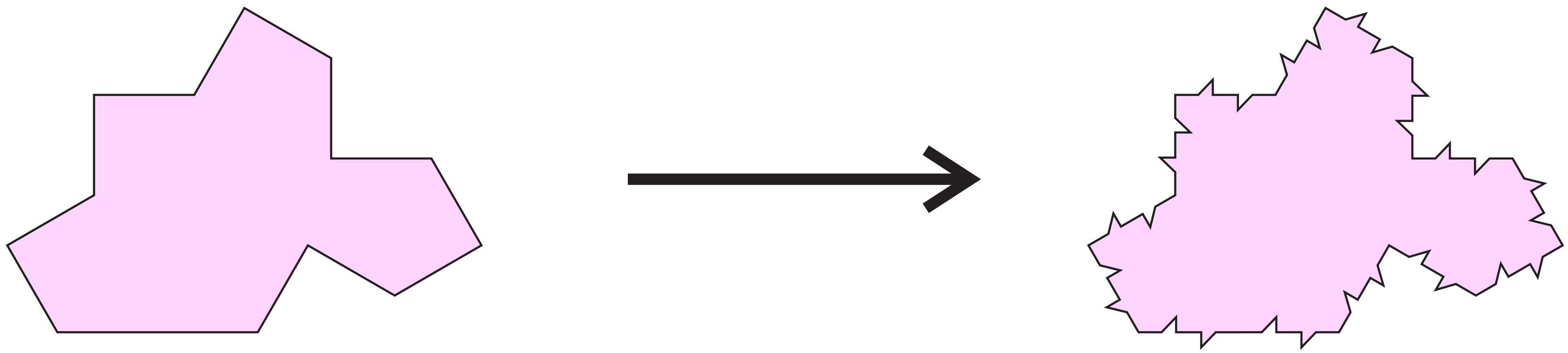
In the periodic tiling both even and odd orientations of the tile are used.

If only even orientations are allowed then the tile still tiles the plane but only non-periodically!

In the periodic tiling both even and odd orientations of the tile are used.

If only even orientations are allowed then the tile still tiles the plane but only non-periodically!

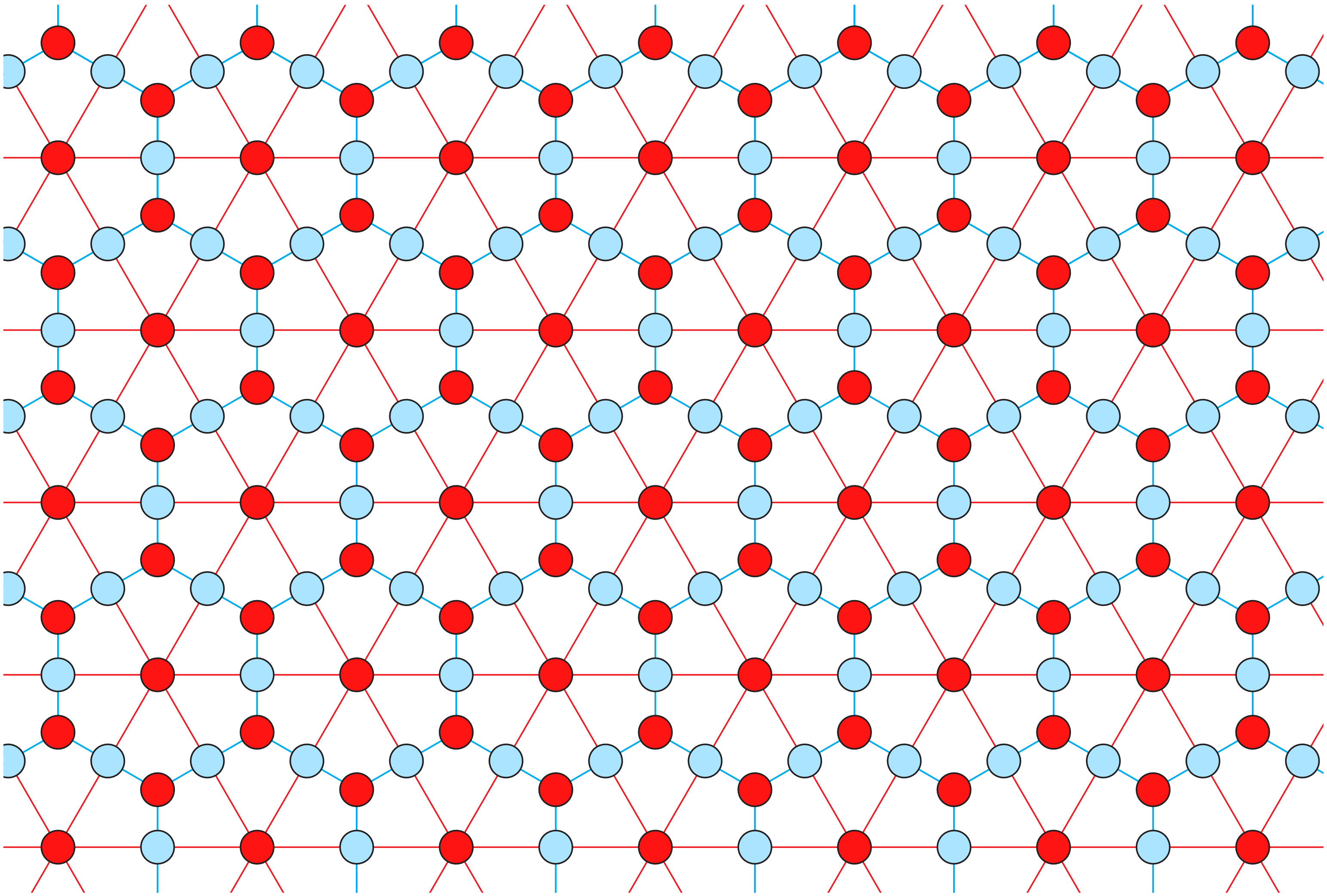
A bumps and dents construction can be used to enforce all tile orientation to have the same parity:



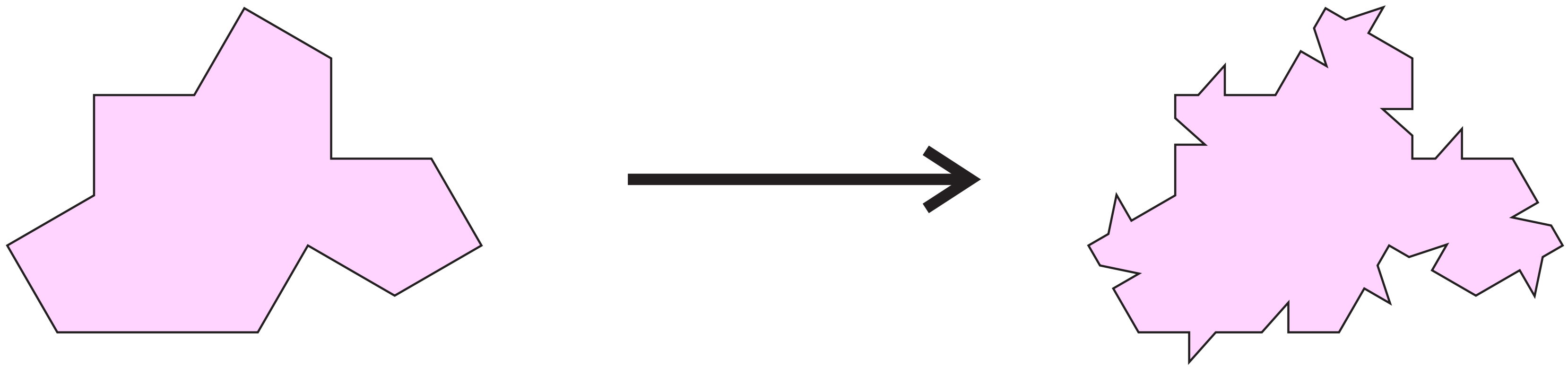
With this tile (**spectre**) then

- there exists a tiling,
- no tiling involves both even and odd variants of the tile,
- there is no valid periodic tiling.

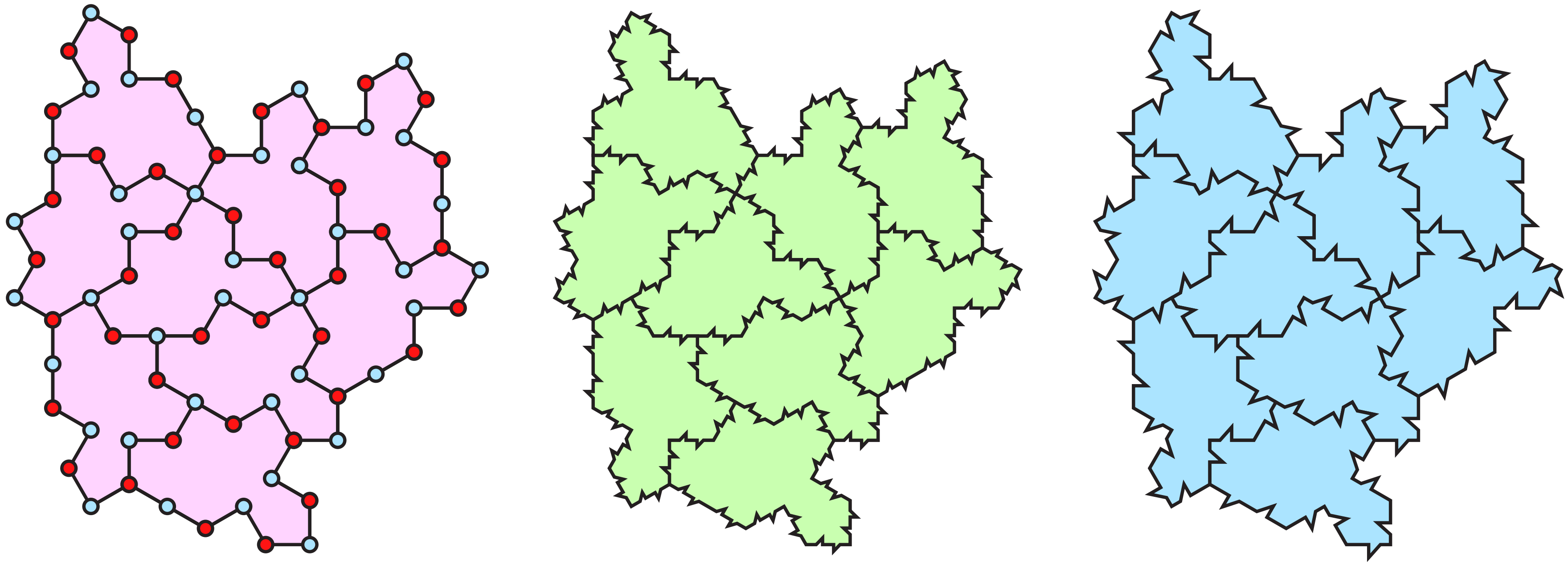
The grid of kites is bipartatite:



This allows to create a **spectre** with simpler bump/dent construction:



A patch tiled with the deformed kite (enforcing all tiles to have even orientations, bipartite coloring shown), and with the two types of **spectres**:



Open problems

Hat is a 13-gon.

Question. What is the smallest n such that there exists an aperiodic n-gon ?
Does there exist an aperiodic pentagon ?

(**Remrk.** Tilings by **convex pentagons** have been recently classified, and there is no single aperiodic convex pentagon.

The conversion

$$\text{Wang tiles} \longrightarrow \text{polygons}$$

with bumps and dents is effective (=algorithmic). The vertices of the resulting polygons can be taken to have rational number coordinates.

So, the undecidability results proved for Wang tiles hold for polygonal prototiles as well:

Theorem. The following decision problems are undecidable:

- "Does a given protoset of polygons with rational coordinates admit a periodic tiling ?",
- "Does a given protoset of polygons with rational coordinates admit a tiling ?".

Proof.

It is not known if there exists a decision algorithm to determine if a given **single polygonal prototile** admits a valid (periodic) tiling.

Question. Are the following decision problems decidable ?

- "Does given single polygon with rational coordinates admit a tiling ?"
- "Does given single polygon with rational coordinates admit a periodic tiling ?"

One may also consider similar questions for **polyominoes** (=tiles that are edge-to-edge attachments of unit squares to each other.)

Theorem. The tiling problem is undecidable for protosets of 5 polyominoes.

In particular, this also implies the undecidability of the tiling problem among sets of 5 polygons.

On the other hand, the tiling problem is known to be decidable for single polyominoes if only translations are allowed, that is, the tiles must be placed in the given orientation.

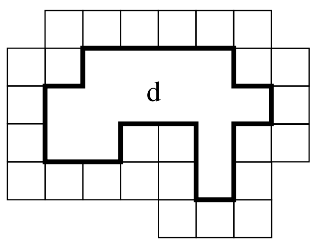
Here is a related question by H. Heesch. Consider a single prototile t that does not admit a tiling of the plane.

The **Heesch number** of t is the maximum number of times the tile can be completely surrounded by copies of t. More precisely, for a topological disk $d \subseteq \mathbb{R}^2$, a **corona** of d is a collection C of tiles, all congruent to t, such that

- the interiors of the elements of C are pairwise disjoint, and disjoint from d, and
- $d \cup \bigcup_{s \in C} s$ is a topological disk whose interior contains d.

In other words, tiles in the corona C surround set d completely.

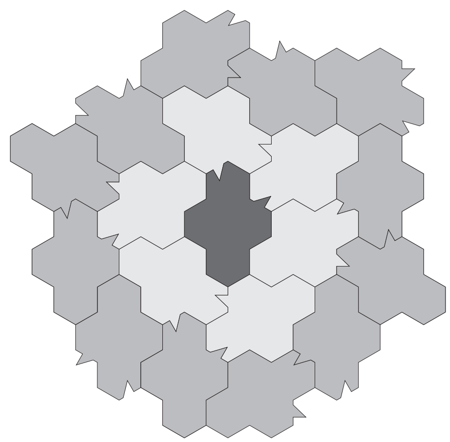
Example. The squares form a corona of the set d in the middle:



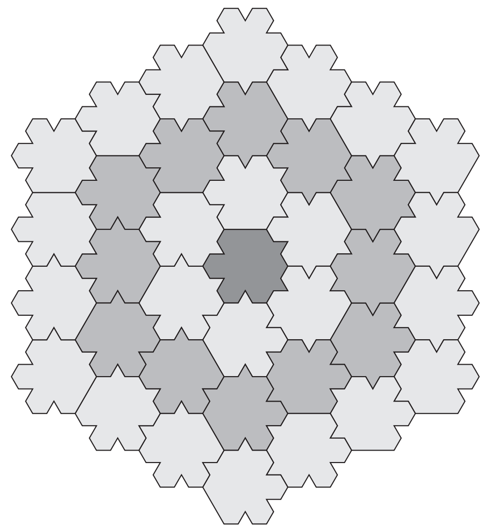
A a second corona of d is a corona of the set that is the union of d and its first corona. Inductively, a $k+1$'st corona is a corona of the topological disk formed by d and its first k coronas.

In the Heesch problem we start with a single copy of t and form its 1st, 2nd, 3rd, etc. coronas. If the k'th corona exists for every k then by the extension theorem the entire plane can be tiled. But if t does not admit a plane tiling then there exists the largest k such that the first k coronas exist. This k is called the **Heesch number** of tile t.

Example. The following figure illustrates two coronas of a tile:



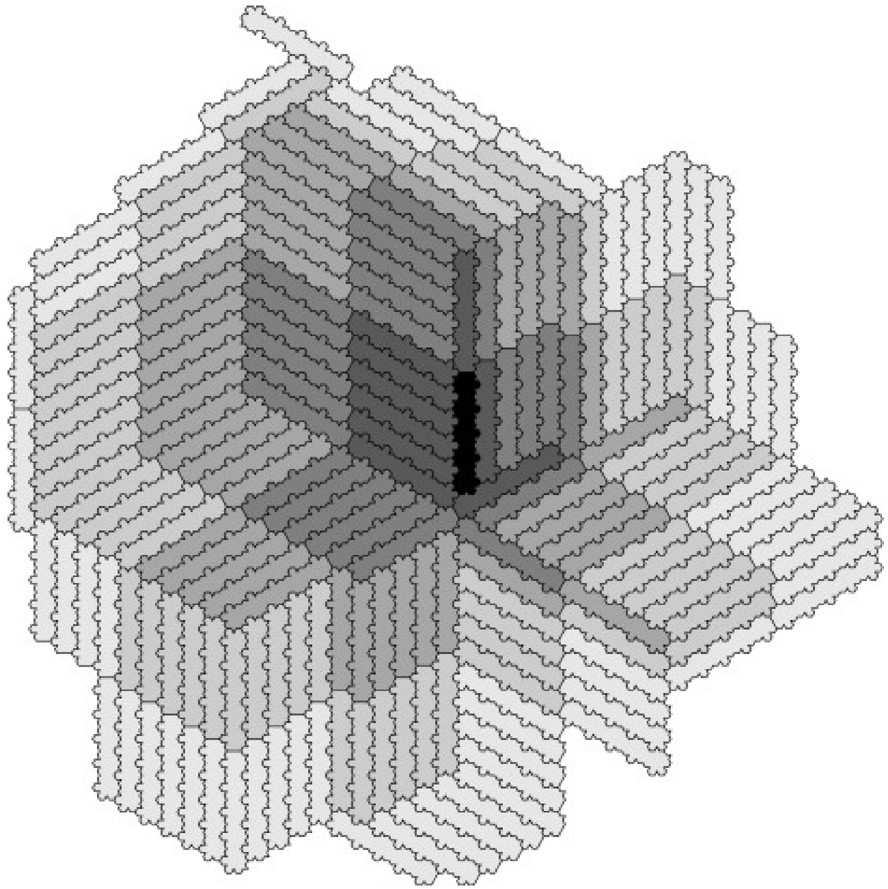
Example. A regular hexagon with incoming arrows on three sides and outgoing arrows on two sides admits three coronas:



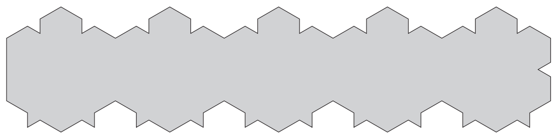
(In the picture, the arrows are represented by bumps and dents.)

Due to the imbalance in the number of incoming and outgoing arrows the full plane cannot be tiled by this tile. (Similar to a prof at the homeworks for Wang tiles.)

Example. Heesch number five (by Casey Mann):

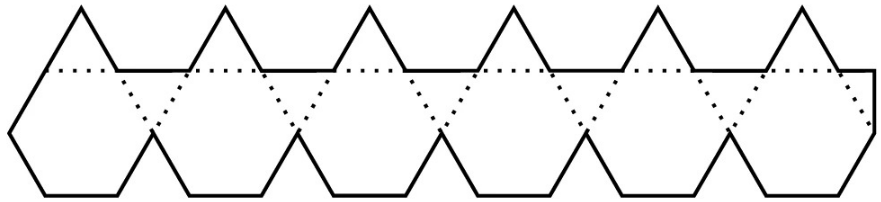


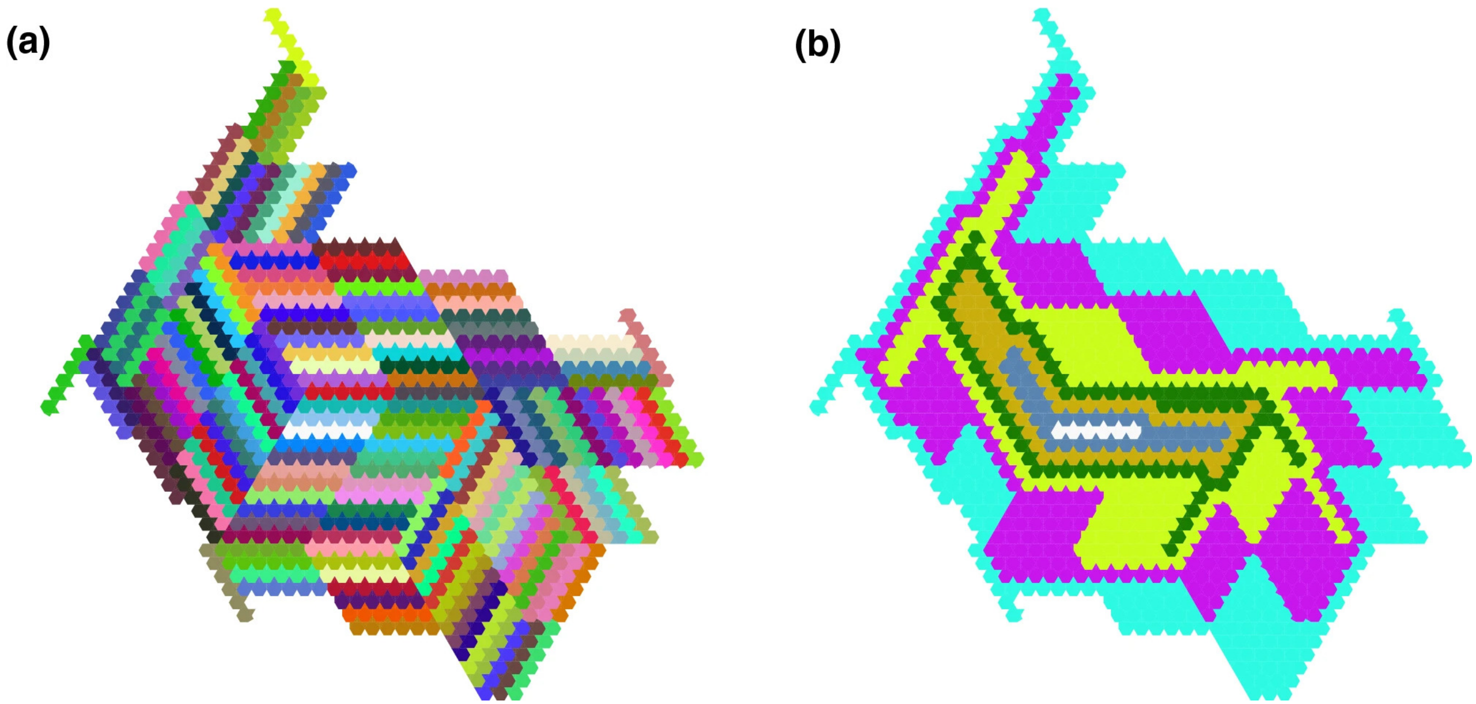
The tile consists of five regular haxagons glued together, with bumps and dents on 10 and 11 sides:



The imbalance in the number of bumps and dents guarantees that no valid tiling of the plane is possible.

Example. Recently, a tile with Heesch number 6 was published:





Heesch's question. Does there exist a number k such that the Heesch number of every tile that does not admit a tiling is at most k ? If such a k exists, what is the smallest such k ?

Note that if the Heesch numbers are bounded by some constant k then there is an **algorithm** (in any reasonable set-up such as edge-to-edge tilings by polygons where one can try all possible coronas) to determine if a given single tile admits a tiling: To test if a tiling exists, all we need to do is to try all possible ways of building $k+1$ coronas. A valid tiling exists if and only if $k+1$ coronas exist.