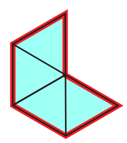
The grid of equilateral triangles is formed by equally spaced infinite lines in three directions, at 120° angles with each other. We number these directions 1, 2 and 3. For each direction $i \in \{1,2,3\}$ we call *i-lines* the infinite lines in direction i that together form the triangular lattice.

The grid of kites is in fact the dual of the $3 \cdot 4 \cdot 6 \cdot 4$ Archimedean tiling, obtained by joining the centers of adjacent tiles of the Archimedean tiling. A kite in the grid is a quadrilateral with two short edges (length 1) and two long edges (length $\sqrt{3}$). The long edges are along the i-lines, while the short edges are on the boundaries of the regular hexagons. The hat tile inherits these edge lengths. Note that when viewed as a polygon the hat has also one edge of length 2 (formed by two consecutive parallel short edges of kites), but in the following we consider this segment to consist of two edges of length 1. Thus the hat has 6 long edges and 8 short edges.

Let us read the edges as vectors, moving around the hat clockwise. We get 6 vectors of length $\sqrt{3}$, in three pairs of opposite vectors, and 8 vectors of length 1. The sum of the vectors is the zero vector (as the sum represents the vector leading from a vertex to itself). In fact, both the short and the long vectors separately sum up to zero:

- (H1) The sum of the six long vectors on the boundary of the hat is the zero vector, as is the sum of the eight short vectors.

Based on (H1) we can deform the hat tile as follows: keeping the orientations of the edges unchanged, scale the lengths of all long edges by some constant $a \geq 0$ and the lengths of all short edges by some constant $b \geq 0$, with $a \neq 0$ or $b \neq 0$. The scaled long edge vectors naturally still sum up to zero, and the scaled short edge vectors sum up to zero as well. The scaled edges define a boundary of a deformed tile. In our proof below we actually only need the scaling factors $a = 1$ and $b = 0$, *i.e.*, we remove the short edges. This results in the deformed tile that is the *chevron*



This is a hexagon with three pairs of equally long and parallel sides. It is also a union of four equilateral triangles with sides $\sqrt{3}$. As the area of each equilateral triangle is the same as the area of a kite in the grid of the hat, we see that

- (H2) the area of the hat is precisely twice the area of the chevron.

Let us argue next that deforming the tiles as above in a valid hat tiling produces a corresponding valid tiling by the deformed tiles. So let $\mathcal{T}$ be a valid monohedral tiling using the hat. Let $\mathcal{V} \subseteq \mathbb{R}^2$ be the set of vertices of $\mathcal{T}$, and let us assume, without loss of generality, that $\vec{0} \in \mathcal{V}$. Let us determine how the tile deformation moves an arbitrary vertex $\vec{v} \in \mathcal{V}$.

A *path* p is a sequence $\vec{v}_0, \vec{v}_1, \ldots, \vec{v}_n \in \mathcal{V}$ of vertices such that for each $i \in \{1,2,\ldots n\}$ some tile in $\mathcal{T}$ has an edge from $\vec{v}_{i-1}$ to $\vec{v}_i$. The path thus tracks edges of the tiles. Denote $\vec{a}_i = \vec{v}_i - \vec{v}_{i-1}$, and let $L \subseteq \{1,2,\ldots n\}$ be the set of indices such that $\vec{a}_i$ is long (of length $\sqrt{3}$, that is), and let $S = \{1,2,\ldots n\} \setminus L$ be the index set of the short edges (of length 1). If $\vec{v}_n = \vec{v}_0$ then the path is a *cycle*. If, moreover, $\vec{v}_i \neq \vec{v}_j$ for all $i \neq j$ except when $\{i,j\} = \{0,n\}$ then the cycle is *simple*. For each cycle it is clear that

$$\sum_{i=1}^{n} \vec{a}_i = \vec{0},$$

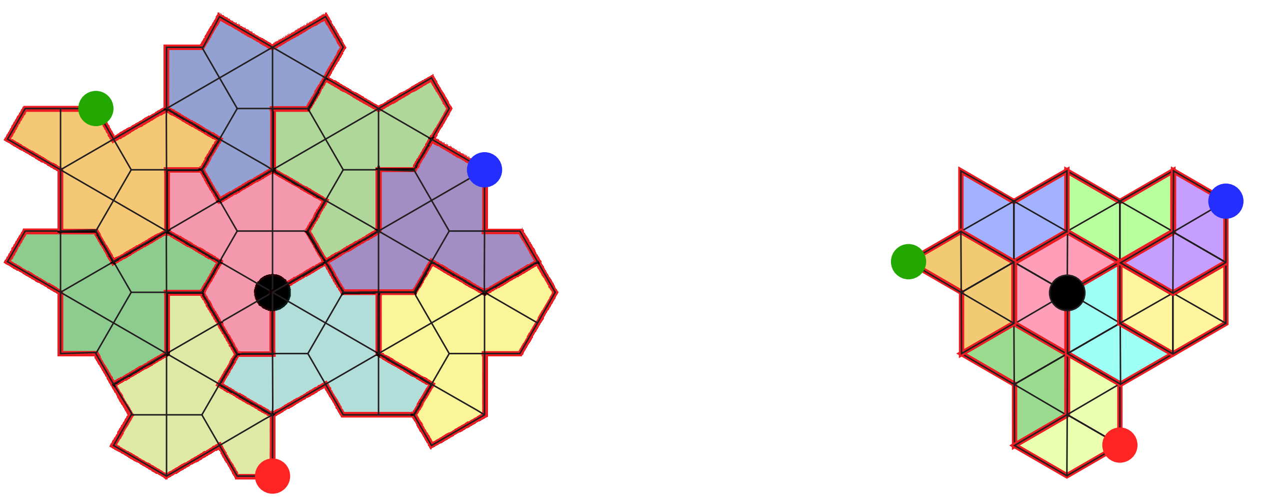
but in fact we also have the property that

$$\sum_{i \in L} \vec{a}_i = \sum_{i \in S} \vec{a}_i = \vec{0}. \tag{2}$$

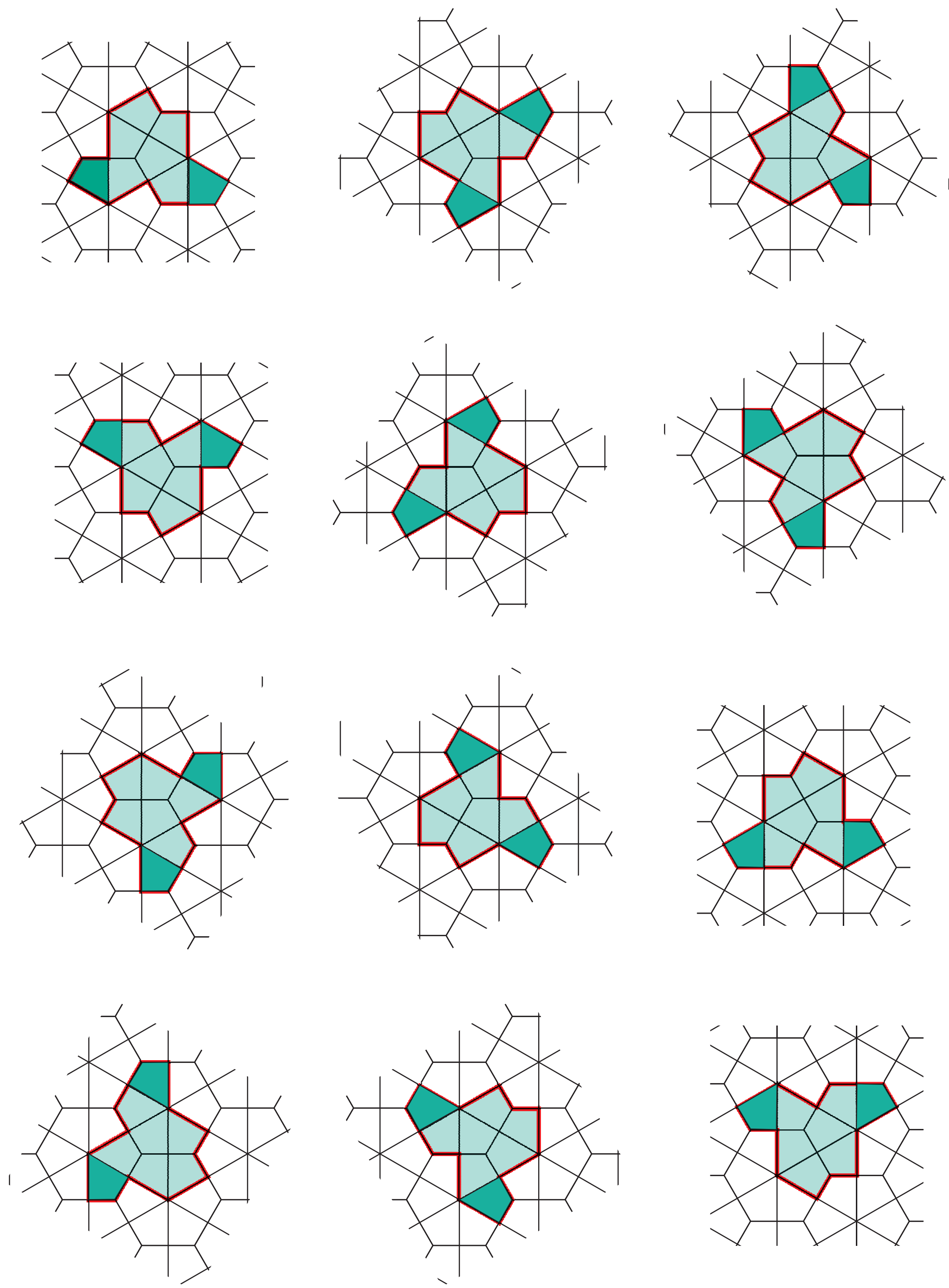
To see this, it is enough to focus on simple cycles since every cycle breaks into a union of simple cycles. Each simple cycle is the boundary of a topological disk that is a union of the tiles of a finite subset $\mathcal{F} \subseteq \mathcal{T}$, and without loss of generality we may assume that the cycle is oriented clockwise. We use induction on the number $k = |\mathcal{F}|$ of enclosed tiles. The claim is trivial if $k = 0$. Assume then $k > 1$, and let $t \in \mathcal{F}$ be an enclosed tile whose edge is on the path. Any edge of t that is tracked by the path is tracked in the clockwise orientation. Let $\vec{b}_1, \ldots, \vec{b}_{14}$ be the edge vectors of t in the opposite, counter clockwise direction. By (H1), both the long vectors and the short vectors among $\vec{b}_j$ add up to zero. We merge the collections of vectors $\vec{a}_i$ and $\vec{b}_j$. For each edge of t that is on the path there is the opposite vector among $\vec{b}_j$. Cancelling such pairs, the remaining vectors can be grouped into simple cycles that enclose subsets of $\mathcal{F} \setminus \{t\}$. By the inductive hypotheses the sums of the long and the short vectors among them both sum up to zero. So overall the sums of the long and the short vectors among $\vec{a}_i$ and $\vec{b}_j$ add up to zero, which implies the claimed equality (2).

It now follows that scaling the long and the short edges of the hat by factors $a \geq 0$ and $b \geq 0$, respectively, transforms the tiling $\mathcal{T}$ into a tiling $\mathcal{T}'$ by the transformed tiles: Any vertex $\vec{v} \in \mathcal{V}$ is reached from $\vec{0}$ by a path $p = \vec{v}_0, \vec{v}_1, \ldots, \vec{v}_n$ in $\mathcal{T}$ where $\vec{v}_0 = \vec{0}$ and $\vec{v}_n = \vec{v}$. The corresponding vertex $\vec{v}'$ in $\mathcal{T}'$ is reached from $\vec{0}$ by the path p' obtained from p by scaling the long and short edges of the path by factors a and b, respectively. This results in the same $\vec{v'}$ regardless of the choice of p: If q is another path from $\vec{0}$ to $\vec{v}$ then the path p followed by q reversed is a cycle, and as proved above, the long and the short edges of the cycle sum up to zero. Thus the scaled long and short edges also sum up to zero, and therefore the paths p' and q' obtained by scaling p and q lead from $\vec{0}$ to the same point $\vec{v'}$. Thus the deformed tiling is well-defined.

The following picture illustrates a piece of a hat tiling and the corresponding tiling by the chevrons. The figure also shows four sample vertices of the hat tiling and the corresponding vertices in the chevron tiling.



Skipping the detailed proof, we note that all valid hat tilings of $\mathbb{R}^2$ must be such that the copies of the hat are aligned on the grid of kites. So in the following we only consider tilings where the hats are aligned on a given fixed grid of kites. This means that there are 12 possible orientations of the hat: six obtained by rotating the hat, and another six obtained by rotating the reflected hat:

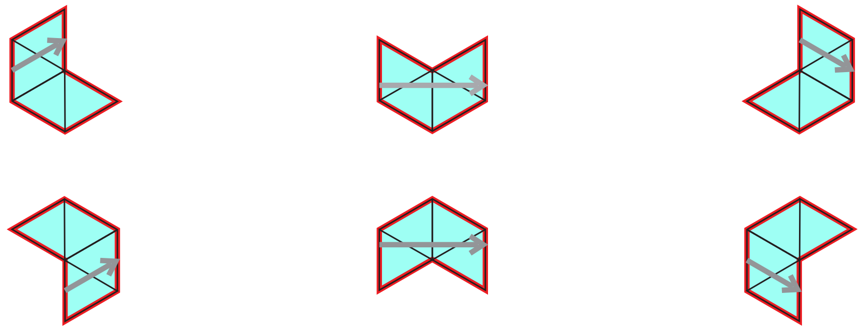


A notable property of the hat tile is that while the kites in the hat come in all 6 available orientations, one opposite pair of kite orientations is used twice. (In the illustration above these excess kites are indicated in darker color.) Contrasting this imbalance with the fact that the grid itself has kites in all six orientations in equal proportions, we see that in valid tilings there are three groups of four orientations of the hat (the columns of the illustration above) such that

- (H3) in a valid tiling, one third of the hats come oriented as in each of the three column in the picture above.

Recall that the long edges of hats are along i-lines for $i \in \{1, 2, 3\}$. Observe that the pairs of parallel long edges of the hat are always on consecutive i-lines. These lines are are at distance 3 from each other. Denote $D = \frac{3}{2}$ so that the i-lines repeat with distance $2D$.

Deforming the hat into a chevron preserves the orientations of the long edges. From the 12 possible orientations of the hat we obtain only six different orientation of the chevron, due to the reflection symmetry of the chevron:



The three columns in this picture correspond to the three columns in the picture of hat orientations before. It follows that

- (H4) a chevron tiling that is obtained by deforming a hat tiling has one third of its chevrons from each of the three columns above.

The chevrons are also aligned with a grid of equilateral triangles, formed by three families of parallel lines in the same directions as the i-lines in the grid of kites. We also call these i-lines, for $i \in \{1, 2, 3\}$. The sides of the triangles are of length $\sqrt{3}$, and so the consecutive i-lines of chevrons are at distance $\frac{3}{2}$ from each other,*i.e.*, the distance is D, half of the distance of i-lines of hats.

Consider, for example, the pair of vertical ($i = 1$) edges of a chevron: if oriented as in the first column above, the edges are on consecutive 1-lines but the rightmost edge is higher (by $\sqrt{3}/2$) than the leftmost edge. (See the grey vector in the picture above.) In the third column the rightmost edge is lower by the same amount. In the middle column the vertical edges are on the same height but they are not on consecutive 1-lines, but at double distance $2D$. We have that, on the average, the vertical position of the two vertical edges is the same, and the average horizontal distance between them is $\frac{4}{3}D$. Due to symmetry, edges in the other two directions behave similarly.

In the following we prove that the hat tile does not admit a two-periodic tiling. Assume the contrary: suppose $\mathcal{T}$ is a two-periodic tiling using the hat. Let $\vec{p}$ and $\vec{q}$ be generators of the periods of $\mathcal{T}$, meaning that $i\vec{p} + j\vec{q}$ are precisely its vectors of periodicity, for $i, j \in \mathbb{Z}$. Denote $\mathcal{P} = \mathbb{Z}\vec{p} + \mathbb{Z}\vec{q}$ for the set of the periods of $\mathcal{T}$. Let $\mathcal{V} \subseteq \mathbb{R}^2$ be the set of vertices of $\mathcal{T}$, and assume that $\vec{0} \in \mathcal{V}$. Note that

$$\vec{v} \in \mathcal{V} \Longleftrightarrow \vec{v} + \vec{p} \in \mathcal{V} \Longleftrightarrow \vec{v} + \vec{q} \in \mathcal{V},$$

which, together with $\vec{0} \in \mathcal{V}$, implies that $\mathcal{P} \subseteq \mathcal{V}$.

We deform the hat and the periodic tiling $\mathcal{T}$ by scaling factors $a = 1$ and $b = 0$, as discussed above. Let $\mathcal{T}'$ be the deformed tiling (by chevrons) and let $\mathcal{V}'$ be its vertex set. Let $f : \mathcal{V} \longrightarrow \mathcal{V}'$ give for each vertex the corresponding vertex in the deformed tiling, *i.e.*, if p is a path in $\mathcal{T}$ from $\vec{0}$ to $\vec{v} \in \mathcal{V}$ then the corresponding deformed path p' leads in $\mathcal{T}'$ from $\vec{0}$ to $f(\vec{v})$. (Note that f is not one-to-one: removing short edges merges the vertices they connect.)

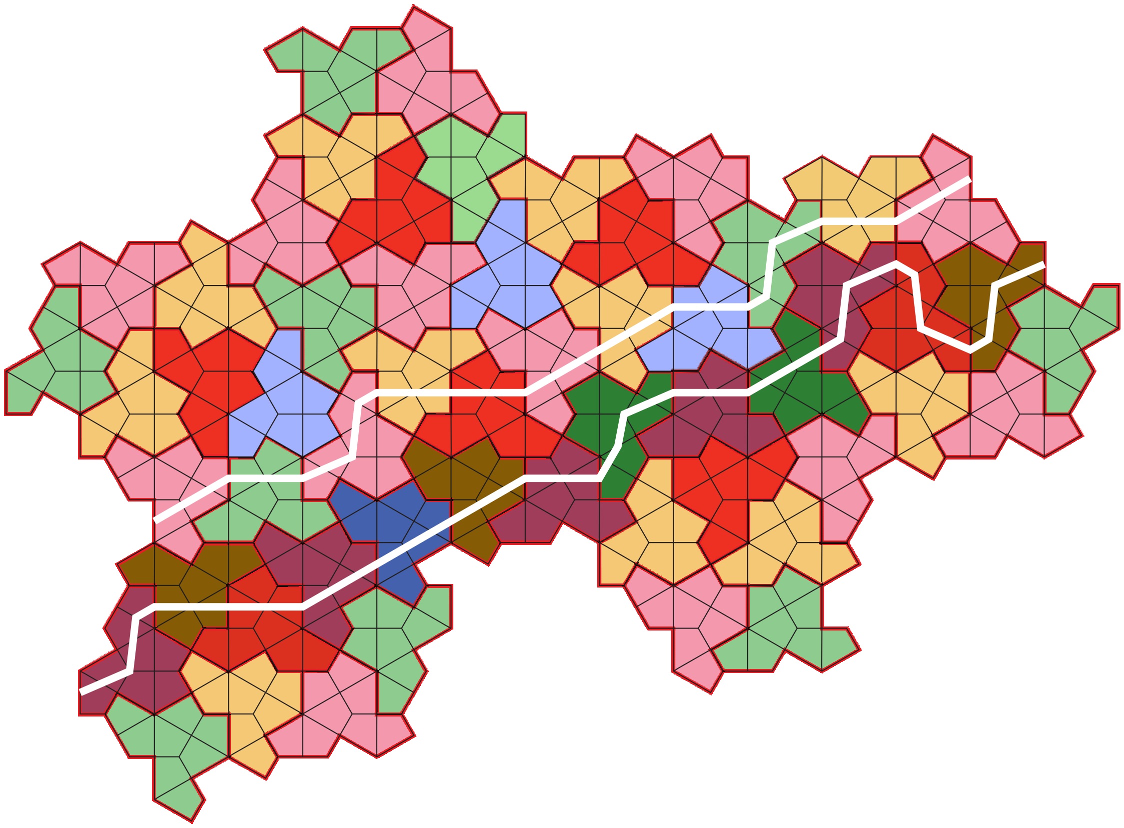
For any $\vec{x} \in \mathcal{V}$, choose a path $p_{\vec{x}}$ in $\mathcal{T}$ from $\vec{0}$ to $\vec{x}$ and let $p'_{\vec{x}}$ be the corresponding deformed path in $\mathcal{T}'$ from $\vec{0}$ to $f(\vec{x})$. Let $\vec{v} \in \mathcal{P} \subseteq \mathcal{V}$ be any period of $\mathcal{T}$, and let $\vec{x} \in \mathcal{V}$ be arbitrary. Since $\vec{v} \in \mathcal{P}$, there is a path q in $\mathcal{T}$ from $\vec{v}$ to $\vec{v} + \vec{x}$ that is a translation by $\vec{v}$ of the path $p_{\vec{x}}$. The path $p_{\vec{v}}$ followed by the path q is a path from $\vec{0}$ to $\vec{v} + \vec{x}$ whose deformation leads in $\mathcal{T}'$ from $\vec{0}$ to $f(\vec{v}) + f(\vec{x})$. We see that $f(\vec{v} + \vec{x}) = f(\vec{v}) + f(\vec{x})$. In particular, we have that $f(\vec{v})$ is a period of $\mathcal{T}'$ since for any vertex $f(\vec{x}) \in \mathcal{V}'$ also $f(\vec{v}) + f(\vec{x})$ is in $\mathcal{V}'$.

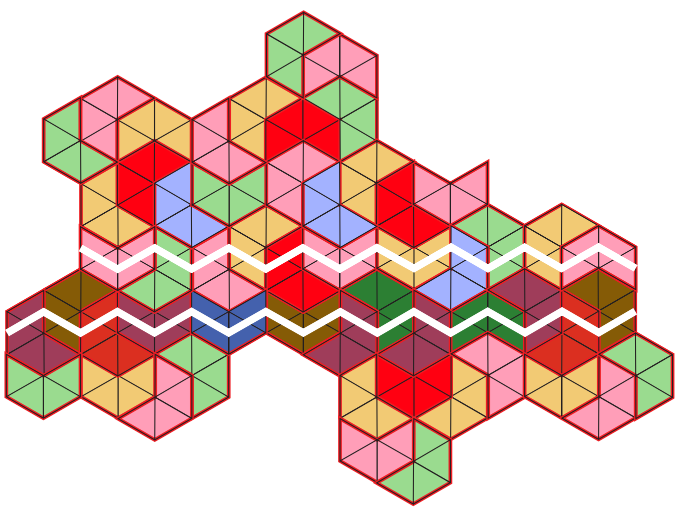
Moreover, it follows that $f(i\vec{p} + j\vec{q}) = if(\vec{p}) + jf(\vec{q})$ for all $i, j \in \mathbb{Z}$ where $\vec{p}, \vec{q}$ are the generating periods of $\mathcal{T}$. This means that f is linear among the periods. Let us denote by $\hat{f} : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ the unique linear map that coincides with f on $\mathcal{P}$, that is, the unique linear function that maps $\vec{p} \mapsto f(\vec{p})$ and $\vec{q} \mapsto f(\vec{q})$. Note that f and $\hat{f}$ do not need to coincide on $\mathcal{V} \setminus \mathcal{P}$. The linear function $\hat{f}$ is one-to-one: it has full rank since otherwise $f(\mathcal{P})$ would be on a single line, which is clearly not the case. (If $f(\mathcal{P}) \subseteq \ell$ for some line ℓ, then for any two hats in $\mathcal{T}$ that are transitive under some translational symmetry of $\mathcal{T}$ the corresponding deformed tiles in $\mathcal{T}'$ are on a common line parallel to ℓ. There are only finitely many transitivity classes under translational symmetries in $\mathcal{T}$ but clearly there is an infinite set of chevrons in $\mathcal{T}'$ that are pairwise not on such a common line, a contradiction.)

Our next goal is to prove that the linear function $\hat{f}$ is a similarity, *i.e.*, distances of points are scaled by the same constant: we argue that there exists a constant c such that for any $\vec{x}, \vec{y} \in \mathbb{R}^2$ it holds that $d(\hat{f}(\vec{x}), \hat{f}(\vec{y})) = d(\vec{x}, \vec{y})/c$.

Consider any of the three directions $i \in \{1, 2, 3\}$. A *de Bruijn segment* on a hat tile in direction i is a line drawn inside the tile connecting the centers of the two long edges of the hat that are parallel to i-lines. Similarly we define de Bruijn segments of chevrons. There is one de Bruijn segment drawn in each tile in each direction i. Here are examples in the vertical direction $i = 1$:

In the valid tiling $\mathcal{T}$, the de Bruijn segments of two tiles sharing a long edge continue across that edge, thus defining infinite *de Bruijn lines* in directions i across the entire tiling. Each tile is crossed by a unique de Bruijn line in each direction i, and the lines in the same direction do not cross each other. Let us call the set of tiles on the same de Bruijn line in direction i an *i-strip*. Similarly we define de Bruijn lines and i-strips in tilings by chevrons. The following picture shows a patch of a tiling by hats, and two de Bruijn lines across the patch in the vertical direction $i = 1$. Tiles of the i-strip of the lower line are rendered dark. The lower patch is the same drawing in the corresponding tiling by chevrons.





Note that any translational symmetry τ of $\mathcal{T}$ must map i-strips onto i-strips, preserving the order of tiles on the strips: if $t_2 = \tau(t_1)$ then the tile following (preceding) t_1 on its i-strip will be mapped on the tile following (preceding, resp.) t_2 on its i-strip.

Fix direction i and consider one i-strip. The strip contains infinitely many tiles. Because $\mathcal{T}$ is two-periodic, there are only finitely many tiles that are not transitive under translational symmetries of $\mathcal{T}$. Thus the strip contains distinct tiles t_1, t_2 such that $t_2 = \tau(t_1)$ for some translational symmetry τ of $\mathcal{T}$. As noted above, this means that the strip is mapped by τ onto itself. As τ cannot change the order of i-strips, it follows that every i-strip is mapped onto itself by τ.

A similar reasoning works with all directions $i \in \{1, 2, 3\}$. For each i, let $\vec{u_i}$ be a unit vector parallel to i-lines, chosen so that $\vec{u_1}$, $\vec{u_2}$ and $\vec{u_3}$ are at $120°$ angles to each other. For each i, let $\vec{v_i}$ be a vector of length $2D$, perpendicular to i-lines, again chosen so that $\vec{v_1}$, $\vec{v_2}$ and $\vec{v_3}$ are at $120°$ angles to each other, *i.e.*, so that $\vec{v_1} + \vec{v_2} + \vec{v_3} = \vec{0}$.

As seen above, for each $i \in \{1,2,3\}$ there is a translational symmetry of $\mathcal{T}$ that maps i-strips onto themselves. Let such a translation in direction $i \in \{1,2,3\}$ be by the vector $\vec{p_i} = x_i\vec{v_i} + y_i\vec{u}_i$. Here x_i is a positive integer indicating how many tiles the translation moves forward on an i-strip. As multiples of periodicity vectors are also periodicity vectors, we can choose the periods $\vec{p_i}$ so that $x_1 = x_2 = x_3$ is the same positive integer k.

Consider then the i-strips on the corresponding chevron tiling $\mathcal{T}'$. These are precisely the i-strips of $\mathcal{T}$ after the deformation. The translation by vector $f(\vec{p_i}) = \hat{f}(\vec{p_i})$ is a symmetry of $\mathcal{T}'$ that maps each i-strip onto itself: it shifts by k the tiles within each i-strip. Averaging over all finitely many i-strips with distinct translational transitivity classes we see – based on observation (H4) – that the vector $\hat{f}(\vec{p_i})$ must be perpendicular to the i-lines, and its length is $k \cdot \frac{4}{3}D$. The length is independent of i, and the directions for $i \in \{1,2,3\}$ are at 120° angles to each other, implying that $\hat{f}(\vec{p_1}) + \hat{f}(\vec{p_2}) + \hat{f}(\vec{p_3}) = \vec{0}$. Linearity of $\hat{f}$ thus means that $\hat{f}(\vec{p_1} + \vec{p_2} + \vec{p_3}) = \vec{0}$, and by the injectivity of $\hat{f}$ then $\vec{p_1} + \vec{p_2} + \vec{p_3} = 0$.

On the other hand, as $\vec{p_i} = k\vec{v_i} + y_i\vec{u}_i$ and $\vec{v_1} + \vec{v_2} + \vec{v_3} = \vec{0}$, we have that $y_1\vec{u}_1 + y_2\vec{u}_2 + y_3\vec{u}_3 = \vec{0}$. Vectors $\vec{u}_i$ are unit vectors at angles 120° to each other, so that we must have $y_1 = y_2 = y_3$. In conclusion, vectors $\vec{p_i}$ have equal lengths and they are at 120° angles to each other, and their images $\hat{f}(\vec{p_i})$ under $\hat{f}$ have also equal lengths and they are at 120° angles to each other. This implies that $\hat{f}$ is a similarity map, i.e., an isometry followed by scaling by some constant $1/c$.

Using the fact that the area of the hat tile is twice the area of the chevron, we can even conclude the value of the similarity factor to be $c = \sqrt{2}$. Indeed, if there are m translational transitivity classes of hats in tiling $\mathcal{T}$, then the area of the parallelogram with sides $\vec{p}$ and $\vec{q}$ is m times the area of the hat tile. Based on $\mathcal{T}'$ then the area of the parallelogram with sides $\hat{f}(\vec{p})$ and $\hat{f}(\vec{q})$ is m times the area of the chevron tile. As $\hat{f}$ is a similarity with scaling c, the ratio of the areas of the parallelograms is c^2, while the ratio of the area of a hat to the area of a chevron is 2. This gives $c^2 = 2$.

Finally we show that scaling by $c = \sqrt{2}$ is not possible. Note that the periodicity vector $\vec{p}$ of $\mathcal{T}$ is a vector between two vertices of the underlying triangle grid formed by i-lines (as the translational symmetry is also a translational symmetry of the underlying grid of kites, so it maps sharp ends of kites to sharp ends – and the sharp ends are located at the vertices of the triangle grid.) On the other hand, the corresponding vector $\vec{p}' = \hat{f}(\vec{p})$ in $\mathcal{T}'$ is a vector between two vertices of the triangular grid of the chevron tiling. The triangles in the first grid have sides twice as long as in the second grid, so they can be subdivided into smaller triangles to get the second grid. Both vectors $\vec{p}$ and $\vec{p}'$ are thus connecting vertices of the same grid of small triangles. Due to the similarity scaling, the lengths of the vectors are related by $|\vec{p}| = \sqrt{2}|\vec{p}'|$.

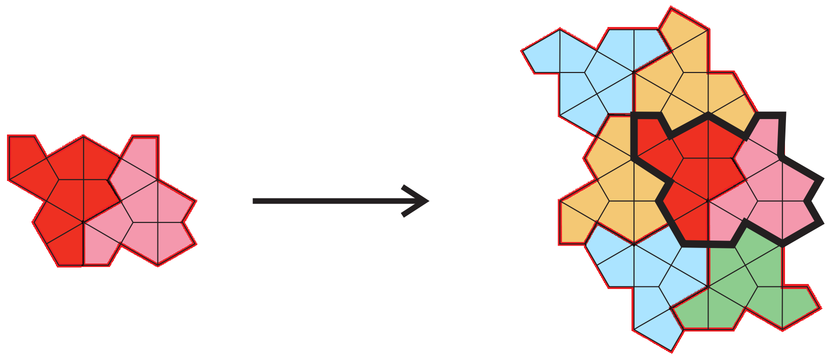
Let us prove that the length ratio $\sqrt{2}$ is not possible between two non-zero vectors connecting vertices of the same triangular grid. For the simplicity of notations, consider the grid of equilateral triangles containing vertices $(0,0)$ and $(1,0)$. The vertices of the grid have coordinates $i(1,0) + j(1/2, \sqrt{3}/2)$ for $i,j \in \mathbb{Z}$. If d is the distance of such a vertex from $(0,0)$ then $d^2 = (i+j/2)^2 + (j\sqrt{3}/2)^2 = i^2 + j^2 + ij$. Thus the squares of the lengths of vectors between vertices come from the set $S = \{i^2+j^2+ij \mid i,j \in \mathbb{Z}\}$. Let us show that for any non-zero $s \in S$ one has $2s \notin S$. If i is odd or j is odd (or both are odd) then $s = i^2 + j^2 + ij$ is odd. If i and j are both even then dividing both by 2 gives that $s/4 \in S$. This means that any even $s \in S$ is divisible by 4, and then $s/4 \in S$. Repeatedly dividing any even $s \in S$ by 4 one hence eventually reaches an odd number. In conclusion: The largest power of 2 that divides any non-zero $s \in S$ is even, and therefore $2s \notin S$.

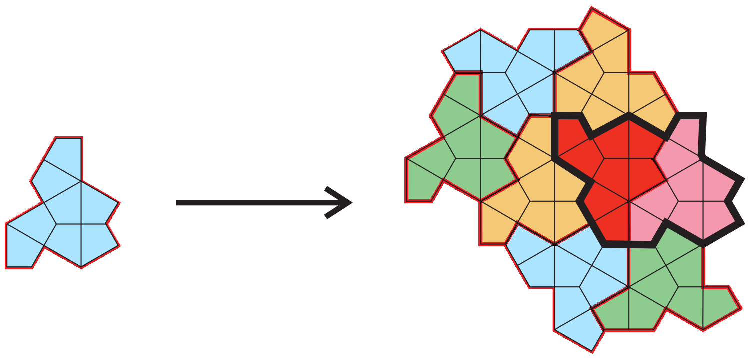
This shows that $|\vec{p}|^2 = 2|\vec{p}'|^2$ is not possible, and we have the following:

Theorem 7.6 *The hat tile does not admit any two-periodic tiling.*

By Theorem 7.5 (the periodicity theorem), the hat tile does not even admit a one-way periodic tiling.

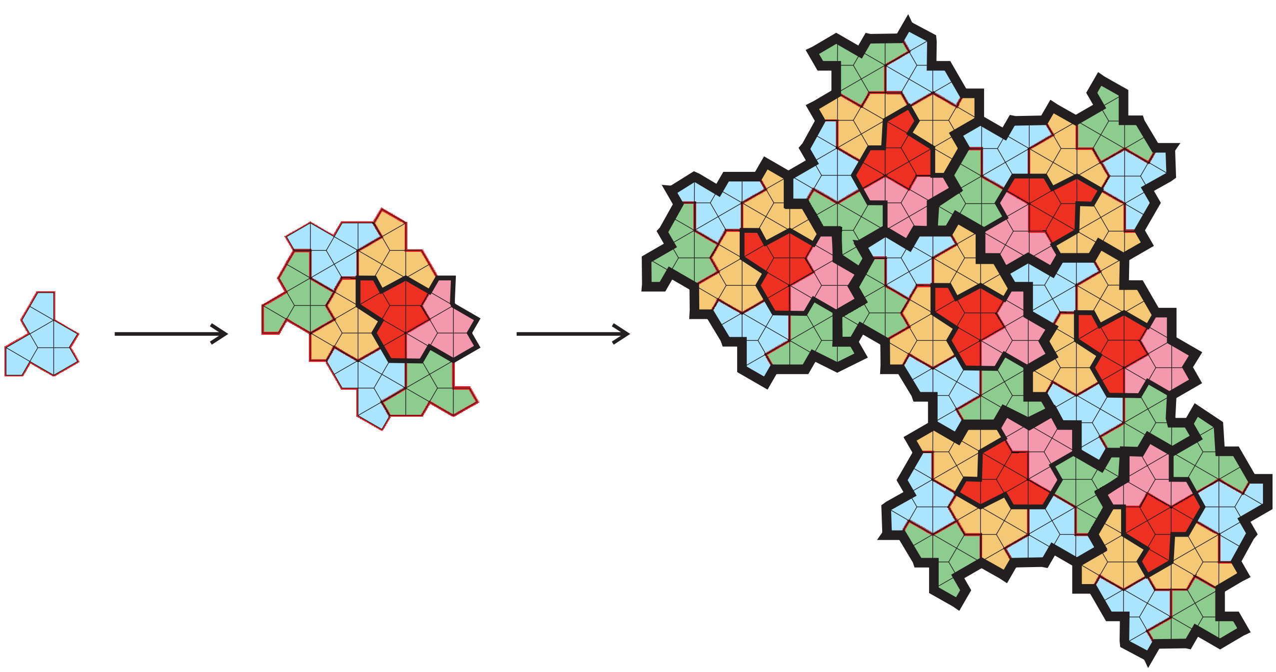
We skip the details of the proof that the hat tile admits a tiling. One can generate a tiling as follows. Define first patch P of two tiles by attaching the hat tile and the flipped over variant of the hat (upper left patch on the figure below). Let Q be a patch consisting of a single hat tile (lower left patch). Then define a substitution on patches P and Q that replaces each P and Q by super-P and super-Q, respectively (upper right and lower right in the figure):





Both supertiles contain one copy of P, shown in red/pink in the figure, and several patches Q. None of the patches are flipped over, so the red tile inherited from Q is the only tile of the super patches that is a flipped over variant of the hat.

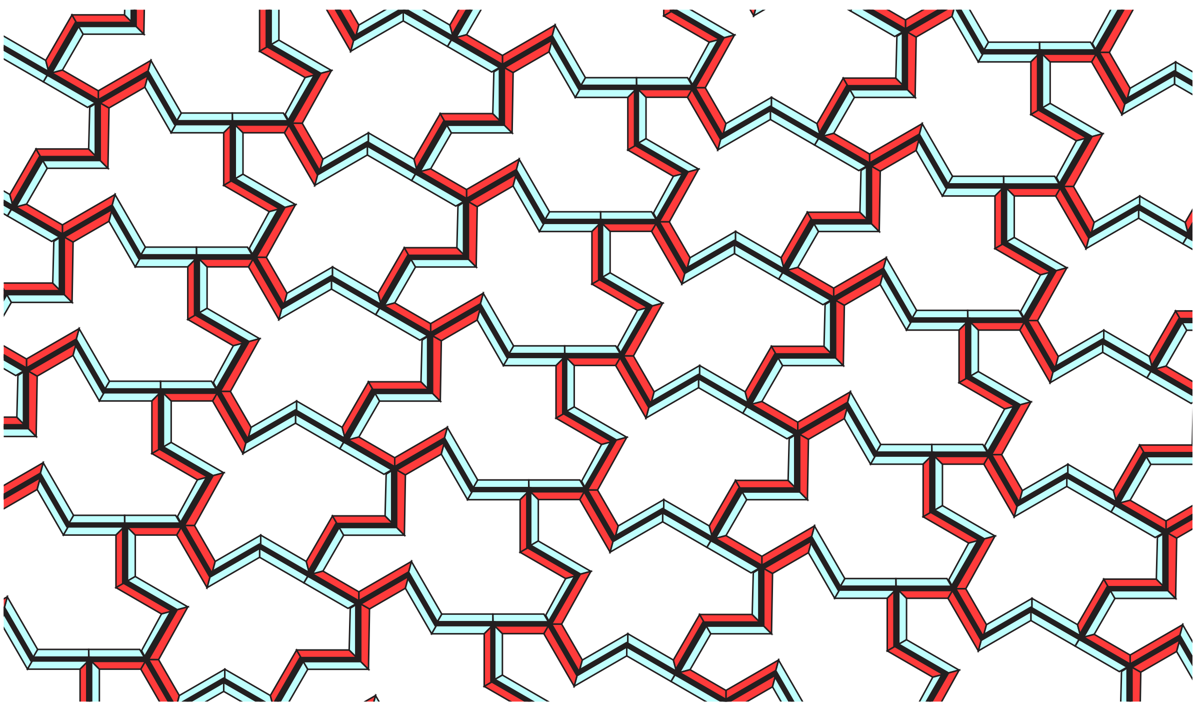
This substitution can be iterated, thus obtaining as a limit a tiling of the plane by hats. The following picture shows the creation of the second level super-P patch:



Note that the tiles in dark red are the only flipped over hat tiles. The fact that the hat tile does not admit a periodic tiling (Theorem 7.6) can also be proved analogously to our previous substitution examples: any valid tiling can be uniquely decomposed into super-patches. There are several cases to consider and we skip the proof.

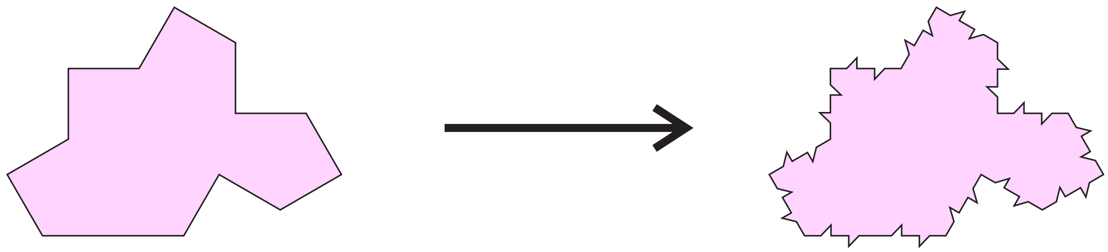
Some final remarks: Deforming the hat with scaling factors $a \geq 0$ and $b \geq 0$ on long and short edges produces "equivalent" tiles that are also aperiodic, except when (i) $a = 0$, or (ii) $b = 0$ or (iii) $b/a = \sqrt{3}$. In cases (i) and (ii) the long and short edges vanish, respectively, and in case (iii) the long and short edges become equally long. In all cases deforming a tiling by the hat becomes a tiling by the deformed tile, so all deformed tiles admit monohedral tilings. Conversely, in all cases except (i), (ii) and (iii) a valid tiling by the deformed tiles is necessarily edge-to-edge in such a manner that long edges meet long edges and short edges meet short edges. In this situation the inverse deformation can be done back from the deformed tile to the hat. If the deformed tile would admit a periodic tiling, the inverse deformation of this tiling would produce a periodic tiling by the hat, which by Theorem 7.6 does not exist. So the deformed tile does not admit a periodic tiling.

Note that in cases (i), (ii) and (iii) the inverse deformation is not well defined on tilings. In particular, in case (iii) the tile admits a valid tiling where two neighboring tiles meet with their "long" and "short" edges against each other. In the periodic tiling below the "long" edges are shown red, short edges are shown as light blue.



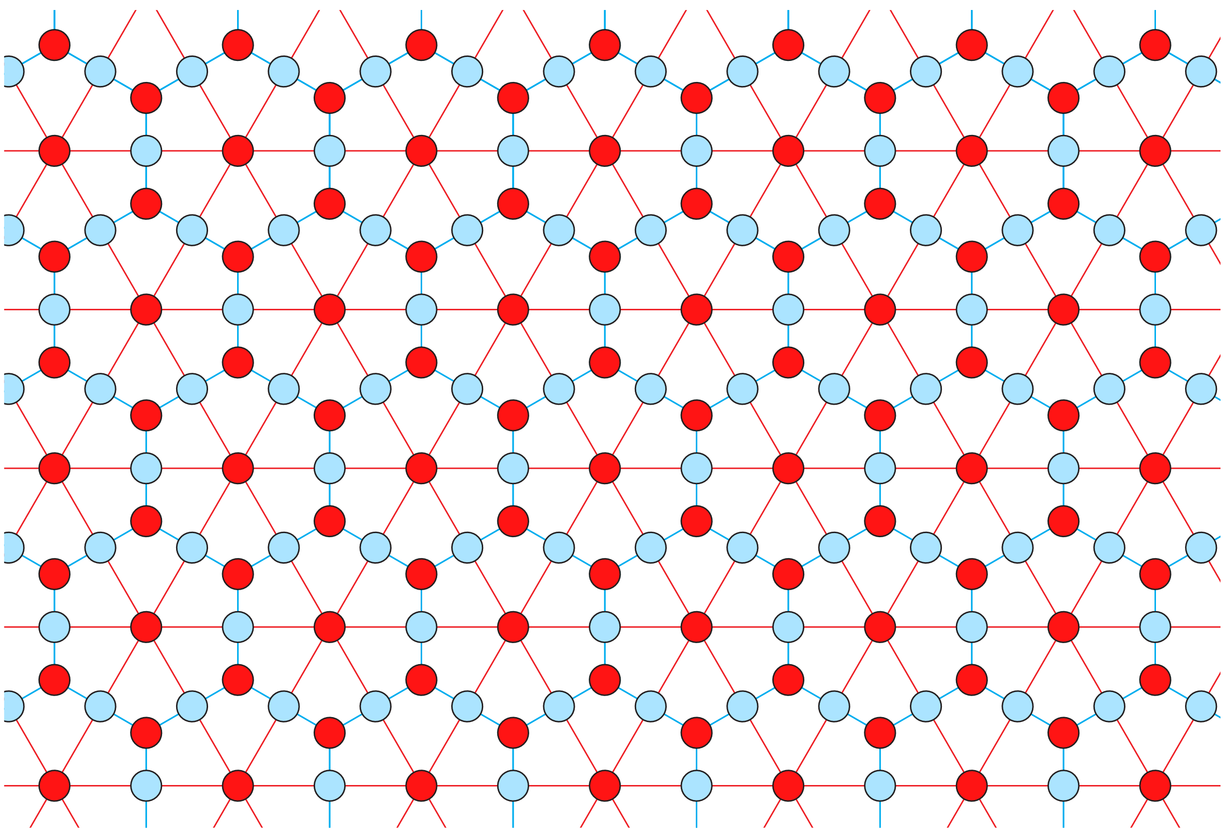
Note that there are several instances where a red edge meets a blue edge. In this situation the inverse deformation breaks the tiling. So this periodic tiling, of course, does not provide a periodic tiling by the hat.

Note that in the periodic tiling above both even and odd (=flipped over) variants of the tile are used. It turns out (proof skipped) that the tile also admits a tiling using only the even variants, and that all tilings using only the even variants are non-periodic! Using a standard bumps and dents construction on the edges we can prevent the even and the odd variants from appearing in the same tiling. In this construction the new shape of the edge is symmetric under a half turn so that the even tiles still match with each other along their edges:

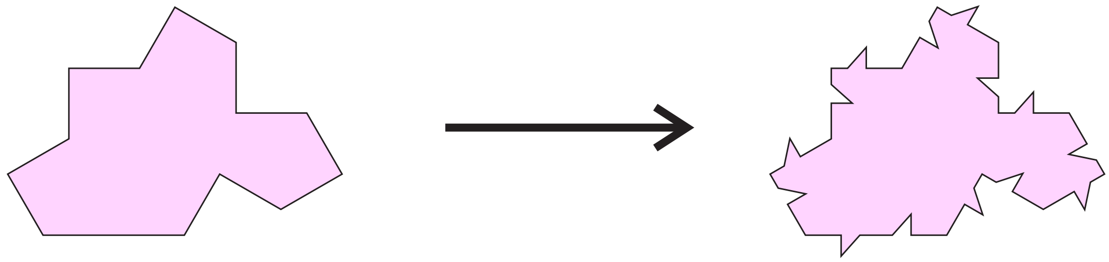


This tile is thus aperiodic monotile in the following strong sense: it admits a tiling, no tiling involves both even and odd variants of the tile, and there is no valid periodic tiling. The tile is called *spectre* (as is any of the analogous variants where differently curved edges are used to prevent even and odd tiles next to each other).

One more interesting observation can be made: The original kite grid is bipartite in the sense that we can color red and color blue those vertices where edges meet at angles that are multiples of 120° and multiples of 90°, respectively, and each edge then connects a red and a blue vertex:

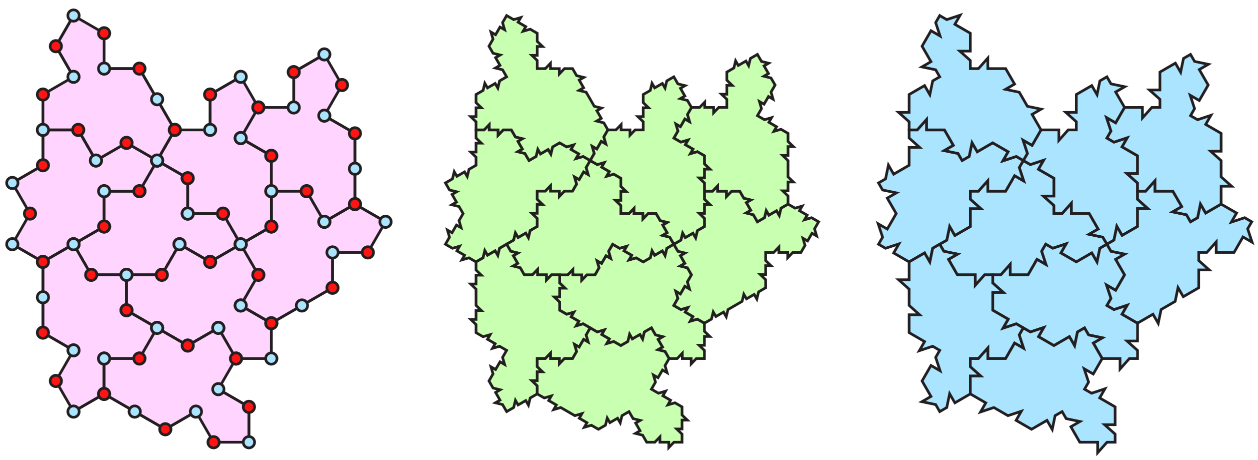


Thus, if we color vertices of the hat tile alternatingly red and blue, in any valid tiling by the hat the meeting vertices have the same color. The same is then true on the deformed hat. Consequently, the bumps and dents construction above can be simplified so that the new shape of the edge does not need to be symmetric under a half turn, but instead the edges of the tile are alternatingly the shape and its half turned variant:



This tile has the same spectre properties: it admits a tiling, no tiling involves both even and odd variants of the tile, and there is no valid periodic tiling.

The following picture shows corresponding patches using the deformed hat (left, including coloring of the vertices), spectre with half turn symmetric edge shapes (middle) and spectre with the simpler edge shapes that are not symmetric but alternate in consecutive edges (right).



7.5 Open problems

The aperiodic monotile "hat" discussed above is a 13-gon. A natural follow-up problem is to try to reduce the number of sides. Quadrilaterals tile the plane periodically, so the smallest possible number of sides on a polygonal aperiodic monotile is at least 5.

Open problem *What is the smallest n such that there exists an aperiodic n-gon ? Does there exist an aperiodic pentagon ?*

The construction from Wang tiles to polygons with bumps and dents is clearly effective, which means that it can be executed mechanically by an algorithm. Consequently, the undecidability results proved for Wang tiles hold for polygonal prototiles as well. In decision problems whose input consist of polygons we must use a finite way of representing the polygons. A natural way is to assume that the input polygons are such that the vertices of the polygons have rational coordinates, and the encoding then is a list of consecutive vertices. The bumps and dents can be done with rational coordinates so we immediately get the following undecidability results, corresponding to Theorems 5.7 and 5.8.

Theorem 7.7 *The following decision problems are undecidable:*

- *"Does a given protoset of polygons with rational coordinates admit a periodic tiling ?",*
- *"Does a given protoset of polygons with rational coordinates admit a tiling ?".*

Proof. Suppose an algorithm A exists for one of the given problems. Then we can solve the analogous problem on Wang tiles: For the given Wang protoset we use the bump/dent construction to form a protoset of polygons, and we give this protoset as input to the hypothetical algorithm A. The answer from algorithm A tells whether the Wang protoset admits a (periodic) tiling. This contradicts Theorem 5.7 or 5.8.

□

It is not known if there exists a decision algorithm to determine if a given single polygonal prototile admits a valid (periodic) tiling. Note that with Wang tiles the question is trivial, as for every k there are only a finite number of non-isomorphic protosets with k tiles, and consequently the decision problems are decidable. But there are infinitely many different polygons, so there is no trivial reason why there would exist an algorithm to tell even if a single tile admits a tiling. The aperiodic monotile "hat" provides the necessary pre-condition for undecidability.

Open problem *Is the following decision problem decidable ?*

- *"Does given single polygon with rational coordinates admit a tiling ?"*

What about the same question restricted to pentagons ?

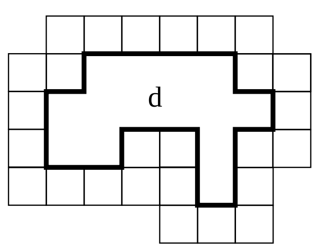
□

It has been shown by N. Ollinger that the tiling problem is undecidable among protosets that contain 5 polyominoes. (A polyominoe is a tile obtained by edge-to-edge attachments of any number of unit squares to each other.) In particular, this implies the undecidability of the tiling problem among sets of 5 polygons. On the other hand, the tiling problem is known to be decidable for single polyominoes if only translations are allowed, that is, the tiles must be placed in the given orientation.

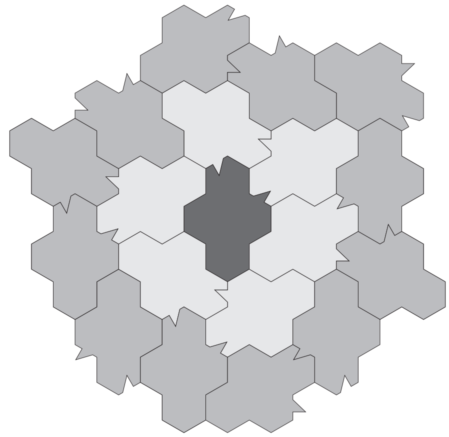
As a related question, consider the following problem by H. Heesch. Given a prototile t that does not admit a tiling of the plane, the Heesch number of t is the maximum number of times the tile can be completely surrounded by copies of t. More precisely, for a topological disk $d \subseteq \mathbb{R}^2$, a *corona* of d is a collection C of tiles, all congruent to t, such that

1. the interiors of the elements of C are pairwise disjoint, and disjoint from d, and
2. $d \cup \bigcup_{s \in C} s$ is a topological disk whose interior contains d.

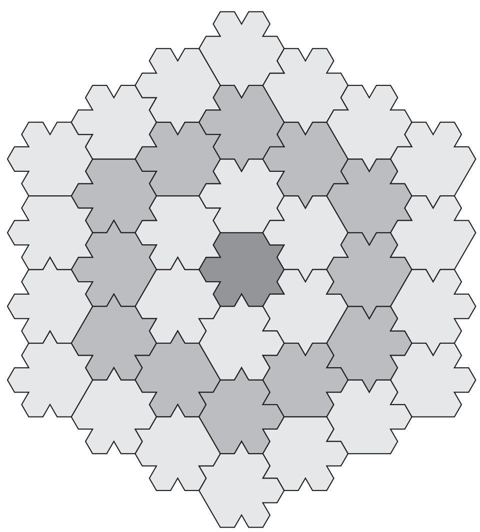
In other words, tiles in the corona C surround set d completely. For example, the squares in the following figure form a corona of the set d in the middle:



We can define a second corona of d as a corona of the set that is the union of d and its first corona. Inductively, a $k+1$'st corona is a corona of the topological disk formed by d and its first k coronas. In the Heesch problem we start with a single copy of t and form its 1st, 2nd, 3rd, etc. coronas. If the k'th corona exists for every k then by Theorem 7.4 the entire plane can be tiled. But if t does not admit a plane tiling then there exists the largest k such that the first k coronas exist. This k is called the *Heesch number* of tile t. The following figure illustrates two coronas of a tile:

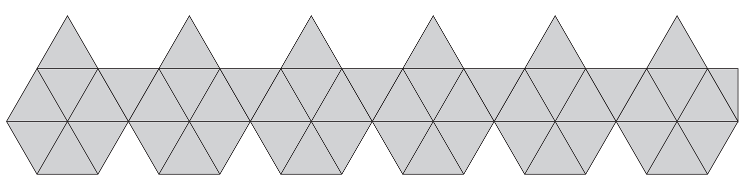


A regular hexagon with incoming arrows on three sides and outgoing arrows on two sides admits three coronas:

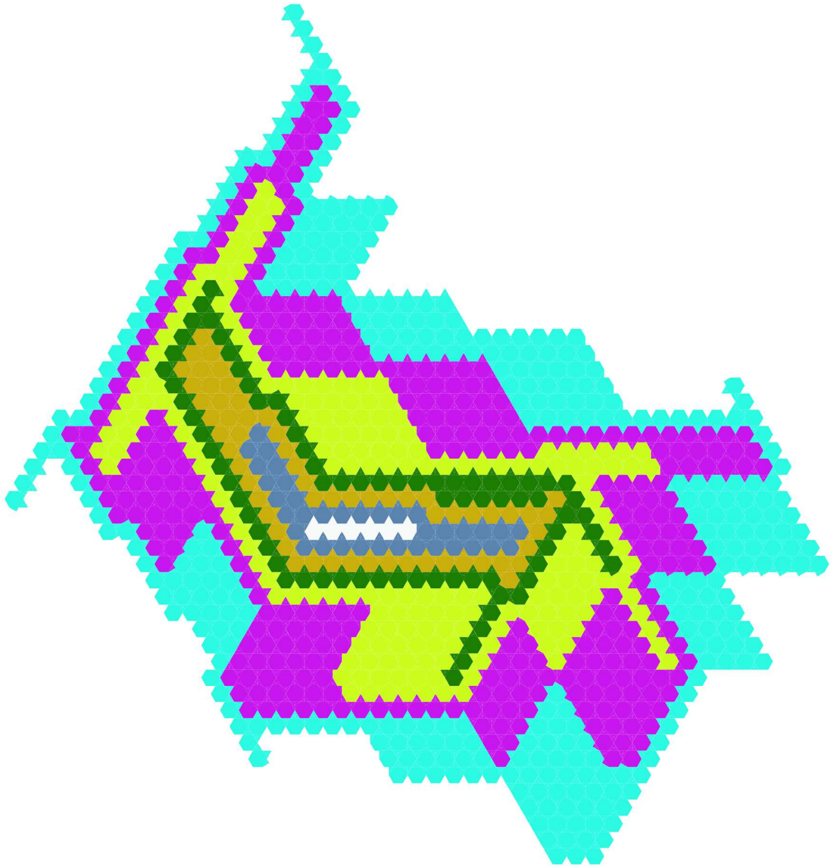


(In the picture, the arrows are represented by bumps and dents.) Due to the imbalance in the number of incoming and outgoing arrows, it is easy to apply the argument used earlier in the homeworks to conclude that the full plane cannot be tiled by this tile.

At the present time the largest known Heesch number is six. This is is a tile reaching that number:



Here's a picture showing the maximum number of coronas :



[Bašić, Bojan – Smaller version of https://en.wikipedia.org/wiki/Heesch]

Open problem *Does there exist number k such that the Heesch number of every tile that does not admit a tiling is at most k ? If such a k exists, what is the smallest such k ?*

□

Note that if the Heesch numbers are bounded by some constant k then there is an algorithm (at least in any reasonable set-up such as edge-to-edge tilings by polygons where one can try all possible coronas) to determine if a given single tile admits a tiling: To test if a tiling exists, all we need to do is to try all possible ways of building $k+1$ coronas. A valid tiling exists if and only if $k+1$ coronas exist.