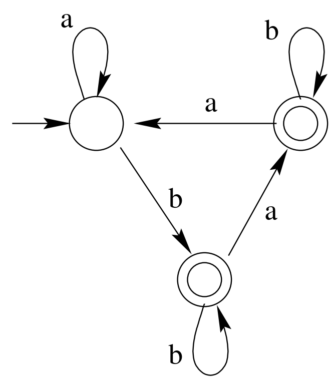
Automata and Formal Languages. Homework 4 (7.10.2024)

1. Let L be the language recognized by the following DFA:



   Construct a DFA that recognizes the language

   (a) $h^{-1}(L)$ where h is the homomorphism $h(0) = bab$, $h(1) = aba$.

   (b) L/baa^*.

2. Prove that if $L \subseteq \Sigma^*$ is regular so is the language

   (a) $\text{Prefix}(L) = \{u \in \Sigma^* \mid \exists v \in \Sigma^* : uv \in L\}$,

   (b) $\text{Suffix}(L) = \{v \in \Sigma^* \mid \exists u \in \Sigma^* : uv \in L\}$,

   (c) $\text{Subword}(L) = \{w \in \Sigma^* \mid \exists u, v \in \Sigma^* : uwv \in L\}$.

3. Show that the family of regular languages is <u>not</u> closed under the following operations:

   (a) $\text{Sort}(L) = \{a^n b^m \mid \text{L contains a word with exactly } n \text{ letters } a \text{ and } m \text{ letters } b\}$,

   (b) $\text{Shuffle}(L) = \{u \mid \text{letters of } u \text{ can be reordered to give a word in } L\}$.

   For example,

$$\begin{aligned}\text{Sort}(\{bab, aba, bba\}) &= \{abb, aab\},\\ \text{Shuffle}(\{bab, aba, bba\}) &= \{abb, bab, bba, aab, aba, baa\}.\end{aligned}$$

4. Prove that the given language L is not regular. You may rely on any results proved in the class or in the homework sessions.

   (a) $L = \{a^{3p} \mid p \text{ is a prime number}\}$,

   (b) $L = \{a^{n^2+1} \mid n \geq 1\}$,

   (c) $L = \{w \in \{a,b\}^* \mid \text{the number of } a\text{'s in } w \text{ and the number of } b\text{'s in } w \text{ are not relatively prime with each other}\}$.

5. Show that the family of regular languages is closed under the following operations:

   (a) $\text{Half}(L) = \{w \mid ww \in L\}$,

   (b) $\text{Merge}(L_1, L_2) = \{u_1 v_1 u_2 v_2 \ldots u_k v_k \mid u_1 u_2 \ldots u_k \in L_1, v_1 v_2 \ldots v_k \in L_2\ \}$.

   (Language $\text{Half}(L)$ contains all words such that ww is in L, and language $\text{Merge}(L_1, L_2)$ contains all words that can be obtained by shuffling together the letters of some words $u \in L_1$ and $v \in L_2$. The merge needs not to be perfect, i.e., the words u_i and v_i in the definition are elements of Σ^* and not necessarily single letters.)

6. Prove that if A_1 and A_2 are DFA with the same input alphabets and with n_1 and n_2 states, respectively, then there is

   (a) a DFA with $n_1 n_2$ states that recognizes $L(A_1) \cup L(A_2)$, and

   (b) a DFA with $n_1 n_2$ states that recognizes $L(A_1) \cap L(A_2)$.

7. Consider the language

$$L = \{a^n b^m c^k \mid n, m, k \geq 0, \text{ and if } n = 1 \text{ then } m = k\ \}.$$

   (a) Prove that L is not regular.

   (b) Prove that L satisfies the pumping lemma of regular languages:

$$\begin{aligned}
&\exists n \in \mathbb{N} \\
&\quad \forall z \in L : |z| \geq n \\
&\qquad \exists u, v, w \in \{a, b, c\}^* : z = uvw, |uv| \leq n, v \neq \varepsilon \\
&\qquad\quad \forall i \in \mathbb{N} \\
&\qquad\qquad uv^i w \in L.
\end{aligned}$$