Automata and Formal Languages. Homework 8 (4.11.2024)

1. Use the pumping lemma to prove that the following languages are not context-free:
   (a) $\{a^p \mid p \text{ is a prime number }\}$,
   (b) $\{a^n b^{n^2} \mid n \geq 0\}$.

2. Use the pumping lemma to prove that the following languages are not context-free:
   (a) $\{w \in \{a, b, c\}^* \mid \ |w|_a = |w|_b = |w|_c \}$,
   (b) $\{v\#u \mid u, v \in \{a, b\}^* \text{ and } u \text{ is a subword of } v \}$.

3. Determine whether the following languages are context-free or not. Prove your claim.
   (a) $\{a^n b^m \mid n \neq m\}$,
   (b) $\{a^n b^m c^k \mid n \leq m \leq k\}$.

4. A context-free grammar is called linear if in all productions $A \longrightarrow \alpha$ of the grammar the righthand side α contains at most one variable, i.e., $\alpha = uBv$ or $\alpha = w$ where $B \in V$ and $u, v, w \in T^*$. Let us call a language linear if it is generated by a linear grammar. Prove the following pumping lemma for linear languages:

   If L is linear then there exists a positive number n such that every word z of length n or greater that belongs to language L can be divided into five segments

   $$z = uvwxy$$

   in such a way that

   $$\begin{cases} |uvxy| \leq n, \text{ and} \\ v \neq \varepsilon \text{ or } x \neq \varepsilon, \end{cases}$$

   and for all $i \geq 0$ the word $uv^i wx^i y$ is in the language L.

   (The only difference to the pumping lemma for CF languages is that the condition $|vwx| \leq n$ has been replaced by the condition $|uvxy| \leq n$ that bounds the distance of w from the beginning and the end of the word z.)

5. Prove that there is a linear language that is not regular, and that there is a context-free language that is not linear. (Hint: the pumping lemma in the previous problem can be used to show, for example, that the language of all words with equally many a's and b's is not linear.)

6. Let us say that language L is deterministic context-free if it is recognized by a deterministic PDA, using the final state acceptance mode. Prove: If

   $$L = \{a^n b^n \mid n \geq 0\} \cup \{a^n b^{2n} \mid n \geq 0\}$$

   is deterministic context-free then

   $$L \cup \{a^n b^n c^n \mid n \geq 0\}$$

   is context-free. (Hint: Construct a PDA that non-deterministically may decide at a suitable moment to start treating input letters c as if they were b's.)

7. Show that

$$\{a^n b^n \mid n \geq 0\} \cup \{a^n b^{2n} \mid n \geq 0\} \cup \{a^n b^n c^n \mid n \geq 0\}$$

is not a context-free language. Using this fact and Problem 6 above, show that there are context-free languages that cannot be recognized by a deterministic PDA.