Automata and Formal Languages. Homework 11 (25.11.2024)

In these exercises, (semi-)algorithms may be described informally, without constructing concrete Turing machines unless specifically asked.

1. (Do not use Rice's theorem in this problem.) Determine (and prove your claim) whether the given decision problem is decidable or undecidable.
   (a) "Does a given Turing machine M accept some word with letter a in it ?"
   (b) "For a given Turing machine M, does there exist an input word w such that M halts within the first three steps on input w ?"
   (c) "For a given Turing machine M, does there exist an input word w such that $w \in L(M)$ but $ww \notin L(M)$?"
2. Function $g : \mathbb{N} \longrightarrow \mathbb{N}$ is *total recursive* or *total computable* if – informally – there exists an algorithm with natural number inputs and outputs such that for every $n \in \mathbb{N}$ the number $g(n)$ is the output corresponding to input n.

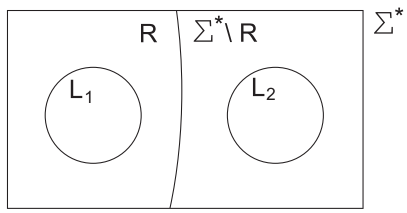
   Let $g_0, g_1, g_2, \ldots$ be an enumeration of all total recursive functions. Prove that the function
   $$g(n) = g_n(n) + 1$$
   is not total recursive. Using this, show that there is no algorithm that computes $g_n(m)$ for any given n and m.
3. Consider Turing machines with only two tape letters a and B, with B being the blank symbol. Prove that there is no algorithm to determine if such a TM halts when started on the blank tape.
4. For any positive integer n, let $TM(n)$ denote the set of all Turing machines that (i) have n non-final states (and one final state) (ii) have two tape symbols (one of which is the blank symbol), and (iii) halt when started on the blank tape. In other words, Turing machines of $TM(n)$ are of the form $(\{q_1, q_2, \ldots, q_n, f\}, \{a\}, \{a, B\}, \delta, q_1, B, f)$ and they halt from the blank initial tape.

   Let $BB(n)$ be the maximum number of moves that any machine in $TM(n)$ makes before halting when started on the initially blank tape.
   (a) Show that $BB(3) \geq 21$. (In fact, it is known that $BB(2) = 6$, $BB(3) = 21$, $BB(4) = 107$ and $BB(5) = 47176870$. The exact value of $BB(6)$ is not known but it is known that $BB(6) \geq 10^{(10^{\cdot^{\cdot^{10}}}}$, a tower of 15 occurrences of 10.)
   (b) Prove that BB is not total recursive (see problem 2 above) and that, in fact, for any total recursive g there exists n such that $BB(n) > g(n)$.

5. Are the families of recursive and r.e. languages closed under concatenation and the Kleene closure * ? Give a proof in each case (four combinations).

6. Are the families of r.e. and recursive languages closed under homomorphisms and inverse homomorphisms ? Give a proof in each case (four combinations).

7. Two disjoint languages L_1 and L_2 over alphabet Σ are *recursively separable* if there exists a recursive language R such that $L_1 \subseteq R$ and $L_2 \subseteq \Sigma^* \setminus R$:



Otherwise, disjoint L_1 and L_2 are *recursively inseparable.*

(a) Prove that if L_1 and L_2 are disjoint and their complements are r.e. then L_1 and L_2 are recursively separable.

(b) Prove that there exist disjoint r.e. languages that are recursively inseparable. Hint: Let

$$\begin{aligned} L_1 &= \{\langle M\rangle \mid M \text{ accepts } \langle M\rangle \}, \\ L_2 &= \{\langle M\rangle \mid M \text{ halts on input } \langle M\rangle \text{ in a non-final state } \}. \end{aligned}$$

Use a diagonal argument to show that there is no Turing machine M_x that halts on all inputs such that $L_2 \subseteq L(M_x)$ and $L_1 \cap L(M_x) = \emptyset$. (Ask the question whether $\langle M_x\rangle$ is accepted by M_x or not.)