Automata and Formal Languages. Homework 12 (2.12.2024)

In these exercises, (semi-)algorithms may be described informally, without constructing concrete Turing machines unless specifically asked.

1. (Do not use Rice's theorem in this problem.) Let $L = \{\langle M \rangle \mid L(M) \text{ is finite } \}$.
   (a) Prove that L is not recursively enumerable. (Hint: reduce the complement of L_u, as in Example 108 in the notes.)
   (b) Prove that the complement of L is not recursively enumerable. (Hint: again, reduce the complement of L_u. Effectively construct a machine that uses its own input to bound how long a given Turing machine M is simulated on its given input w.)
2. Which of the following questions are undecidable by the Rice's theorem.
   (a) "Does a given Turing machine M accept some word with letter a in it ?",
   (b) "Does a given Turing machine M recognize $\{a^n b^n \mid n \geq 1\}$?"
   (c) "Does a given Turing machine M eventually read a blank tape symbol when started on a given input word w ?"
   (d) "Does given Turing machine M accept at least one word w such that ww is not accepted by the same machine M ?"
3. Consider the semi-Thue system with the following two rewrite rules:

$$\begin{aligned} abb &\longrightarrow bab \\ baa &\longrightarrow aba \end{aligned}$$

   (a) Show that $a^4 b^4 \Longrightarrow^* (ab)^4$.
   (b) Show that every derivation sequence eventually terminates, that is, show that there does not exist an infinite sequence $w_1 \Rightarrow w_2 \Rightarrow w_3 \Rightarrow \ldots$. Use this fact to show that every word with equally many a's and b's derives a word in the language $(ab)^* + (ba)^*$.
4. Determine (and prove) if the following decision problems are decidable or not:
   (a) "Given a semi-Thue system T, does there exist at least one non-empty word w such that $w \Longrightarrow^* \varepsilon$?"
   (b) "Given a semi-Thue system T and a word w, does $w \Longrightarrow^+ w$?"
5. Construct a type-0 grammar that generates the language $\{ww \mid w \in \{a,b\}^* \}$.

6. A type-0 grammar $G = (V, T, P, S)$ is *context-sensitive* if $|u| \leq |v|$ in every production $u \longrightarrow v$ of P. A language L is a context-sensitive language (CSL) if it is generated by a context-sensitive grammar.

   (a) Show that every context-free language that does not contain the empty word ε is context-sensitive.

   (b) Show that the language $L = \{a^n b^n c^n \mid n \geq 1\}$ is context-sensitive.

   (c) Show that every context-sensitive language is recursive, by showing that the membership problem is decidable.

7. Show that the family of context-sensitive languages (see the previous problem) is closed under the following operations:

   (a) union (If L_1 and L_2 are CSL then also $L_1 \cup L_2$ is a CSL)

   (b) concatenation (If L_1 and L_2 are CSL then also $L_1 L_2$ is a CSL),

   (c) positive closure (If L is a CSL then L^+ is a CSL).